

Reliability Assessment under Incomplete Information:

An Evaluative Study

by


Ruth Hernandez Ruiz

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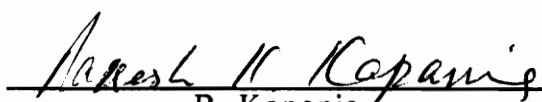
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APPROVED:



E. Nikolaidis, Chairman

R. T. Haftka

R. Kapania

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Ruth Hernandez Ruiz

E. Nikolaidis, Chairman

Aerospace and Ocean Engineering

(ABSTRACT)

Traditionally, in reliability design, the random variables acting on a system are assumed independent. This assumption is usually poor because in most real life problems the variables are correlated. The available information, most of the time, is limited to the first and second moments. Very few methods can handle correlation between the variables when the joint probability density function is unknown. There are no reports that provide information of the accuracy of these methods.

This work presents an evaluative study of reliability under incomplete information, comparing three existing methods for calculating the probability of failure: The method presented by Ang and Tang which assumes the correlation between the variables to be invariant; Kiureghian and Liu's method which accounts for the change in correlation and; Rackwitz's method under the assumption of independence. We have also developed a new algorithm to generate random samples of correlated random variables when the marginal

distributions and correlation coefficients of these variables are specified. These samples can be used in Monte Carlo simulation which is a tool for comparison of the three methods described above. This Monte Carlo simulation approach is based on the assumption of normal joint probability density function as considered by Kiureghian and Liu. To examine if this approach is biased towards Kiureghian and Liu, a second Monte Carlo simulation approach with no assumption about the joint probability density function is developed and compared with the first one.

Both methods that account for correlation show a clear advantage over the traditional approach of assuming that the variables are independent. Moreover, Kiureghian and Liu's approach proved to be more accurate in most cases than Ang and Tang's method.

In this study, it is also shown that there is an error in calculating the safety index for correlated variables when either one of the methods in study is implemented, because the joint probability density function of the random variables is neglected.

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This thesis is dedicated to my parents, my brother Jesus, and Jaime.

Nomenclature

[C] Covariance matrix of the original variables;

[C'] Covariance matrix of the reduced variables;

$Cov(X_i, X_j)$ Covariance between variables X_i and X_j ;

$Cov(Z_i, Z_j)$ Covariance between variables Z_i and Z_j ;

d Distance from the origin to the limit state of the system;

[D] Diagonal matrix of the standard deviations;

[D'] Diagonal matrix of the equivalent standard deviations;

$f_X(x)$ Probability density function of \mathbf{X} ;

$f_{X_i, X_j}(x_i, x_j)$ Joint probability density function of X_i and X_j ;

$F_{X_i}(x_i)$ Original cumulative distribution of X_i ;

g Limit state function;

$J(x_i, x_j)$ Jacobian of x_i and x_j ;

[L] Matrix of the Cholesky decomposition of [R];

\mathbf{M} Vector of the mean values;

\mathbf{M}' Vector of the equivalent means;

p_f Probability of failure;

R Resistance;

R_i Reliability;

[R] Correlation matrix;

S Load;

[T] Orthogonal transformation matrix;

v_i Correlated normal random variables;

W	Vector of independent normal random variables;
x_i^*	Most probable failure point in the standard space;
$x_i^{(j)}$	Checking point in the original space in j^{th} iteration;
X	Vector of basic random variables;
y_i^*	Most probable failure point in the space of uncorrelated reduced variables;
$y_i^{(j)}$	Checking point in the space of uncorrelated reduced variables in j^{th} iteration;
Y	Vector of uncorrelated transformed variables;
$z_i^{(j)}$	Checking point in the reduced space in j^{th} iteration;
Z	Vector of reduced normal variables;
α	Direction of cosines;
β	Safety index;
δ_i	Coefficient of variation;
∇g	Gradient vector of g ;

$[\lambda]$	Diagonal matrix of the eigenvalues of $[C']$;
μ	Mean value;
μ_x^N	Equivalent normal mean;
$\mu_{y x}$	Conditional mean;
ρ_{ij}	Correlation coefficient;
ρ_{oij}	Transformed correlation coefficient;
σ	Standard deviation;
σ_x^N	Equivalent standard deviation;
$\sigma_{y x}$	Conditional mean;
ϕ	Probability density function of the standard normal random variable;
Φ	Cumulative distribution of the standard normal random variable.

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Chapter 1

Introduction

1.1 Structural Reliability Analysis: Identification of the Problem

Uncertainties are involved in almost every problem where we have to design a structure. Problems of structural design must be resolved in the face of uncertainty and as a consequence risk, is unavoidable. For example when we design an airplane, we do not know the exact values of the applied loads. In some cases these uncertainties may be large, and therefore if we ignore them we can design a structure that is too expensive or even unsafe. This is why we use probabilistic methods. Probability is used to enhance realism and improve consistency in the treatment of uncertainty. Probabilistic methods use random

variables to account for the uncertainties and they assess the safety of a structure by estimating its failure probability. Most methods in reliability assessment are only applicable to cases where the random variables are independent, (e.g., Cornell⁵, Rackwitz and Fiessler²³). However, in reality random variables are usually correlated. There are two methods that account for correlation: Ang and Tang² and Kiureghian and Liu¹³. Both methods account for the correlation coefficients of the random variables only because in real life we do not have information for the joint probability distributions of the random variables. Little information is available on what is the error associated with the estimation of the failure probability using the marginal probability densities and the correlation coefficients of the random variables only. Moreover, nobody has evaluated Ang and Tang's and Kiureghian and Liu's methods.

1.2 Objective

In this work, we are going to show the importance of taking correlation into account and describe two existing methods for evaluating the reliability of a structure when the information available about the variables is incomplete:

1. Ang and Tang's approach
2. Kiureghian and Liu's approach

Both these methods can deal with correlated random variables under incomplete information. When the variables are both correlated and non-normal, they should be transformed first into equivalent normals and then to independent variables. In this case the correlation matrix of the original random variables should also be transformed into the correlation matrix of the equivalent normal random variables, but Ang and Tang keep the original correlation matrix through the whole procedure. The transformation of the correlation matrix is introduced by Kiureghian and Liu.

Our objective is to evaluate the two methods.

Specifically, we will try to answer the following questions:

1. How accurate are the two methods?
2. Is Kiureghian and Liu's method, which is more computational intensive than Ang and Tang's, more accurate than the later method?
3. How accurate are Kiureghian and Liu's and Ang and Tang's methods compared with a second order method that neglects the correlation between the random variables?
4. How large can be the error in estimating the probability of failure if we use the marginal probability distributions and correlation coefficients of the random variables?

Several examples were chosen and solved using Kiureghian and Liu and Ang and Tang's algorithms. We also developed a new algorithm to generate random samples with arbitrary distributions and given correlation coefficients. These samples can be used for Monte Carlo simulation. Note that these samples were first generated assuming the same joint probability density function as Kiureghian and Liu. This study compares Kiureghian and Liu's and Ang and Tang's methods with Monte Carlo simulation and also with a second order method that neglects the correlation between random variables. Then, the assumption that the joint distribution of the transformed variables is normal was relaxed and a second approach for Monte Carlo simulation was developed. We used this approach to corroborate the results obtained using the first method for Monte Carlo simulation and to assess the error resulting from not knowing the joint probability distributions of the random variables.

1.3 Outline

In this work an evaluative study of reliability under incomplete information is presented.

First, we review some basic concepts in structural design, the first order second moment method and three important studies on this field. These studies are:

1. Cornell
2. Hasofer and Lind
3. Rackwitz and Fiessler

Next, we will show the importance of accounting for correlation between the variables and we will present two existing algorithms that can handle this type of problem when the information about the random variables is incomplete:

1. Ang and Tang
2. Kiureghian and Liu

Following these theoretical descriptions we will describe two new Monte Carlo simulation algorithms that will allow us to compare and check the accuracy of the two methods that account for correlation between the variables and a second order method that neglects correlation.

To evaluate the above methods, we select some examples, apply the methods to these examples and compare the results.

Chapter 2

Structural Design

2.1 *Introduction*

In classical structural design, the variables are deterministic. This simplifies the problem, but can lead to non-optimal or even unsafe designs. Probabilistic methods account for more information than deterministic methods, and as a result they yield more economical and safe designs than deterministic methods.

The random variables $\mathbf{X} = (X_1, X_2, \dots, X_n)$ used in modeling the uncertainties of a system are called basic variables.

All the elements in a structure can be modelled by failure modes, each of which is called failure element. For each failure element i , $i = 1, 2, \dots, n$ there is a

function g_i , called performance or limit state function, such that the element fails if $g_i \leq 0$ and survives if $g_i > 0$.

For example, consider the most basic problem in reliability, where the load on a structure can be expressed as S , and the strength (resistance) can be expressed as R . A structural element will fail if its resistance, R , is less than S . The probability of failure, p_f , of a structural element can be stated as:

$$p_f = P(R - S \leq 0) = P[g(R,S) \leq 0] \quad [2.1.1]$$

or in general

$$p_f = P[g(\mathbf{X}) \leq 0] \quad [2.1.2]$$

where g is the limit state function or performance function and $P[.]$ is the probability that the function $g(\mathbf{X})$ is less or equal than zero.

For an element i , the probability of failure can be calculated using the following equation:

$$p_{f,i} = \int_{g(x) < 0} f_{\mathbf{X}}(x) dx \quad [2.1.3]$$

where $f_{\mathbf{X}}(x)$ denotes the joint probability density function of \mathbf{X} , and the integral is over the failure set. \mathbf{X} is the set of random variables and x is a component of the set.

The reliability is:

$$R_i = 1 - p_{f,i} \quad [2.1.4]$$

If we have more than two random variables, which is usually the case in a real life problem, it may be difficult to evaluate the integral in Eq. [2.1.3]. Approximate methods have been developed to evaluate $p_{f,i}$ and R_i .

The safety index β , is used to measure structural safety. The probability of failure of a structure is defined then by the following equation:

$$p_f = \Phi(-\beta) \quad [2.1.5]$$

where $\Phi(\cdot)$ is the cumulative distribution of the standard normal (i.e normal with zero mean and unit standard deviation). $\Phi(\cdot)$ is shown in Fig.(1). Clearly, the safety index is a monotonically decreasing function of the failure probability. For typical engineering structures (buildings, bridges, offshore platforms...), the safety index ranges from 3 to 8.

For example, consider a structure where the limit state function could be represented as before:

$$g(R,S) = R-S$$

Assuming the variables to be independent normal, the mean and standard deviation of the function g can be expressed as follows:

$$\mu_g = \mu_R - \mu_S \quad [2.1.6]$$

$$\sigma_g^2 = \sigma_R^2 + \sigma_S^2 \quad [2.1.7]$$

The probability of failure could be expressed as:

$$p_f = P((R - S) \leq 0) = \Phi\left(\frac{0 - \mu_g}{\sigma_g}\right) \quad [2.1.8]$$

The ratio $\frac{\mu_g}{\sigma_g}$ is equal to the safety index, β .

When the vector \mathbf{X} of random variables is normally distributed, we could calculate the reliability of the system using Eq. [2.1.8]. If this was not the case, some type of transformation (e.g., Rosenblatt²⁴) should be applied to transform the variables into normal.

2.2 *Second Moment Method*

As mentioned before, the evaluation of Eq. [2.1.3] requires knowledge of the probability distributions. This information is often not available.

The second moment method^{2,6} is an approach to estimate reliability when the information available is limited to the mean and variance of the variables, i.e first

and second moments respectively. Since the joint probability distribution function of the random variables is unknown, we assume that the random variables are independent normal.

Let us again consider the case where $g = R - S$. By introducing the reduced normal variates, Z_R and Z_S defined as :

$$Z_R = \frac{R - \mu_R}{\sigma_R} \quad [2.2.1]$$

$$Z_S = \frac{S - \mu_S}{\sigma_S} \quad [2.2.2]$$

The limit state function $g(R,S) = 0$ becomes:

$$\sigma_R Z_R - \sigma_S Z_S + \mu_R - \mu_S = 0 \quad [2.2.3]$$

which is presented in Fig.(2). The distance from the line $g(R,S) = 0$, to the origin is the safety index , given as :

$$d = \beta = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} \quad [2.2.4]$$

Generally, the resistance, R , and load, S , are nonlinear functions of n random variables and the performance function should be defined as:

$$g(\mathbf{X}) = g(X_1, X_2, \dots, X_n) \quad [2.2.5]$$

where $\mathbf{X} = (X_1, X_2, \dots, X_n)$ is the set of random variables. The limit state of the system is $g(\mathbf{X}) = 0$.

The performance function could be expanded in a Taylor series at a point x^* , which is on the failure surface $g(x^*) = 0$

$$g(X_1, X_2, \dots, X_n) = g(x_1^*, x_2^*, \dots, x_n^*) + \sum_{i=1}^N (X_i - x_i^*) \left(\frac{\partial g}{\partial X_i} \right)^* \quad [2.2.6]$$

$$+ \sum_{j=1}^n \sum_{i=1}^n (X_i - x_i^*) (X_j - x_j^*) \frac{(\partial^2 g)}{(\partial X_i \partial X_j)^*} + \dots$$

The derivatives are evaluated at the most probable failure point x^* , (M.P.F.P.), which is the point on the failure surface, $g(\mathbf{X}) = 0$, with minimum distance to the origin. This distance will determine how far or close to the origin is the limit state surface and therefore delimiting the safe region $g(\mathbf{X}) > 0$, and the failure region $g(\mathbf{X}) < 0$. The M.P.F.P. represents the point at which the joint probability density of the random variables is larger than at any point on the limit state surface, [Shinozuka²⁵].

Noting that $g(x_1^*, x_2^*, \dots, x_n^*) = 0$ and truncating the approximation at the first order term, the mean value and standard deviation of the function $g(\mathbf{X})$ represented in the reduced variate space, which is the space where the variables

are normally distributed with zero mean and unit standard deviation, could be expressed as:

$$\mu_g \approx - \sum_{i=1}^n z_i^* \left(\frac{\partial g}{\partial Z_i} \right)^* \quad [2.2.7]$$

$$\sigma_g^2 \approx \sum_{i=1}^n \left(\frac{\partial g}{\partial Z_i} \right)^2 \quad [2.2.8]$$

respectively. Subscript * indicates that the derivatives are evaluated at the M.P.F.P..

As mentioned before, the safety index could be expressed as the ratio $\frac{\mu_g}{\sigma_g}$.

Therefore,

$$\beta = \frac{\mu_g}{\sigma_g} = \frac{- \sum_{i=1}^n z_i^* \left(\frac{\partial g}{\partial Z_i} \right)^*}{\sqrt{\sum_{i=1}^n \left(\frac{\partial g}{\partial Z_i} \right)^2}} \quad [2.2.9]$$

It should be noted that the first order approximation should be evaluated about the M.P.F.P.. In the first studies on structural reliability assessment via first order approximation (Cornell⁵, Ang and Cornell¹), the evaluation was done about the mean values.

When the limit state functions are nonlinear, a second order approximation is frequently used but other higher order approximations may be applied [Veneziano²⁶, Fiessler, et al.⁸, Horne and Price¹¹ , Berthelley and Rackwitz³].

If the random variables are not normal it is necessary to transform them into equivalent normal random variables [Paloheimo and Hannus²⁰, Rackwitz²², Rosenblatt²⁴], and apply the procedure that was described in the previous paragraphs. According to Rosenblatt, a non-normal distributed random variables u , can be converted to equivalent normal random variables x , through the relation:

$$x_1 = F_1^{-1}[\Phi(u_1)]$$

$$x_2 = F_2^{-1}[\Phi(u_2)/x_1] \quad [2.2.10]$$

etc

where $F_i(.)$ is the conditional cumulative distribution function and $\Phi(.)$ is the standard normal cumulative distribution.

In this section we are going to review some commonly used methods to assess the reliability of an element.

2.2.1 Cornell

Cornell⁵ focused on the evaluation of the safety factors based on the given uncertainties so that these safety factors are consistent with the prescribed levels of risk. The minimum information required to evaluate safety is, the expected value of each design variable, and a measure of uncertainty.

Under the code format proposed by Cornell, predictions on the resistance and load are made together with the measure of uncertainties of these predictions. The safety factors are determined based on the required degree of safety.

For each quantity we take as expected value the corresponding previous deterministic value. No probability distributions assumptions are made at any point and coefficients of variation are introduced to express the dispersion or uncertainty in the best estimates.

Consider the limit state function $g = R - S$, where R and S are normal variables. The design is considered satisfactory when the resistance, R is larger than the load, S . Therefore, survival is expressed as $P[R > S]$, where $P[.]$ denotes the "probability of an event".

The measure of safety will be the safety index β , which can be evaluated as follows:

$$P[R > S] = 1 - \Phi(-\beta) \quad [2.2.11]$$

$$\beta = \frac{\mu_R - \mu_S}{\sqrt{(\delta_R \mu_R)^2 + (\delta_S \mu_S)^2}} \quad [2.2.12]$$

where $\Phi(\cdot)$ is the cumulative distribution of the standard normal random variables, μ_R, μ_S are the mean values of the load and the resistance respectively, δ_R, δ_S are the coefficients of variation.

It is assumed that the expected values and coefficients of variation are known.

This approach is summarized in Fig.(5).

2.2.2 Hasofer and Lind

Hasofer and Lind⁹ also used the second-moment reliability approximation. Consider a structure having a deterministic resistance, R , and is subjected to a load, S . Failure occurs when $S > R$. The probability of the failure event is then $P[S > R]$. When this probability is acceptably small, the design is acceptable. Since, usually there is not enough information available on the tail distribution of S , this probability criterion can be replaced by:

$$R > \mu_s + \beta \sigma_s \quad [2.2.13]$$

where, μ_s and σ_s are the mean value and standard deviation of the load respectively, and β is again the safety index, or reliability index. The greatest value of β satisfying Eq. [2.2.13] is the reliability index of the design.

For the general case where there are any number of random variables, the reliability index is found using the following procedure:

Let $\mathbf{X} = (X_1, X_2, \dots, X_n)$ be the vector of random variables also called basic variables.

The failure criterion will be $g(\mathbf{X}) < 0$, such that the space of \mathbf{X} is divided into a safe region where $g(\mathbf{X}) > 0$, and a failure region, where $g(\mathbf{X}) < 0$.

In the following, we explain the algorithm for the calculation of the probability of failure which is summarized in Fig.(6):

1. In the case of correlated variables, the first step is to make an orthogonal transformation of the variables \mathbf{X} , to a new set \mathbf{W} of uncorrelated variables.

For each pair of variables, the covariance between them is expressed as:

$$E[(X - \mu_X)_i(X - \mu_X)_j] = Cov(X_i X_j) \quad i, j = 1, 2, \dots, n \quad [2.2.14]$$

Then:

$$\text{Let } E[(\mathbf{X} - \mathbf{M}_X)(\mathbf{X} - \mathbf{M}_X)^T] = [\mathbf{C}] \quad [2.2.15]$$

where \mathbf{M} is the vector of the mean values and $[\mathbf{C}]$ is the covariance matrix of \mathbf{X} . $[\mathbf{C}]$ is positive definite, and therefore there exists an orthogonal matrix $[\mathbf{T}]$ such that the diagonal elements of $[\mathbf{T}][\mathbf{C}][\mathbf{T}]^{-1}$ will be the eigenvalues of $[\mathbf{C}]$.

$$\text{Let } \mathbf{W} = [\mathbf{T}] \mathbf{X} \quad [2.2.16]$$

$$\text{Then } E[(\mathbf{W} - \mathbf{M}_W)(\mathbf{W}_W - \mathbf{M}_W)^T] = \mathbf{T} \mathbf{C} \mathbf{T}^{-1} \quad [2.2.17]$$

will be a diagonal matrix, and the transformed variables, \mathbf{W} will be the uncorrelated variables.

If the given variables \mathbf{X} are uncorrelated, $\mathbf{W} = \mathbf{X}$

2. Then, the reduced variables are introduced as:

$$Y_i = \frac{W_i - \mu_{W_i}}{\sigma_{W_i}} \quad i = 1, 2, \dots, n \quad [2.2.18]$$

3. The next step is to calculate the safety or reliability index which is defined as the distance from the origin to the failure surface in the space of the reduced random variables Fig.(3). This distance is equal to the length of vector OA which is normal to the curve $g(\mathbf{Z})=0$. Point A is called design or most probable failure point. The probability of failure is related to the reliability index as follows:

$$p_f = \Phi(-\beta) \quad [2.2.19]$$

The calculation of the safety index requires an iterative algorithm that can be described as follows:

Defining $g(\mathbf{y})=0$ as the limit state function in the reduced space, a first approximation to the point closest to the origin on $g(\mathbf{y})=0$ is made. Once this vector, $\mathbf{y}^{(j)}$, is selected, we look for a better approximation $\mathbf{y}^{(j+1)}$. To do this, we expand $g(\mathbf{y}^{(j+1)})$ by:

$$g(\mathbf{y}^{(j+1)}) = g(\mathbf{y}^{(j)}) + (\mathbf{y}^{(j+1)} - \mathbf{y}^{(j)})^T \nabla g(\mathbf{y}^{(j)}) = 0 \quad [2.2.20]$$

where $\nabla g(\mathbf{y}^{(j)})$ is the gradient of $g(\mathbf{y})$ at $\mathbf{y}^{(j)}$.

Since we try to find the distance of the origin to $g(\mathbf{y}^{(j+1)})$, $g(\mathbf{y}^{(j+1)}) = 0$, and we set:

$$g(\mathbf{y}^{(j)}) + (\mathbf{y}^{(j+1)} - \mathbf{y}^{(j)})^T \nabla g(\mathbf{y}^{(j)}) = 0 \quad [2.2.21]$$

$\mathbf{y}^{(j+1)}$ should satisfy:

$$\mathbf{y}^{(j+1)} = A \nabla g(\mathbf{y}^{(j)}) \quad [2.2.22]$$

where A is a scalar.

Then, by substitution of Eq. [2.2.22] in Eq. [2.2.21]:

$$g(\mathbf{y}^{(j)}) + \nabla g(\mathbf{y}^{(j)})^T (A \nabla g(\mathbf{y}^{(j)}) - \mathbf{y}^{(j)}) = 0 \quad [2.2.23]$$

$$g(\mathbf{y}^{(j)}) + \nabla g(\mathbf{y}^{(j)})^T \nabla g(\mathbf{y}^{(j)}) A - \nabla g(\mathbf{y}^{(j)})^T \mathbf{y}^{(j)} = 0$$

$$A = \frac{\nabla g(\mathbf{y}^{(j)})^T \mathbf{y}^{(j)} - g(\mathbf{y}^{(j)})}{\nabla g(\mathbf{y}^{(j)})^T \nabla g(\mathbf{y}^{(j)})}$$

Therefore from substitution of the above equation in Eq. [2.2.22], $\mathbf{y}^{(j+1)}$ is obtained from the expression:

$$\mathbf{y}^{(j+1)} = \frac{[\mathbf{y}^{(j)}]^T (\nabla g(\mathbf{y}^{(j)})) - g(\mathbf{y}^{(j)})}{[\nabla g(\mathbf{y}^{(j)})]^T \nabla g(\mathbf{y}^{(j)})} \nabla g(\mathbf{y}^{(j)}) \quad [2.2.24]$$

Since the direction of cosines is defined as:

$$\alpha = - \frac{\nabla g_Y(\mathbf{y}^{(j)})}{|\nabla g_Y(\mathbf{y}^{(j)})|} \quad [2.2.25]$$

the expression [2.2.22] becomes:

$$\mathbf{y}^{(j+1)} = [\mathbf{y}^{(j)}]^T \alpha^{(j)} + \frac{g(\mathbf{y}^{(j)})}{|\nabla g(\mathbf{y}^{(j)})|}] \alpha \quad [2.2.26]$$

This procedure is repeated until convergence is achieved.

2.2.3 Rackwitz and Fiessler

Rackwitz and Fiessler^{16,23} developed a first order reliability method algorithm to calculate the reliability of a design.

First order reliability methods have proved to be efficient tools to substitute for multidimensional integration for the calculation of reliability measures. The accuracy of these methods is in all but extreme cases sufficient for engineering purposes. In first order reliability methods, the problem of finding the reliability of a structure is that of finding the reliability index, β .

First we need to transform non-normal distribution functions into normal distribution functions. Given the means μ , and standard deviations σ , of the variables in the original space, it is necessary to obtain the equivalent means and standard deviations to work in the reduced space. These quantities can be obtained as follows:

$$\mu_i^N = x_i^* - \sigma_i^N \Phi^{-1}[F_X(x_i^*)] \quad [2.2.27]$$

$$\sigma_i^N = \frac{\phi\{\Phi^{-1}[F_X(x_i^*)]\}}{f_X(x_i^*)} \quad [2.2.28]$$

where x^* is the design point. The procedure to find x^* is presented later in this section. $F(\cdot)$ and $f(\cdot)$ are the distribution function and density function of the

non-normal distribution, respectively, $\phi(\cdot)$ is the standard normal density and $\Phi(\cdot)$ is the standard normal normal cumulative distribution.

The reduced variables Y_i , are introduced as:

$$Y_i = \frac{X_i - \mu_i^N}{\sigma_i^N} \quad i = 1, 2, \dots, n \quad [2.2.29]$$

The performance function in the original space is defined as:

$$g(\mathbf{X}) = g(X_1, X_2, \dots, X_n) \quad [2.2.30]$$

In terms of the normalized variables, this expression becomes:

$$g(\mathbf{X}) = g(Y_1\sigma_{X_1} + \mu_{X_1}, \dots, Y_n\sigma_{X_n} + \mu_{X_n}) \quad [2.2.31]$$

The failure surface in the standard space, and in the reduced space is shown in Fig.(4).

In the following we explain how to find the design point, x^* . The procedure is summarized in Fig.(7).

We start with an initial guess of $x^{(0)}$, $x^{(0)}$. We obtain the standard means and standard deviations from Eqs. [2.2.27] and [2.2.28], and the corresponding reduced variables $y^{(0)}$ from Eq. [2.2.29]. The point $y^{(1)}$ closest to the origin in the reduced space is determined.

Determination of the points closest to the origin in the reduced space requires an iteration algorithm. Rackwitz and Fiessler proposed the algorithm described in Eqs. [2.2.20-2.2.26].

From backward substitution, we obtain the closest point to the origin in the original space:

$$x_i^{(j+1)} = \sigma_i y_i^{(j+1)} + \mu_i \quad [2.2.32]$$

$$= \sigma_i y_i^{(j+1)} + x_i^{(j)} - \sigma_i \Phi^{-1}(F_{X_i}(x_i^{(j)}))$$

$$= x_i^{(j)} + \sigma_i (y_i^{(j+1)} - \Phi^{-1}(F_{X_i}(x_i^{(j)})))$$

$$= x_i^{(j)} + \sigma_i (y_i^{(j+1)} - y_i^{(j)})$$

Therefore, for the first iteration, the corresponding point in the original space becomes:

$$x_i^{(1)} = y_i^{(1)} \sigma_i^{(0)} + \mu_i^{(0)} \quad i = 1, 2, \dots, n \quad [2.2.33]$$

The safety index, β , is obtained from the equation:

$$x_i = \mu_{x_i} - \alpha_i \beta \sigma_{x_i} \quad i = 1, 2, \dots, n \quad [2.2.34]$$

The same procedure is repeated, first calculating the new equivalent means and standard deviation and then a new point $y^{(2)}$. By doing so, a sequence of points $x^{(0)}, x^{(1)}, \dots, x^{(j)}$, is created.

The key idea of the first order second moment methods is the linearization of the performance function through a polynomial expansion. In the first studies (Cornell), this linearization was made about the mean values. Hasofer and Lind, and Rackwitz and Fiessler linearized about the M.P.F.P..

Hasofer and Lind assume the variables to be normally distributed and introduce the feature of transforming the variables to uncorrelated when the covariance is known. Rackwitz and Fiessler transform non-normal variables into normal by calculating the equivalent means and standard deviations.

Chapter 3

Correlated Random Variables

3.1 Description of the problem

In the previous chapter, several methods have been presented to calculate the reliability of a structure when the given information is limited to the mean and covariance matrices.

Many times, in order to simplify the problem, the variables are assumed to be independent. Nevertheless, in real life this is rarely the case. Variables are often correlated and the assumption of independence can yield inaccurate results.

The following example shows, how important it is to account for the correlation of the variables.

Example. Consider the simplest case with only two variables, X_1 and X_2

In this example, X_2 is normally distributed, with a mean value, $\mu_{X_2} = 40$ lb, and standard deviation, $\sigma_{X_2} = 6$ lb. X_1 is also normally distributed with a mean value $\mu_{X_1} = 50$ lb and standard deviation $\sigma_{X_1} = 5$ lb. The variables are correlated with a correlation coefficient of $\rho_{X_1, X_2} = 0.5$ Therefore:

$$X_2 = N(40,6)$$

$$X_1 = N(50,5)$$

The probability of failure is defined as:

$$p_f = P((X_1 - X_2) \leq 0)$$

For normal variates, the probability of failure can be expressed as:

$$p_f = \Phi\left(\frac{\mu_g}{\sigma_g}\right)$$

where $g = X_1 - X_2$

a) Assuming independent variables:

$$\mu_g = \mu_{X_1} - \mu_{X_2} = 50 - 40 = 10\text{lb}$$

$$\sigma_g^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 = 25 + 36 = 61, \sigma_g = 7.8$$

$$p_f = \Phi\left(-\frac{10}{7.8}\right) = \Phi(-1.282) = 0.1003$$

b) Now, let us take into account that the variables are correlated with correlation coefficient, $\rho_{X_1, X_2} = 0.5$

$$\mu_g = \mu_{X_1} - \mu_{X_2} = 50 - 40 = 10\text{lb}$$

$$\sigma_g^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 - 2\rho_{X_1, X_2}\sigma_{X_1}\sigma_{X_2}$$

$$\sigma_g^2 = 25 + 36 - (2 \times 0.5 \times 5 \times 6) = 31, \sigma_g = 5.57$$

$$p_f = \Phi\left(-\frac{10}{5.57}\right) = \Phi(-1.796) = 0.0360$$

The error due to the assumption of independent random variables for this case is:

$$\text{Relative \% Error} = 178.6 \%$$

When the joint probability density functions (PDF) of the random variables are known, we can handle the problem. A dependent random vector \mathbf{X} may be transformed to uncorrelated through a Rosenblatt transformation. The problem arises when we have arbitrary distributions and the only information available is limited to the correlation coefficient between the variables. This is frequently the case and therefore we need to find a method that can handle the problem. In this section, two methods are presented:

1. Ang and Tang's approach
2. Kiureghian and Liu's approach

The first method reduces to the one developed by Rackwitz²², when the variables are independent normals. Kiureghian and Liu's approach includes the feature of transforming not only the means and standard deviations of the variables to normal but also the correlation coefficients. A new Monte Carlo simulation algorithm is also presented and is considered the exact method.

3.2 Ang and Tang Approach

The algorithm presented by Ang and Tang² to calculate the probability of failure is based on the one developed by Rackwitz²².

Here, we are presenting the case where the variables are correlated and non-normals.

A performance function is defined as:

$$g(\mathbf{X}) = g(X_1, X_2, \dots, X_n) \quad [3.2.1]$$

where $\mathbf{X} = (X_1, X_2, \dots, X_n)$ is the vector of the random variables and $g(\mathbf{X})$ determines the performance of the system, and therefore the limit state or limiting performance of the system will be:

$$g(\mathbf{X}) = 0 \quad [3.2.2]$$

$[g(\mathbf{X}) > 0]$ defines the "safe state" and $[g(\mathbf{X}) < 0]$, the "failure state".

The position of the failure surface may be represented by the minimum distance from the surface $g(\mathbf{X}) = 0$ to the origin in the space of the random variables. In the reduced space, the distance of any point of the surface $g(\mathbf{Z}) = 0$ to the origin may be determined as follows:

$$d = (\mathbf{Z}^T \mathbf{Z})^{1/2} \quad [3.2.3]$$

where $\mathbf{Z} = (Z_1, Z_2, \dots, Z_n)$ is a point on the failure surface.

By minimizing the quantity d , we obtain the point on the failure surface (z_1^*, \dots, z_n^*) , having the minimum distance to the origin in the reduced space.

In the case where the random variables are not normals, the probability of failure is evaluated following the equivalent normal distributions by the "Normal Tail Approximation" (Paloheimo²⁰, Rackwitz²²) through Eqs. [2.2.27-2.2.28].

When the variables are correlated, the original variates may be transformed to a set of uncorrelated variables by the following procedure:

Let the covariance matrix of the original variates, X_1, X_2, \dots, X_n be:

$$[C] = \begin{bmatrix} \sigma_{X_1}^2 & Cov(X_1, X_2) & Cov(X_1, X_3) & \dots & Cov(X_1, X_n) \\ Cov(X_2, X_1) & \sigma_{X_2}^2 & Cov(X_2, X_3) & \dots & Cov(X_2, X_n) \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ Cov(X_n, X_1) & Cov(X_n, X_2) & Cov(X_n, X_3) & \dots & \sigma_{X_n}^2 \end{bmatrix} \quad [3.2.4]$$

where the elements, $Cov(X_i, X_j)$ are the respective covariances between the pairs of variables X_i and X_j .

The corresponding covariance between a pair of reduced variates Z_i and Z_j is:

$$Cov(Z_i, Z_j) = \rho_{X_i, X_j} \quad [3.2.5]$$

Therefore, the covariance matrix of the reduced variates (Z_1, \dots, Z_n) is:

$$[C'] = \begin{bmatrix} 1 & \rho_{21} & \rho_{13} & \dots & \rho_{1n} \\ \rho_{21} & 1 & \rho_{23} & \dots & \rho_{2n} \\ \cdot & \cdot & \cdot & \dots & \cdot \\ \rho_{n1} & \rho_{n2} & \rho_{n3} & \dots & 1 \end{bmatrix} \quad [3.2.6]$$

Now, the set of uncorrelated transformed variates can be obtained through an orthogonal transformation as follows:

$$\mathbf{Y} = [T]^T \mathbf{Z} \quad [3.2.7]$$

where $\mathbf{Z} = \{Z_1, Z_2, \dots, Z_n\}$, is the set of reduced variables, $\mathbf{Y} = \{Y_1, Y_2, \dots, Y_n\}$ is the set of uncorrelated transformed variates and $[T]$ is the orthogonal transformation matrix composed of the eigenvectors corresponding to the eigenvalues of the correlation matrix $[C']$. $[T]$ satisfies:

$$[T]^T [C'] [T] = [\lambda] \quad [3.2.8]$$

where $[\lambda]$ is a diagonal matrix of the eigenvalues of $[C']$.

Since $[T]$ is orthogonal, $[T]^{-1} = [T]^T$, the variables \mathbf{Z} and the variables \mathbf{X} are related to \mathbf{Y} by:

$$\mathbf{Z} = [T] \mathbf{Y} \quad [3.2.9]$$

and

$$\mathbf{X} = [D] \mathbf{Z} + \mathbf{M}_X \quad [3.2.10]$$

where $[D]$ is a diagonal matrix of the standard deviations and \mathbf{M}_X is the vector of the mean values. Therefore:

$$\mathbf{X} = [D][T] \mathbf{Y} + \mathbf{M}_X \quad [3.2.11]$$

With the orthogonal transformation of Eq. [3.2.9], the safety index in Eq. [2.2.9] can be expressed as:

$$\beta = \frac{-\nabla g^{*T} \mathbf{Z}^*}{(\nabla g^{*T} [C'] \nabla g^*)^{1/2}} \quad [3.2.12]$$

where ∇g^* is the gradient vector at the most probable failure point $(z_1^*, z_2^*, \dots, z_n^*)$.

In the space of the transformed variates, ∇g can be expressed as, $\frac{\partial g}{\partial Y_i}$, such that:

$$\frac{\partial g}{\partial Y_i} = \sum_{j=1}^n \frac{\partial g}{\partial Z_j} \frac{\partial Z_j}{\partial Y_i} \quad [3.2.13]$$

and therefore, the components of the most probable failure point in the reduced space are:

$$y_i^* = -\alpha_i^* \beta \sigma_{Y_i} \quad i = 1, 2, \dots, n \quad [3.2.14]$$

where α_i^* is the direction cosines defined as:

$$\alpha_i^* = \frac{\left(\frac{\partial g}{\partial Y_i}\right) \lambda_i}{\sqrt{\sum_i \left[\left(\frac{\partial g}{\partial Y_i}\right) \lambda_i\right]^2}} \quad [3.2.15]$$

The algorithm for the case where the variables are correlated , non-normals follows:

1. Assume the failure point x_i^* .
2. Obtain the equivalent means and standard deviations of the given variables through Eqs. [2.2.27] and [2.2.28].
3. Calculate the eigenvalues of the covariance matrix, and obtain the orthogonal transformation matrix [T].
4. Obtain the reduced variates from Eq. [3.2.10].
5. Substitute the expressions of these variables in the performance function.
6. Calculate the partial derivatives.
7. Calculate the reduced variates of the most probable failure point by:

$$z_i^* = \frac{x_i^* - \mu_{x_i}}{\sigma_{x_i}} \quad i = 1, 2, \dots, n \quad [3.2.16]$$

8. Obtain the transformed variates through the relation:

$$Y^* = [T] Z^* \quad [3.2.17]$$

9. Calculate the direction cosines of the variables from Eq. [3.2.15] and the new checking point in the reduced space from Eq. [3.2.14].

10. Substitute the values obtained in the previous step in the failure equation and solve for β .

11. obtain the new checking points and repeat process until β converges.

In Ang and Tang's method, the correlation coefficient is invariant when the original variables are transformed into reduced variables which can be a weak approach. In the following, we describe Kiureghian and Liu's method which account for the change in the correlation coefficient.

3.3 *Kiureghian and Liu Approach*

The work presented here, is based on the first order reliability approximation to solve multivariate problems, where the variables are correlated and the information of the vector of basic variables $\mathbf{X} = [X_1, X_2, \dots, X_n]^T$, is limited to the means μ , standard deviations σ and correlation coefficients ρ .

A performance function, $g(\mathbf{X})$, describes the limiting state of the structure in terms of \mathbf{X} .

For an outcome $\mathbf{X} = \mathbf{X}$, the structure fails if $g(\mathbf{X}) \leq 0$ and it survives if $g(\mathbf{X}) > 0$. Therefore, the limit state surface is defined as:

$$g(\mathbf{X}) = 0 \quad [3.3.1]$$

which is the boundary between failure and survival.

This work is developed in the reduced space. For the cases where the variables are non-normal, they are transformed to normal through the relation:

$$\mathbf{Y} = [\Gamma][D]^{-1}(\mathbf{X} - \mathbf{M}) \quad [3.3.2]$$

where $[D]$ is a diagonal matrix with elements the standard deviations of the variables, \mathbf{M} is the vector of the mean values of the variables and $[\Gamma] = [L]^{-1}$, where $[L]$ is a lower matrix obtained through Cholesky¹⁴ decomposition of the matrix of the correlation coefficients. Eq. [3.3.2] is derived later in this section.

The proposed models to solve a reliability problem require an iterative algorithm which is said to be globally convergent if converge to a solution for any starting point. The chosen algorithm should follow certain rules. Kiureghian and Liu^{13,15} states the following rules for the algorithm to be used for the formulation of distribution models and their transformations:

1. **Generality** : The method can be used for different types of problems.
2. **Robustness** : The method should give accurate results, (close to 'exact' solution).
3. **Simplicity** : The information given and the computational effort needed for the transformation should be in harmony.
4. **Consistency** : The distribution model must be formulated in basis of the given information.
5. **Invariance** : The safety index should be the same regardless the method chosen.
6. **Operability** : The distribution model must be able to incorporate all the given information and include n number of variables.

Kiureghian and Liu¹³ suggest a model for three different cases of given information:

- Marginal Distributions Known.
- Marginal and Partial Joint Distributions Known.
- Less than Marginal Distributions Known.

Our work is based on the first of the cases, where the information given is restricted to the mean and covariance matrices and the set of marginal cumulative distribution functions.

Let us consider two variables, X_1 and X_2 . The objective is to find a bivariate distribution model which is consistent with the available information. The choice is arbitrary as long as it follows the rules listed above.

Let Z_1 and Z_2 be standard normal variates corresponding to the marginal transformations of X_1 and X_2 as follows:

$$Z_i = \Phi^{-1}[F_{X_i}(x_i)] \quad i = 1,2 \tag{3.3.3}$$

Where $\Phi[.]$ is the standard normal cumulative probability and $F_X(x_i)$ is the cumulative distribution of the original variable.

The key idea of Kiureghian and Liu's method is the following:

Since there is no information about the joint distribution of X_1 and X_2 , we assign a joint distribution of X_1 and X_2 such that through a Rosenblatt transformation, Z_1 and Z_2 are jointly normal.

The joint probability density of Z_1 and Z_2 is:

$$f_{Z_1 Z_2}(z_1 z_2) = |J(x_1 x_2)|^{-1} f_{X_1 X_2}(x_1 x_2) \quad [3.3.4]$$

where $f_{X_1 X_2}(x_1 x_2)$ is the joint probability density function of X_1 and X_2 , and $J(x_1 x_2)$ is the Jacobian of x_1 and x_2 :

$$J(x_1 x_2) = \begin{bmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_2}{\partial x_1} \\ \frac{\partial z_1}{\partial x_2} & \frac{\partial z_2}{\partial x_2} \end{bmatrix} \quad [3.3.5]$$

In our case [Papoulis²¹]:

$$J(x_1, x_2) = \frac{f_{X_1}(x_1) f_{X_2}(x_2)}{\phi(z_1) \phi(z_2)} \quad [3.3.6]$$

where $f_{X_1}(x_1)$ and $f_{X_2}(x_2)$ are the probability densities of X_1 and X_2 , respectively, and $\phi(\cdot)$ is the probability density of a standard Normal random variable. Therefore,

$$f_{Z_1 Z_2}(z_1 z_2) = \frac{\phi(z_1) \phi(z_2)}{f_{X_1}(x_1) f_{X_2}(x_2)} f_{X_1 X_2}(x_1, x_2) \quad [3.3.7]$$

The joint probability density function of X_1 and X_2 is:

$$f_{X_1, X_2}(x_1, x_2) = \phi(z_1, z_2, \rho_{0,12}) \frac{f_{X_1}(x_1) f_{X_2}(x_2)}{\phi(z_1) \phi(z_2)} \quad [3.3.8]$$

where $f_{X_i}(x_i)$ is the marginal density of X_i defined as, $f_{X_i}(x_i) = \frac{d}{dx_i} F_{X_i}(x_i)$ $z_i = \Phi^{-1}[F_{X_i}(x_i)]$ and $\phi_2(Z_1, Z_2, \rho_{0,12})$ is the bivariate standard normal density, which means that the variables have zero means, unit standard deviations and new correlation coefficients $\rho_{0,12}$.

This bivariate distribution can be expressed as:

$$\phi_2(z_1, z_2, \rho_{0,12}) = \frac{1}{2\pi\sqrt{1 - \rho_{0,12}^2}} \exp\left[-\frac{z_1^2 - 2\rho_{0,12}z_1z_2 + z_2^2}{2(1 - \rho_{0,12}^2)}\right] \quad [3.3.9]$$

The relation between $\rho_{0,12}$, and the original correlation coefficient ρ_{12} of X_1 and X_2 is expressed as :

$$\begin{aligned} \rho_{12} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\frac{x_1 - \mu_1}{\sigma_1} \right] \left[\frac{x_2 - \mu_2}{\sigma_2} \right] \phi_2(z_1, z_2, \rho_{0,12}) \frac{f_{X_1}(x_1) f_{X_2}(x_2)}{\phi(z_1) \phi(z_2)} dx_1 dx_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\frac{x_1 - \mu_1}{\sigma_1} \right] \left[\frac{x_2 - \mu_2}{\sigma_2} \right] \phi_2(z_1, z_2, \rho_{0,12}) dz_1 dz_2 \quad [3.3.10] \end{aligned}$$

The above equation can be solved for $\rho_{0,12}$ for each pair of marginal distributions with known ρ_{12} . This equation is difficult to evaluate and Kiureghian and Liu¹³ developed a set of semi-empirical formulae for selected marginal distributions, to relate $\rho_{0,12}$ and ρ_{12} , such that:

$$F = \frac{\rho_{0,12}}{\rho_{12}} \quad [3.3.11]$$

The formulas for F were obtained by least-square fitting of polynomial expressions to exact values obtained by numerical integration of Eq.[3.3.10]. F can be either a constant or function of ρ_{12} and the coefficient of variation $\delta_i = \frac{\sigma_i}{\mu_i}$ depending on the cases.

Here, there are some properties that the relation $\rho_{12} - \rho_{0,12}$ should satisfy:

1. When the variables are independent $\rho_{12} = \rho_{0,12} = 0$
2. For arbitrary marginals, $\rho_{12} \leq \rho_{0,12}$ When both marginals are normals, $\rho_{12} = \rho_{0,12}$.
3. The algebraic signs of $\rho_{0,12}$ and ρ_{12} are the same.
4. When one of the variables is normal, F is independent of ρ_{12} .
5. F is invariant of the linear transformation of X_1 and X_2 .
6. For variables with distributions reducible to a standard form through a linear transformation, e.g uniform, type I distributions, F is independent of the mean and standard deviation.

7. For variables with distributions irreducible to standard form through a linear transformation, e.g, lognormal, type II distributions, F is a function of the coefficient of variation of these variables.

Eq. [3.3.8] can be generalized for n number of variables, and be expressed in the form:

$$f_{\mathbf{X}}(\mathbf{x}) = \phi_n(\mathbf{z}, [R_0]) \frac{f_{X_1}(x_1)f_{X_2}(x_2)\dots f_{X_n}(x_n)}{\phi(z_1)\phi(z_2)\dots\phi(z_n)} \quad [3.3.12]$$

where $\phi_n(\mathbf{z}, [R_0])$ is the n-dimensional standard normal density with correlation matrix $[R_0]$ and $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function of $\mathbf{X} = (X_1, X_2, \dots, X_n)$.

Solving for $\rho_{0,12}$ we obtain the correlation coefficient in the reduced space and the transformation to the standard space for our distribution model can be expressed by the following, in general, non-linear transformation:

$$Y = \mathbf{T}(\mathbf{X}) = [\Gamma_0] \quad \mathbf{Z} = [\Gamma_0] \left\{ \begin{array}{l} \Phi^{-1}[F_{X_1}(X_1)] \\ \cdot \\ \cdot \\ \cdot \\ \Phi^{-1}[F_{X_n}(X_n)] \end{array} \right. \quad [3.3.13]$$

To search for the closest point to the origin in the reduced space , we should linearize the transformation at each iteration point by:

$$\mathbf{Y} = \mathbf{T}_i(\mathbf{X}) = [\Gamma_0][D']^{-1}(\mathbf{X} - \mathbf{M}') \quad [3.3.14]$$

where $\mathbf{M}' = [\mu_1^N, \dots, \mu_n^N]^T$, is a vector of the equivalent means and $[D']$ is the diagonal matrix of the equivalent standard deviations. \mathbf{M}' and $[D']$ are evaluated at each iteration point while $[\Gamma_0]$ remains invariant because the correlation matrix of \mathbf{Z} is invariant.

$$\mu_i^N = x_i - \sigma_i^N \Phi^{-1}[F_{X_i}(x_i^*)], \quad i = 1, \dots, n \quad [3.3.15]$$

$$\sigma_i^N = \frac{\phi\{\Phi^{-1}[F_{X_i}(x_i^*)]\}}{f_{X_i}(x_i^*)}, \quad i = 1, \dots, n \quad [3.3.16]$$

where μ_i^N and σ_i^N are the equivalent mean and standard deviation respectively.

The algorithm for the case where the marginal distributions are known and the variables are correlated, can be summarized:

1. If the variables are not Normal, the correlation coefficients are transformed through Eq. [3.3.11]. By Cholesky decomposition of the matrix of the correlation coefficients, $\rho_{0,12}$, we obtain $[\Gamma_0]$.
2. Choose an initial point in the space of the random variables.

3. Obtain the equivalent means and standard deviations from Eqs. [3.3.15] and [3.3.16].
4. Make the transformation in Eq. [3.3.14] and obtain the uncorrelated Normal variables \mathbf{Y} .
5. Then, calculate the direction of cosines defined as the unit vector normal to the surface having its direction towards the unsafe set and is expressed as:

$$\alpha = -\frac{\nabla g(\mathbf{y})}{|\nabla g(\mathbf{y})|} \quad [3.3.17]$$

where the gradient is obtained by:

$$\nabla g(\mathbf{y}) = [L]^T [D] \nabla g(\mathbf{x}) \quad [3.3.18]$$

6. The nearest point to the origin can be calculated by:

$$\mathbf{y}^{(j+1)} = \left[\mathbf{y}^{(j+1)T} \alpha^{(j)} + \frac{g(\mathbf{y}^{(j)})}{|\nabla g(\mathbf{y}^{(j)})|} \right] \alpha^{(j)} \quad [3.3.19]$$

7. Then, the safety index , β is:

$$\beta = \sqrt{\mathbf{y}^T \mathbf{y}} \quad [3.3.20]$$

8. Calculate a new point in the original space through the relation:

$$\mathbf{x} = \mathbf{M} + [D][L] \mathbf{y} \quad [3.3.21]$$

9. Go to step two and repeat the process until β converges.

The most important feature of this method is that it transforms the correlation coefficients ρ to ρ_0 . Note that when transforming the original random variables X_i into the reduced random variables Z_i , the equivalent standard normal means and standard deviations are calculated at each iteration point.

For the case in which the variables are independent and normals, the algorithm reduces to the one developed by Rackwitz and Fiessler²³.

This method avoids the assumption that the correlation coefficient does not change but it is more computationally intensive.

Chapter 4

Approach

Given a limit state function where the variables were correlated, both Ang and Tang's and Kiureghian and Liu's methods for evaluating the safety index were applied to several examples. The safety index was also calculated under the assumption of independence among the variables. The safety indices obtained by each of the three methods were compared to the ones obtained from a 10000 point Monte Carlo simulation, which was assumed to be exact.

4.1 Monte Carlo Simulation

Simulation is the process of replicating the real world based on a set of assumptions and conceived models of reality. Monte Carlo simulation involves

"sampling" at "random" to simulate artificially a large number of experiments and to observe the result. A sample from a Monte Carlo simulation is similar to a sample of experimental observations. Monte Carlo solutions may be the only means for checking or validating an approximate method of probability calculations.

For structural reliability, each random variable X_i is sampled randomly to give a sample value x_i . The function $g(\mathbf{X})$ is then evaluated. If $g(\mathbf{X}) \leq 0$ the structure fails. The experiment is repeated N times, each time with a randomly chosen vector \mathbf{x} of x_i values.

If N trials are conducted, the probability of failure is given approximately by:

$$p_f = \frac{n(g \leq 0)}{N} \quad [4.1.1]$$

where $n(g \leq 0)$ is the number of trials for which $g \leq 0$.

The number N of trials required is related to the desired accuracy of p_f . A Monte Carlo solution generally requires a large number of repetitions, particularly for problems involving rare events, therefore its application could be costly.

To apply Monte Carlo techniques to problems it is necessary:

1. To develop systematic methods for numerical "sampling" of the basic variables \mathbf{X} .
2. To select an economical and reliable simulation technique.
3. To consider the effect of the complexity of calculating $g(\mathbf{X})$ and the number of basic variables on the technique used.
4. To be able to determine the number of samples required to obtain a good estimate of p_f .

The main element of a Monte Carlo simulation procedure is the generation of random numbers from a specified distribution. There are different methods for generating such random numbers.

We are interested in generating correlated random numbers. Ang and Tang² developed a method for the generation of "Jointly Normal Random Numbers".

First, a number x is generated from a normal random variable X with mean μ and standard deviation σ given by:

$$x_1 = \mu + \sigma \sqrt{-2 \ln u_1} \cos 2\pi u_2 \quad [4.1.2]$$

where u_1 and u_2 are independent standard uniformly distributed random numbers.

Given this value of x , the conditional mean of a random variable Y can be obtained by:

$$\mu_{Y|x} = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X} (x - \mu_X) \quad [4.1.3]$$

and the conditional standard deviation of Y is :

$$\sigma_{Y|x} = \sigma_Y \sqrt{1 - \rho^2} \quad [4.1.4]$$

where ρ is the known correlation coefficient.

From this conditional mean and standard deviation, a value y is generated from the normal distribution. This way, a pair of values (x,y) is obtained from the bi-variate normal distribution.

This method works when the means, standard deviations and correlation coefficients of two normal random variables are given, and we want to generate two correlated random numbers. Nevertheless, this method cannot be applied in case that the random variables are not normal or if we have more than two correlated random variables. In the latter case the method does not guarantee that we can generate a third or a fourth random number with the desired correlation coefficient.

4.1.1 Approach 1

In this work a new algorithm is developed that overcomes the problem mentioned before and can be applied to generate N correlated random numbers, that follow any arbitrary distribution.

This algorithm is shown in Fig.(10) and involves the following steps:

1. Given the correlation matrix of the original variables $[R]$, the corresponding transformation (Kiureghian and Liu¹³) of each of the components of this matrix is made to obtain the correlation matrix of the reduced normal random variables $[R_o]$.
2. Then the Cholesky decomposition of the reduced covariance matrix $[C']$ is performed. Since the variables are standard normal, the covariance matrix will look like the matrix [3.2.6].
3. Generate m independent random standard Normal numbers, y_i .
4. Find the corresponding correlated normal random variables, z_i , through the following expression:

$$z_i = Ay_i \quad [4.1.5]$$

where A is the Cholesky decomposition of the covariance matrix of the reduced normal variables, $[C']$.

5. Obtain samples of the original random variables, x_i , through a Rosenblatt transformation such that:

$$F_{X_i}(x_i) = \Phi(z_i) \quad [4.1.6]$$

Then, $x_i = F_{X_i}^{-1}(\Phi(z_i))$

where x_i are samples from the desired distribution, F_{X_i} is the cumulative distribution in the original space and $\Phi(z_i)$ is the cumulative distribution of the standard normal random variables.

6. From each set of numbers obtained in the previous step the value of $g(x)$ is obtained.
7. The probability of failure is given by:

$$p_f = \frac{n_f}{N} \quad [4.1.7]$$

where $n_f = n(g \leq 0)$.

Approach 1 for generating random numbers is based on the assumption used by Kiureghian and Liu on the joint probability density function of the original

random variables. Specifically, for each set of arbitrary distributed random variables for which we only know the marginal probability distributions and correlation coefficients, Kiureghian and Liu select a joint probability distribution for these random variables on the basis of the following assumption; if we transform the original random variable using Rosenblatt transformation, then the transformed random variables will be jointly normal. However, Kiureghian and Liu method uses the normal tail approximation to transform the original random variable into normal instead of Rosenblatt transformation. Therefore, Approach 1 for generating random numbers should not be biased towards Kiureghian and Liu method. However, we also developed an alternative approach which is independent of the assumption that the transformed random variables are jointly normal to further evaluate methods fairly.

4.1.2 Approach 2

The second approach to generate random samples with arbitrary distributions to be used for Monte Carlo simulation is based on the "Central Limit Theorem" which states that the distribution of the sum of a large number of random variables approaches the normal distribution.

The algorithm can be described as follows:

1. Generate a set of random independent samples drawn from different probability distributions. Some of these distributions are not normal. The number of samples should be relative large.
2. Combine these samples in two ways to obtain two correlated normal random numbers v_1 and v_2 . v_1 is obtained by adding some of the samples and v_2 is obtained by adding a different combination of these samples. Some of the samples in the two sums are common so that v_1 and v_2 are correlated. According to the Central Limit Theorem v_1 and v_2 should be normal. However their joint probability distribution is not necessarily normal.
3. Repeat the procedure m times and calculate the means and standard deviations of v_1 and v_2 and their correlation coefficient.
4. Obtain the corresponding correlated standard normal random variables, z_i by:

$$z_i = \frac{v_i - \mu_{v_i}}{\sigma_{v_i}} \quad [4.1.8]$$

5. Follow steps 5 to 7 from section (4.1.1).

Note that the variables obtained applying Rosenblatt transformation to the original random variables are not jointly normal. Therefore, approach 2 should not be biased towards Kiureghian and Liu method.

Chapter 5

Numerical Examples

5.1 Problem description

For each of the examples selected, a limit state function g is given. The information about the variables is limited to:

1. Distribution types: The variables that we considered, followed one of the following types of distributions:
 - a. Normal
 - b. Lognormal
 - c. Extreme 1 Largest

- d. Extreme 2 Largest
 - e. Exponential
 - f. Uniform
2. Mean values
 3. Standard Deviations
 4. Correlation coefficients for each pair of variables. For the sake of simplicity, the range between $\rho = -0.5$ and $\rho = 0.5$ was selected.

As we have already mentioned, the main difference between Ang and Tang's and Kiureghian and Liu's approaches is the transformation of the correlation matrix performed by Kiureghian and Liu. This transformation is made by multiplying the original correlation coefficient of each pair of random variables by a parameter F . The values of F are found for a given distribution of the random variables, Kiureghian and Liu¹³. Depending on the type of distributions there is a percentage error associated with the evaluation of F which is proportional to the coefficient of variation. For the types of distribution where this error is high, we kept the coefficient of variation low (below 0.5); this way the maximum error never exceeded 4.5 %.

To perform Monte Carlo simulation, as it was mentioned in the previous chapter, first we generated standard normal random variables and then transformed them into random variables that follow the specified probability distributions of the original random variables. To do so, we used the expressions provided by Kiureghian and Liu to obtain the parameter F . Since these expressions are approximate, we checked if the probability density functions and correlation coefficients of the obtained samples of the original random variables were close to the specified ones. In all cases we found that the histograms and correlation coefficients of the samples were consistent with those of the original random variables.

To implement the analysis and compare Ang and Tang, Kiureghian and Liu and the method that neglects the dependency between random variables, a linear performance function of the form $g = X_1 - X_2$ has been chosen. Fifteen examples in which the random variables X_1 and X_2 , have various distributions have been performed. The errors in the safety index obtained from the methods are compared.

In each of the Tables (1-15), first the expression for the limit state equation is given. Then, we present the distribution type of each variable and the respective mean and standard deviation of the random variables. The correlation coefficient between the variables and the corresponding F are presented. Next, defining percentage error as the ratio of the difference between the approximate and exact values and the exact value, and bias as the ratio between corresponding

estimates from exact and approximate methods, the safety index and probability of failure for each method are shown, with the corresponding percentage error and bias between each approach and Monte Carlo simulation using 10000 sample points. The safety index, probability of failure and percentage error calculated using Rackwitz's algorithm, assuming that the random variables are independent are also presented. Note that when the variables are independent the three methods yield to the same results.

5.2 Results

5.2.1 Comparison of the Methods by Monte Carlo Approach 1.

The highest errors found by comparing the safety index for each approach is -14.40 % for Kiureghian and Liu, -16.38 % for Ang and Tang and -40.88% when the variables are assumed to be independent, Tables (16-17). The corresponding errors in probability of failure are 21.02%, 18.46% and 56.41% respectively, which means that in these cases, the three methods overestimate the probability of failure. It is observed that the method that neglects dependency usually yields erroneous results.

The mean and standard deviation of the error and of the bias across all the cases considered, have been calculated too and the results are tabulated with their

corresponding coefficient of variation. In all the cases we observed that the standard deviation is smaller for Kiureghian and Liu, which indicates that there is less dispersion of the results for this method. Specifically, the coefficient of variation of the bias is 0.078 for the estimates obtained from Ang and Tang and 0.059 from Kiureghian and Liu. Similarly, the standard deviation of the percentage error in safety index is 3.95 for Kiureghian and 4.37 for Ang and Tang, Table (17).

At this point, it has been showed that either one of the methods where the correlation coefficient between the variables is taken into account for the study of reliability, works better than the assumption of independence. To see which method is better for the study of reliability, several histograms of the error in the safety index are shown in Figs.(11-13). The corresponding percentile plots are presented in Figs.(14-15).

Therefore, in the 75 cases that have been compared, 73.33% of the results are better when Kiureghian and Liu's approach is implemented. Moreover, it is observed from Figs.(11-12) that the dispersion of the error from Kiureghian and Liu's method is smaller than that of Ang and Tang's. Also in Fig.(12), 24 of the examples are in a range of error between 0% and -2.5% and just 10 of the examples have error higher than (+ or -) 10%. In Fig.(11) we see that the error range oscillates between -18% and 12.5% and there are 13 of the examples with an error greater than (+ or -) 10%. In Fig.(14), it can be seen that in 90% of the cases, the error in the safety index was between -12% and 7% for Kiureghian

and Liu, and between -15% and 9% for Ang and Tang. Fig.(15) shows the distribution of the absolute percentage error for 90% of the cases, where we observe that for 50% of the cases, the error for Kiureghian and Liu is smaller than 3% while for Ang and Tang is of 6%. Therefore the chances are that Kiureghian and Liu's method will work better in a problem than Ang and Tang.

5.2.2 Monte Carlo Simulation, Approaches 1 and 2

As mentioned before, two approaches using Monte Carlo simulation were developed. Approach 1 assumes that the joint distribution between the reduced variables obtained using Rosenblatt transformation is normal. In Approach 2, random variables that have given marginal probability distributions and correlation coefficient, are generated in a way that their joint probability density function is not normal.

We chose an example with a performance function $g = X_1 - X_2$, and following approach 2, which was described in section (4.1.2), we generated two series of 100,000 normally distributed numbers corresponding to each one of the variables. The first step was to verify that the numbers were not jointly normal. To do so, we calculated the theoretical jointly normal distribution to compare with the joint distribution obtained from the numbers generated, and test normality by the coefficient of determination¹⁸ (R^2). R^2 is the summation of the squares of the relative frequencies obtained from the samples generated minus the

mean value, divided by the summation of the squares of the theoretical frequencies minus the mean value. Values of R^2 differing significantly from 1.0 indicate that the hypothesis of normality should be rejected. In real life, the value of R^2 for which we can assume that the random variables are normal depends on the type of random variables. For random variables describing material properties, this value depends on the type of materials. For instance, in case of materials used for metallurgy, values lower than 0.98 are rejected. For other materials in other fields 0.90 could hold for normality.

The R^2 method was implemented to prove that the joint probability density function of the variables obtained from the second approach for Monte Carlo simulation was not normal. We also solved the same problems using Approach 1 and compared the results.

We considered an example where $g = X_1 - X_2$ and X_1 and X_2 followed the extreme 2 largest distribution. Their correlation coefficient was 0.421. As it was explained in the previous two paragraphs, we generated random samples whose marginal probability distributions were normal using Approach 2. Then we checked if they were jointly normal. The value of R^2 obtained from the normality check method, using Approach 2 was 0.8, which showed that the random variables were not jointly normal.

Figs.(16-17) show the distributions obtained from the generated samples and from the theoretical values. In these plots the values obtained of the joint

probability density function are shown in 3-dimensions (x,y,z). All the values are concentrated in a 10*10 area covering the range where all the values obtained are included. The z axis shows the cumulative count. Figures 16 and 17 are the top view surfaces of the generated samples and the theoretical values respectively. It is observed that the joint distribution of the samples obtained from Approach 2 is not normal.

For this example, the safety index obtained using the second approach of Monte Carlo simulation was compared with the safety index calculated by the first approach of Monte Carlo simulation, Kiureghian and Liu, and Ang and Tang methods. The results are shown in Table (18), where M-C 1 and M-C 2 are the notations used for Monte Carlo simulation approaches 1 and 2 respectively.

It was observed that Kiureghian and Liu's method gave results that were closer to those for Monte Carlo simulation (approaches 1 and 2). Table (18) shows that the results obtained from Monte Carlo simulation Approach 2 are higher than the ones obtained from the other methods.

Based on the above results we can conclude the following:

We have presented two algorithms to calculate reliability under incomplete information when the variables are correlated. These two algorithms were compared to Monte Carlo simulation (Approach 1), where, as in Kiureghian and Liu's method, the joint probability density function was considered to be

normally distributed. As it has been already mentioned, when these three methods were compared to Monte Carlo (Approach 2) in Table (18) we found a discrepancy in the results.

There are two sources of error when using Ang and Tang's method:

1. The assumption that the correlation coefficient do not change when transforming the random variables.
2. The fact that the available information is incomplete. Specifically, we know only the marginal densities but not the joint probability density function.

In Kiureghian and Liu's method we transform the correlation coefficient of non-normal variables based on the assumption that the joint probability density function of the transformed random variables is normal. Therefore, Kiureghian and Liu removed the first source of error. However, their method uses incomplete information because it does not account for the joint probability density of the random variables.

None of the existing algorithms is able to solve this error which appears to be more important than the one due to the change in correlation. Unfortunately, there is little information available. It is expected that a new method that accounts for the joint distributions of the random variables should be considerably more accurate than the above methods.

In most of the cases studied, Kiureghian and Liu's approach worked better. However, the error in the results of this method is considerable in some cases. This drawback is due to the assumption that the correlation coefficient defines completely the joint probability density function.

Elishakoff and Hasofer⁷ have also shown that this assumption might lead to significant errors in the estimates of the failure probability. The following example demonstrates this.

Example Consider the problem where the performance function is defined as $g = X_1 - X_2$.

The solution of the problem of determining reliability is well known when X_1 and X_2 form an independent normal vector.

Now suppose that X_1 and X_2 do not form a bivariate normally distributed random vector but still, the variables X_1 and X_2 are normally distributed variables with zero correlation. So despite the fact that X_1 and X_2 are not jointly normal, they are individually normal, and they have zero correlation.

For certain combinations of parameters the approximate estimate of the reliability is higher than the exact reliability, implying that in a real situation the reliability will be lower; this shows a detrimental effect of this assumption, which Elishakoff and Hasofer attribute to human error. For the example that

Elishakoff and Hasofer⁷ present, the error can vary between -1.96% and 8.19% for the safety index and 3.25% and -40% respectively for the probability of failure.

This example shows that even if we know the marginal probability distributions of the random variables and their correlation coefficients, the error may be large.

Chapter 6

Conclusions

In reliability analysis, there are random variables that most of the time are correlated.

In this study we evaluated two methods that account for correlation between random variables. These methods are Kiureghian and Liu Ang and Tang and they account for correlation using the correlation coefficients of the random variables only. These methods were compared with Monte Carlo simulation and a second order method that neglects the correlation of the random variables. Specifically, two different Monte Carlo simulation approaches were used which were based on different assumptions. The main conclusions from this work are:

1. Neglecting correlation is commonly used in traditional studies but the effect can be detrimental.

2. In general, Kiureghian and Liu's approach seems to be more accurate than Ang and Tang's.

3. In some cases Kiureghian and Liu's approach results in considerable errors (the error in the safety index might exceed 10%). As it was shown by using Monte carlo simulation approaches 1 and 2 in section (5.2.2), this is due to the fact that the marginal probability distributions of the random variables and their correlation coefficients are not sufficient to completely determine the statistics of the random variables. The same conclusion was reached by Elishakoff and Hasofer, where it was shown that in case that we do not know the joint probability distribution, but only the marginals and the correlation coefficient, we might not be able to estimate the reliability accurately.

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Note that some of the references listed below, are not mentioned in this thesis, but are presented because they are pertinent to this work.

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Figures

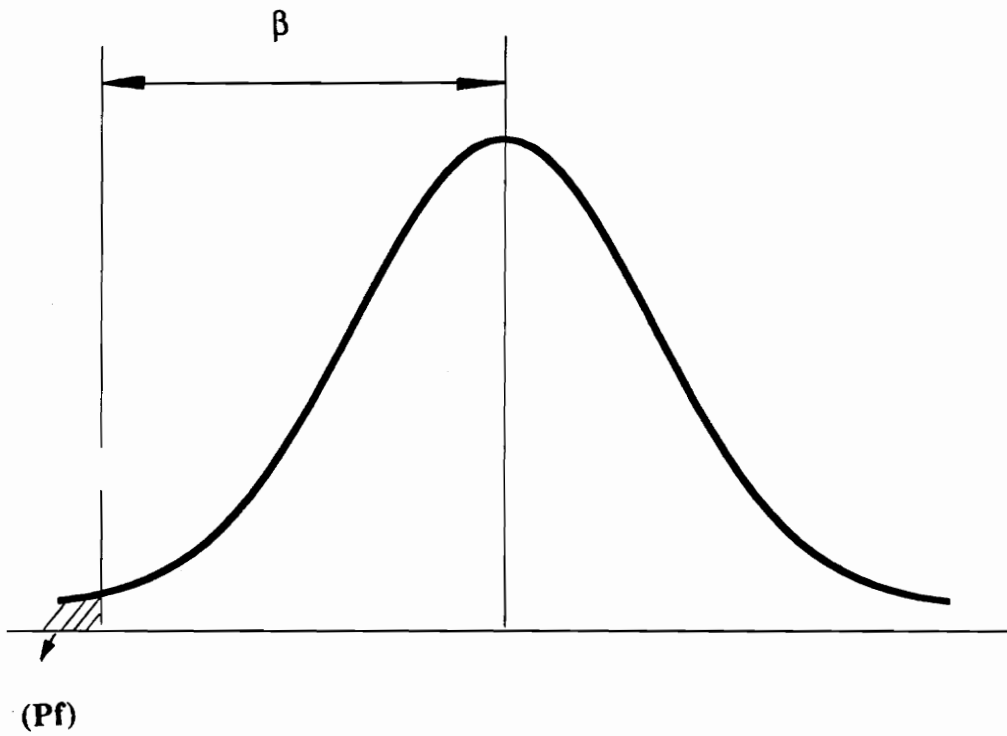


Figure 1. Distribution of Safety Margin.

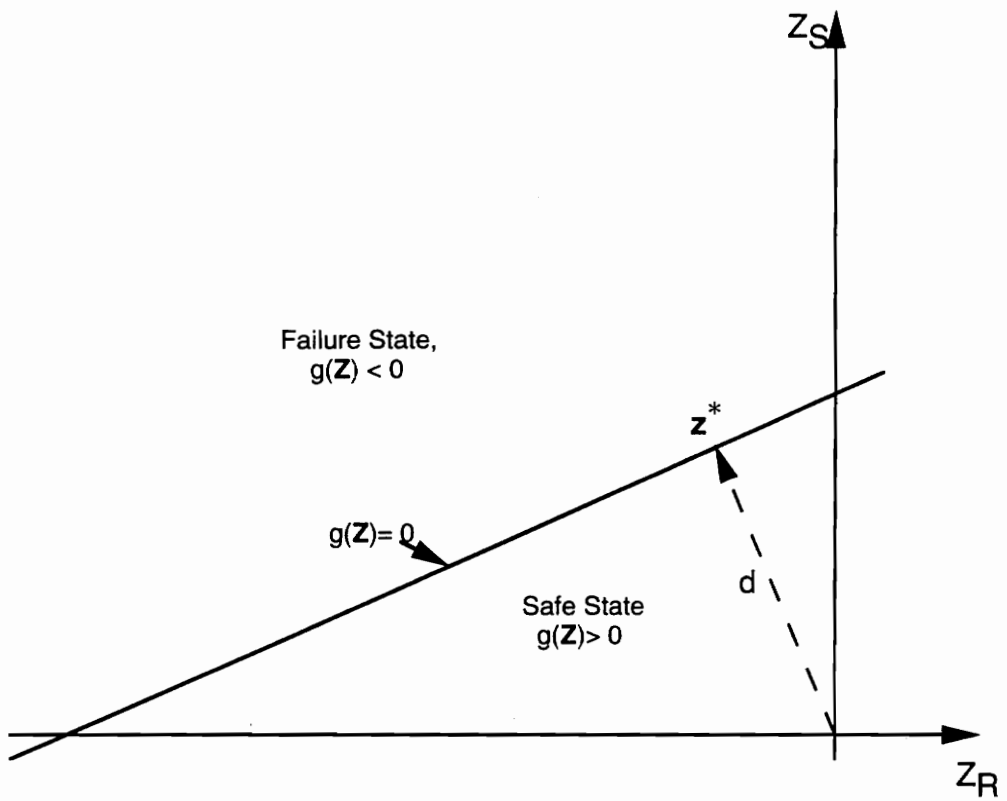


Figure 2. Linear Limit State Function. Safe and Failure States in the Space of Reduced Variates.

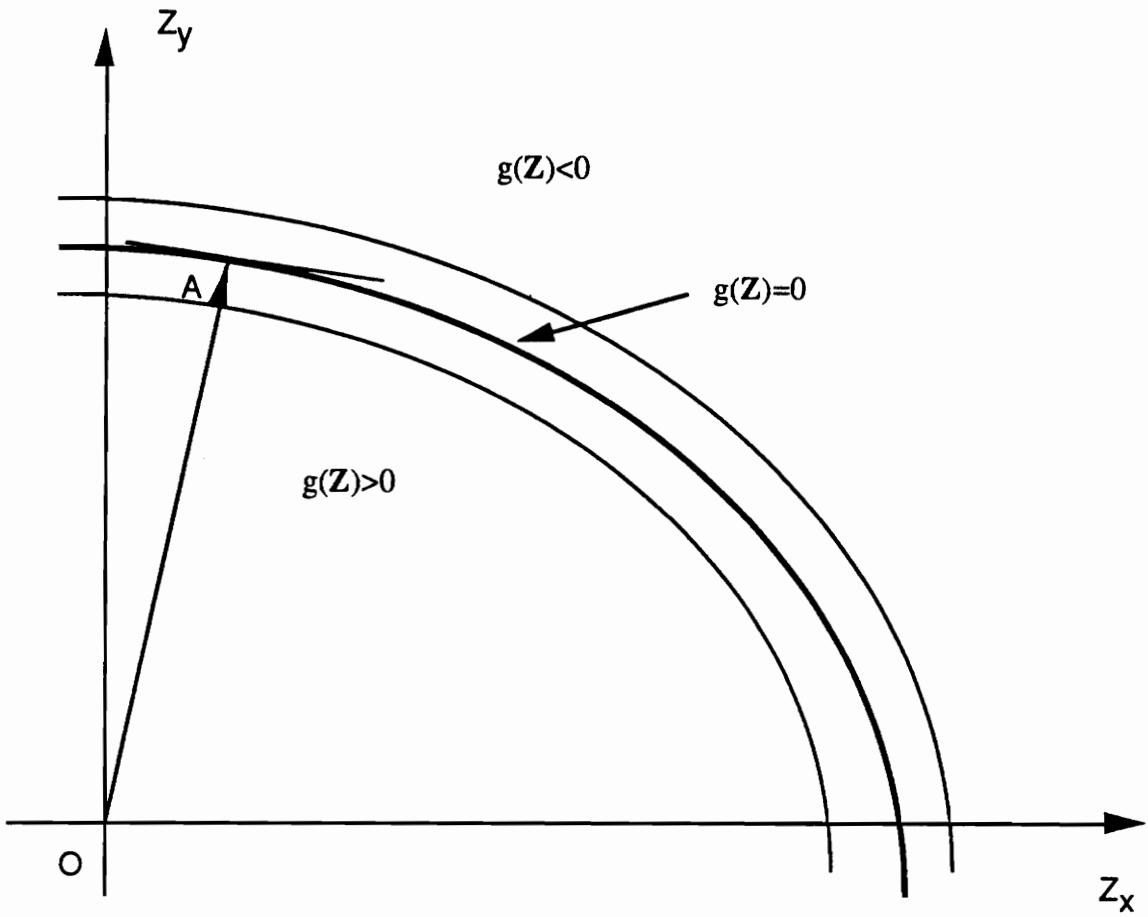


Figure 3. Non-linear Limit State Function. Safe and Failure States in the Space of Reduced Variates.

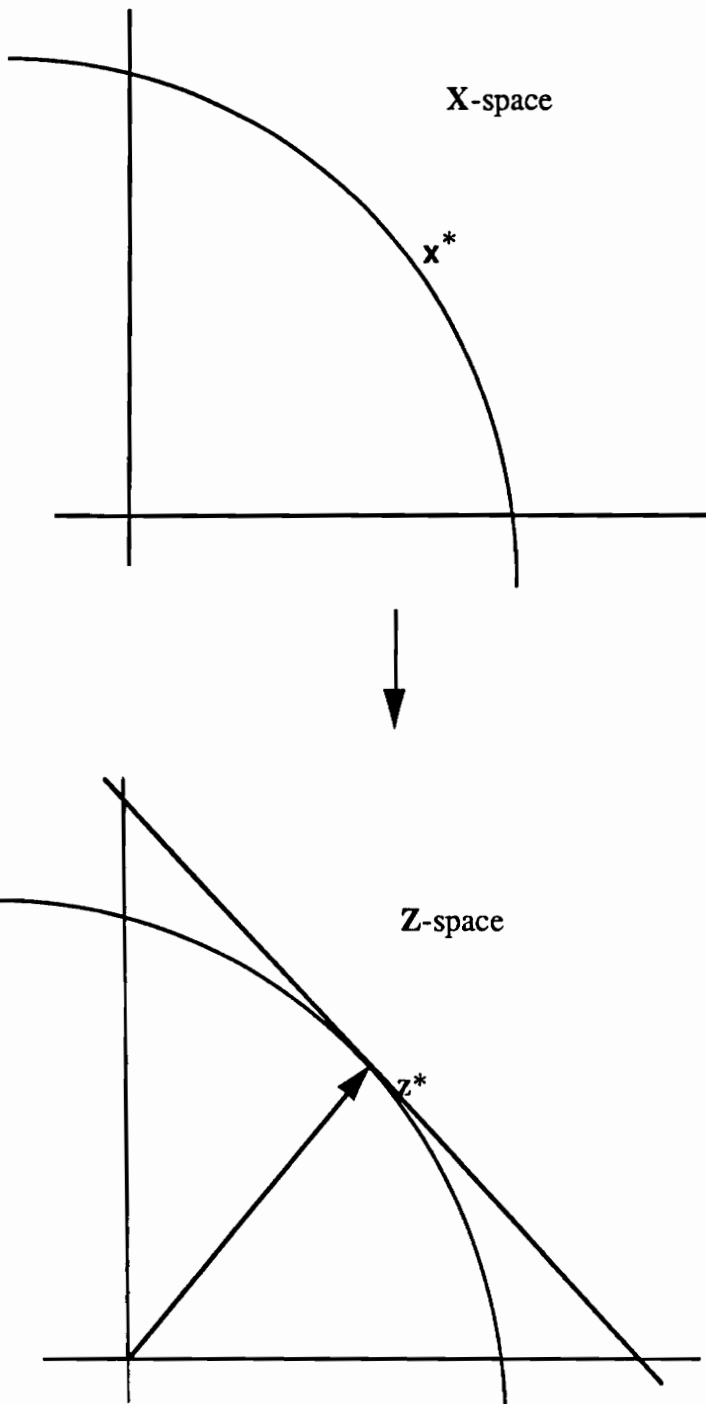


Figure 4. Failure Surfaces in X and Z space.

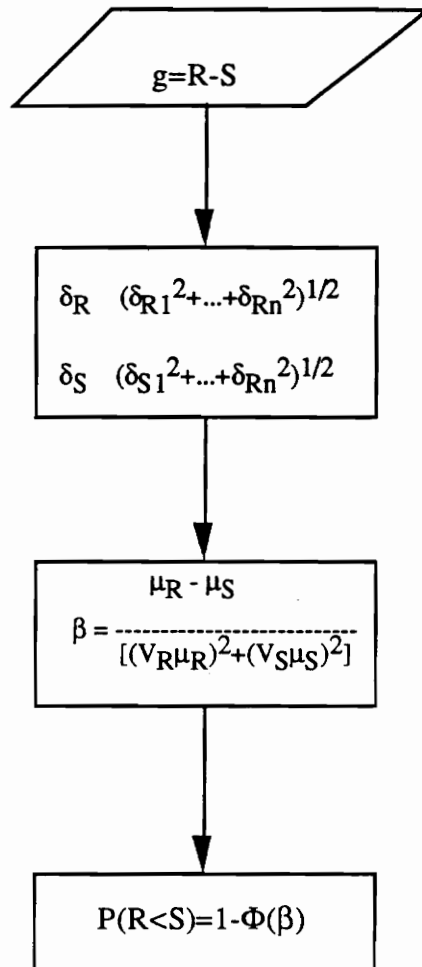


Figure 5. Flow Chart for the Cornell Algorithm.

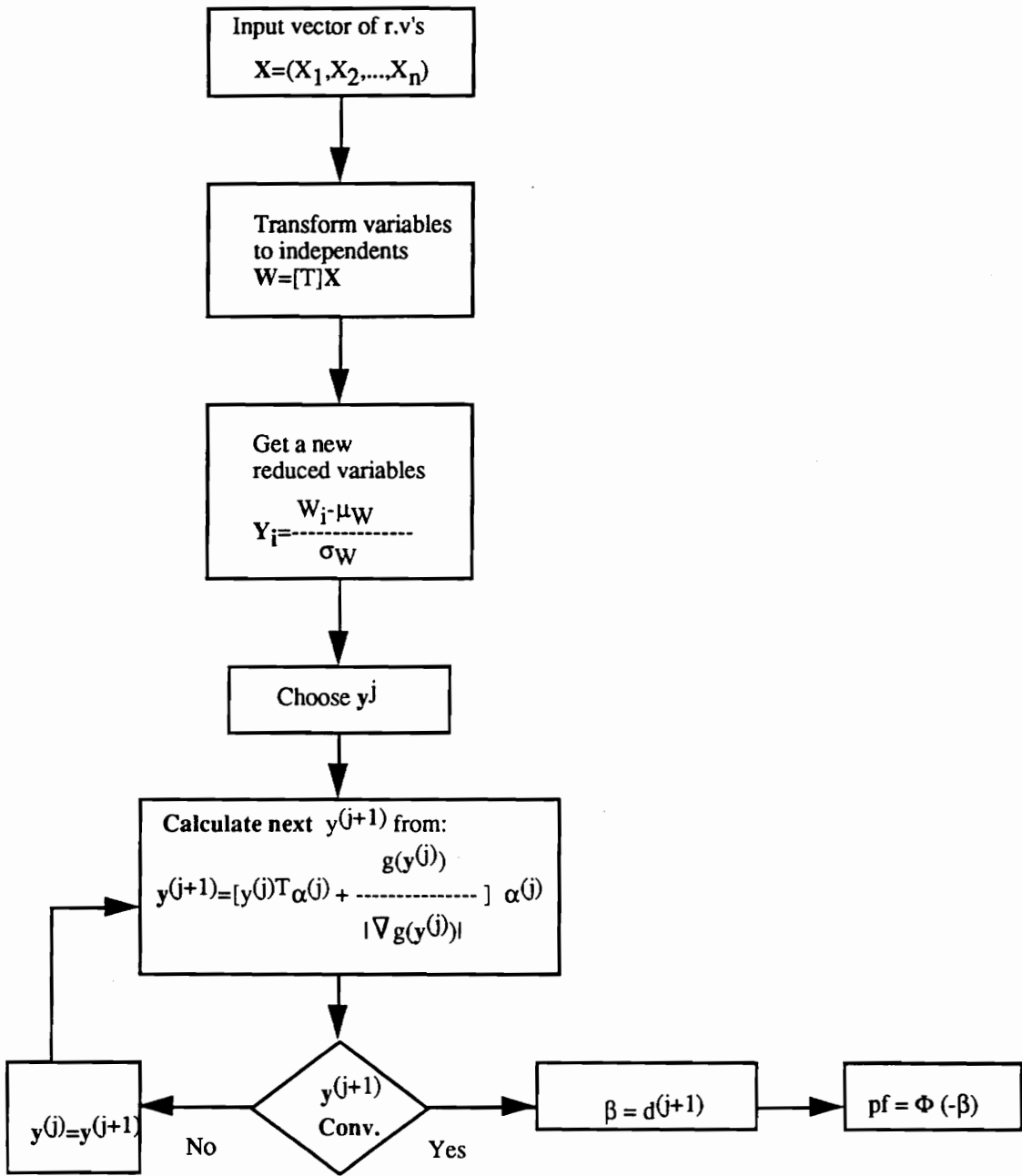


Figure 6. Flow Chart for the Hasofer and Lind Algorithm.

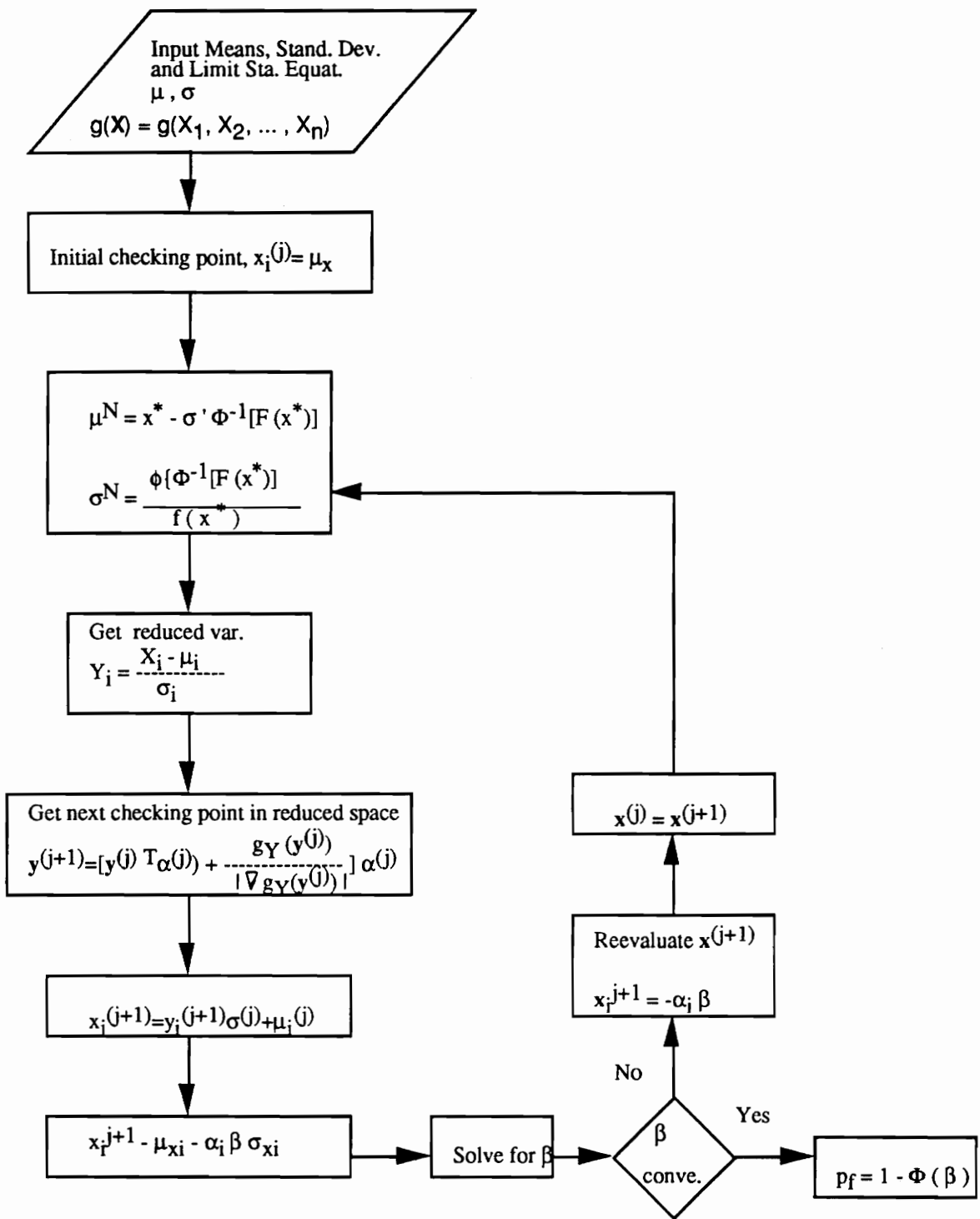


Figure 7. Flow Chart for the Rackwitz and Fiessler Algorithm.

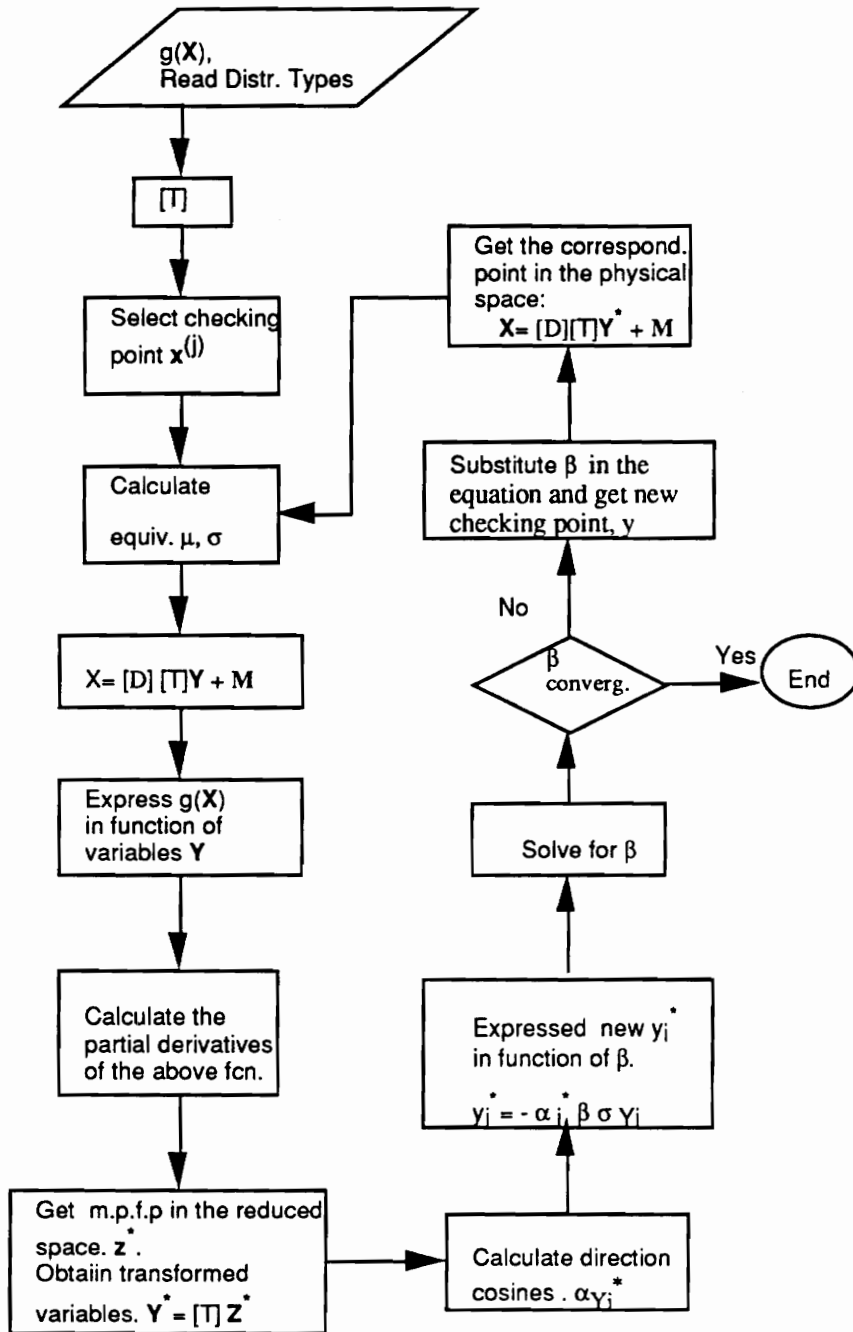


Figure 8. Flow Chart for the Ang and Tang Algorithm.

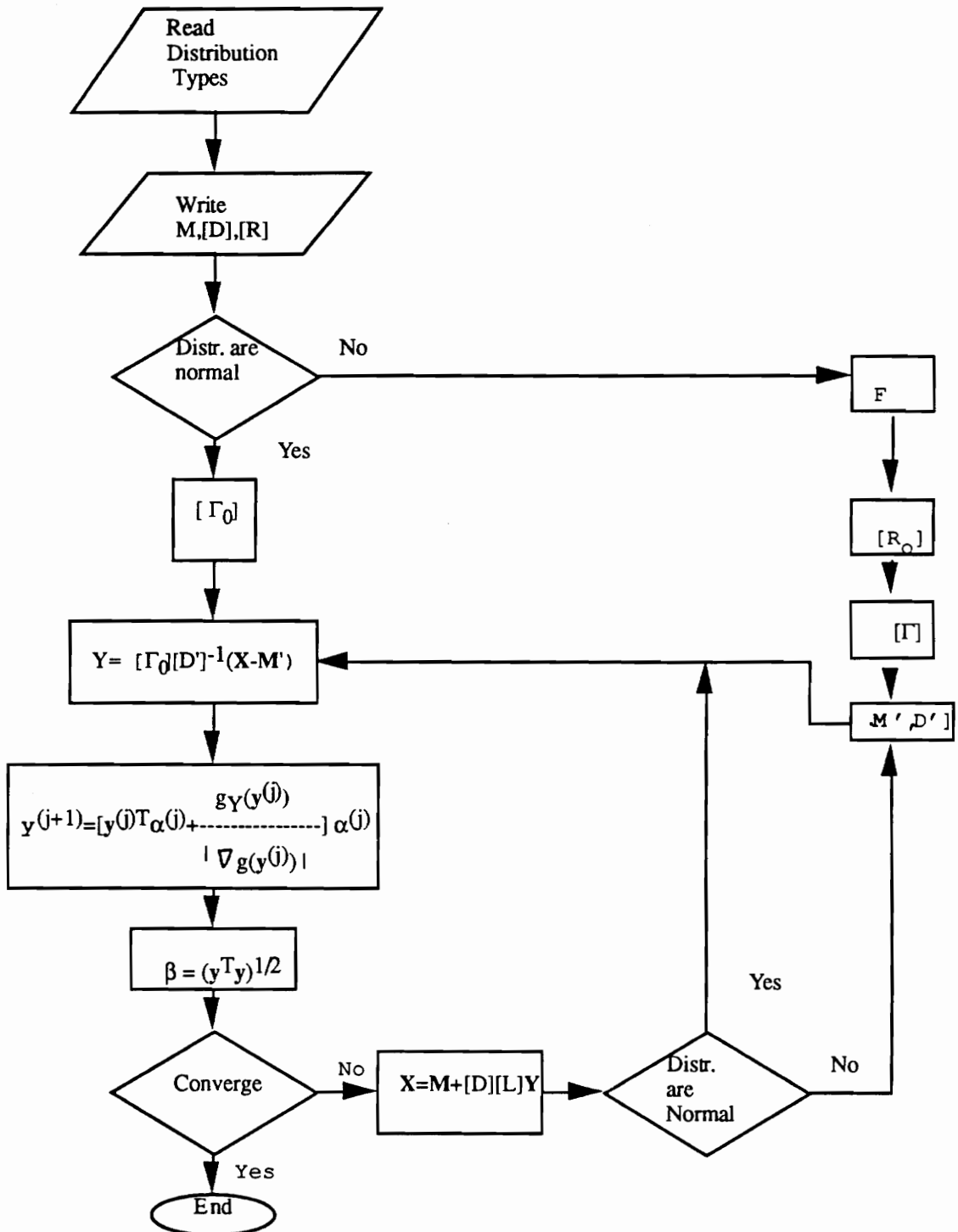


Figure 9. Flow Chart for the Kiureghian and Liu Algorithm.

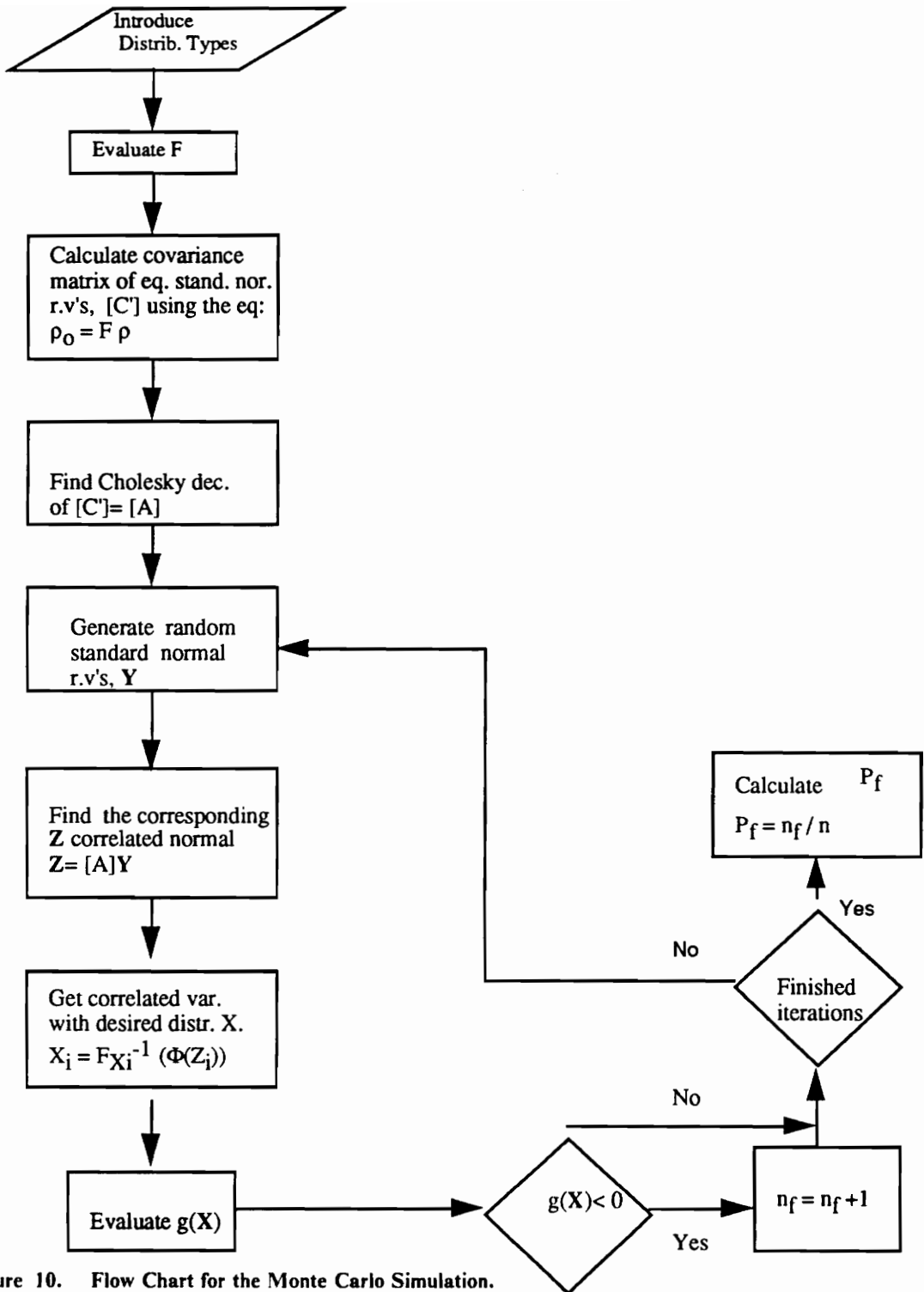


Figure 10. Flow Chart for the Monte Carlo Simulation.

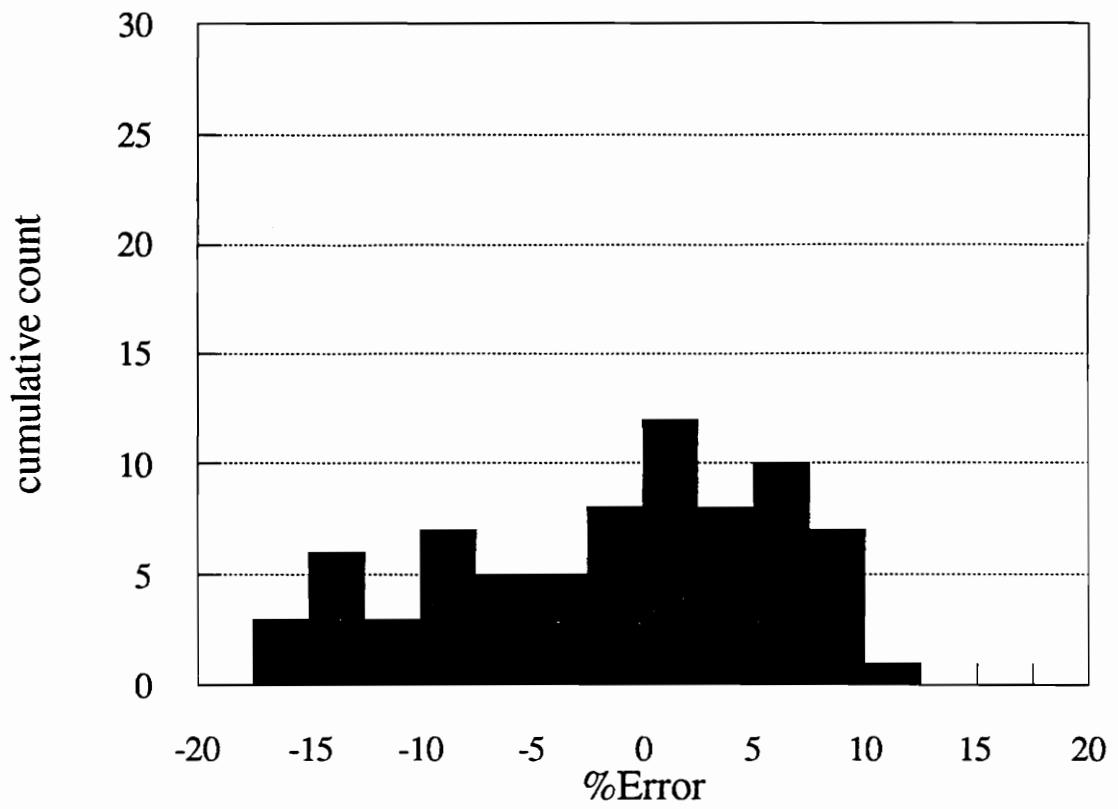


Figure 11. Ang and Tang: Percentage Error in Safety index.

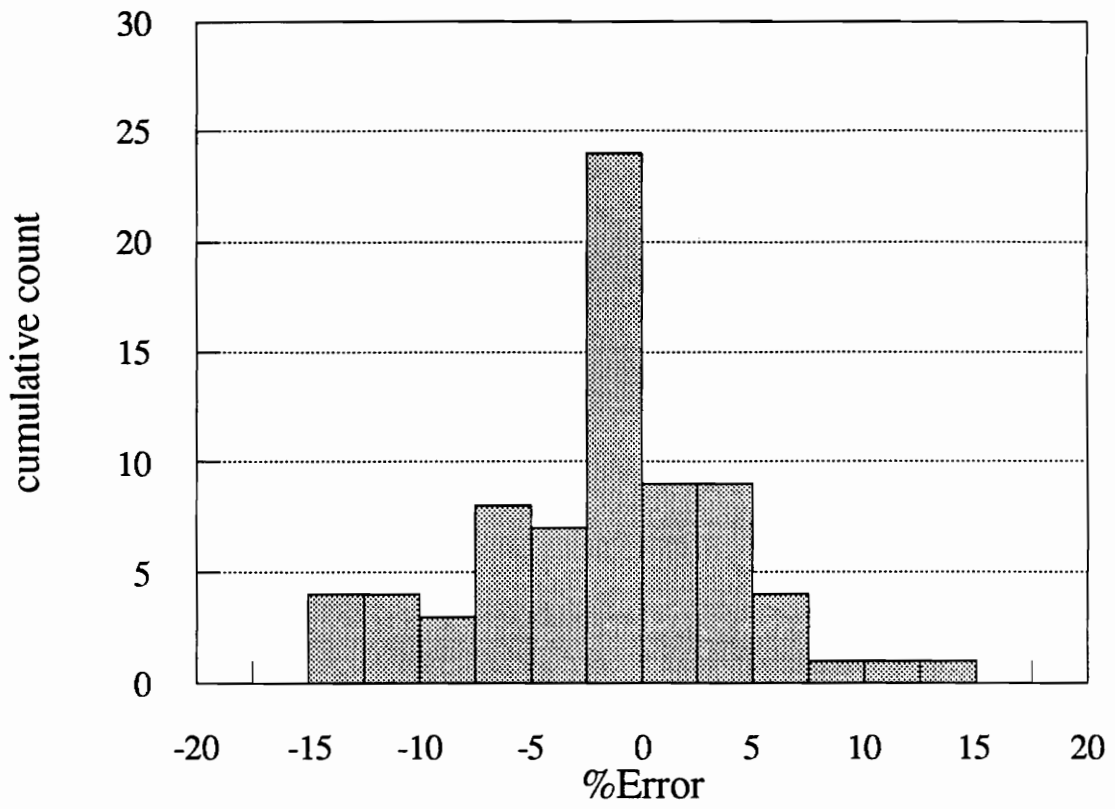


Figure 12. Kiureghian and Liu: Percentage Error in Safety index.

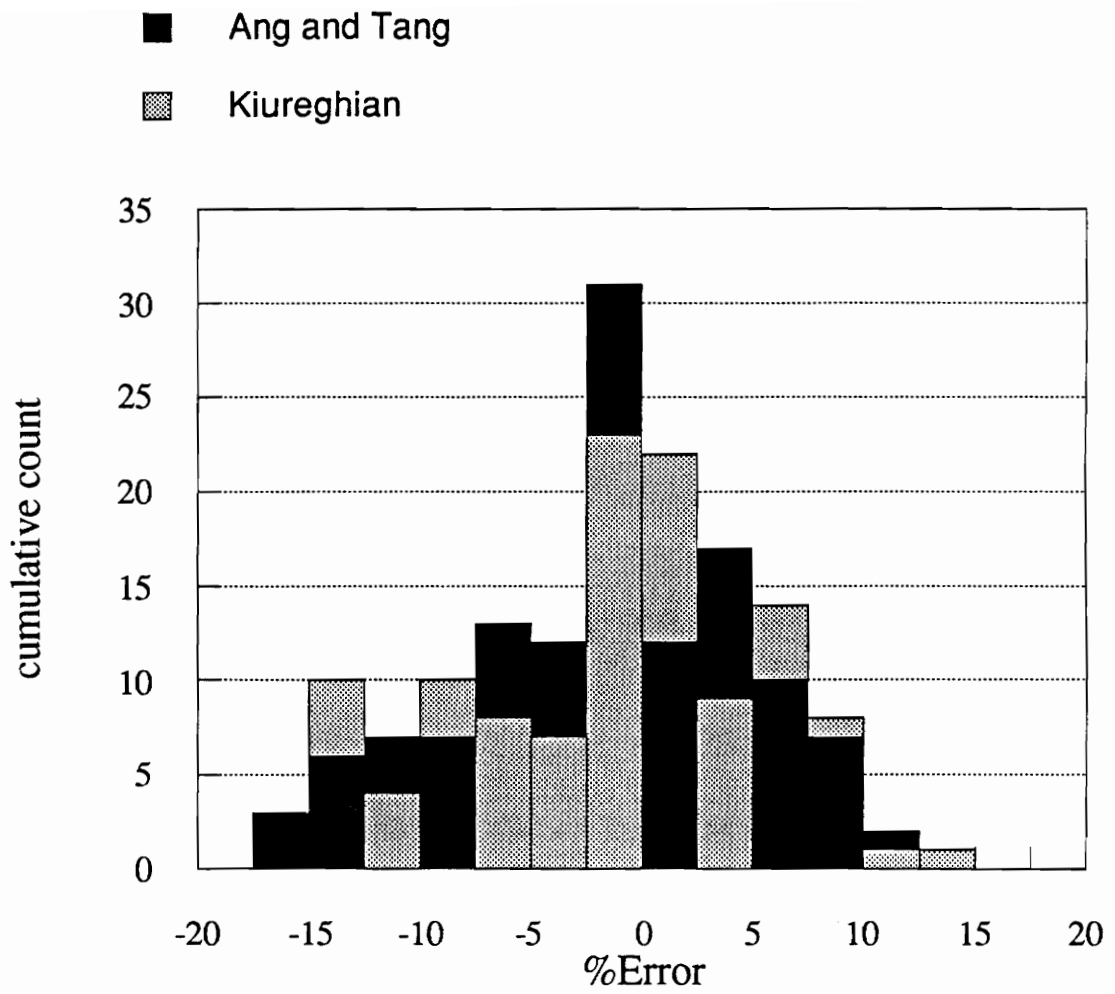


Figure 13. Ang and Tang, Kiureghian and Liu: Percentage Error in Safety Index.

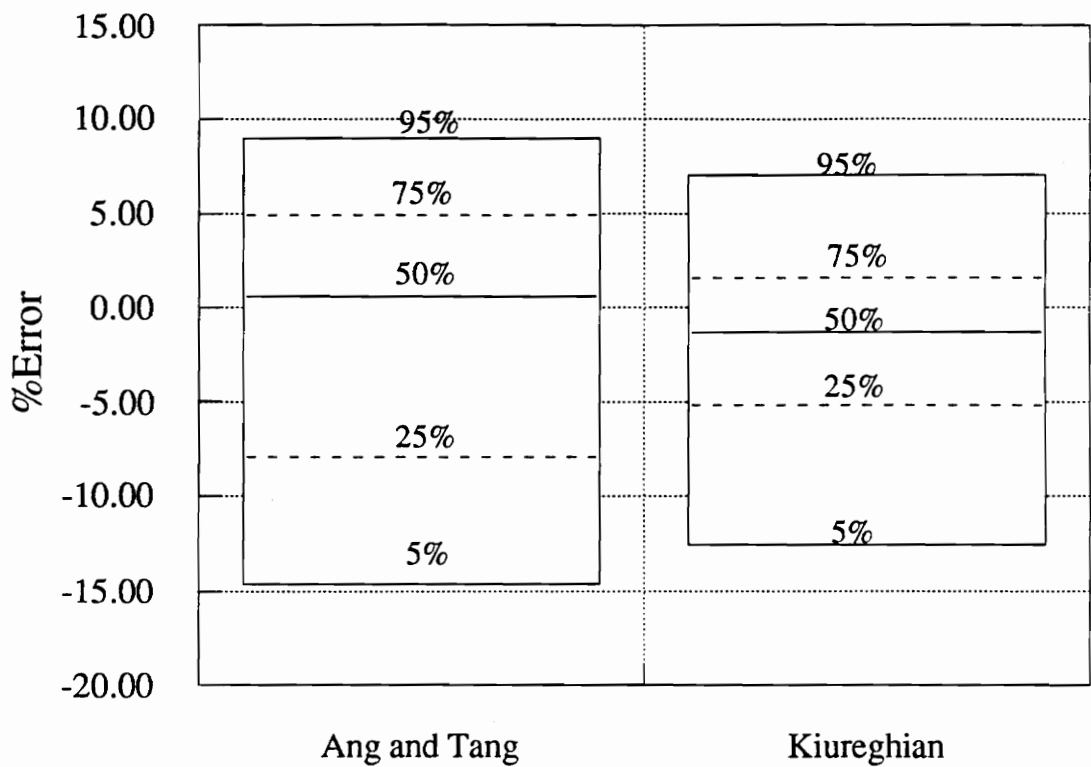


Figure 14. Percentile Error in Safety Index.

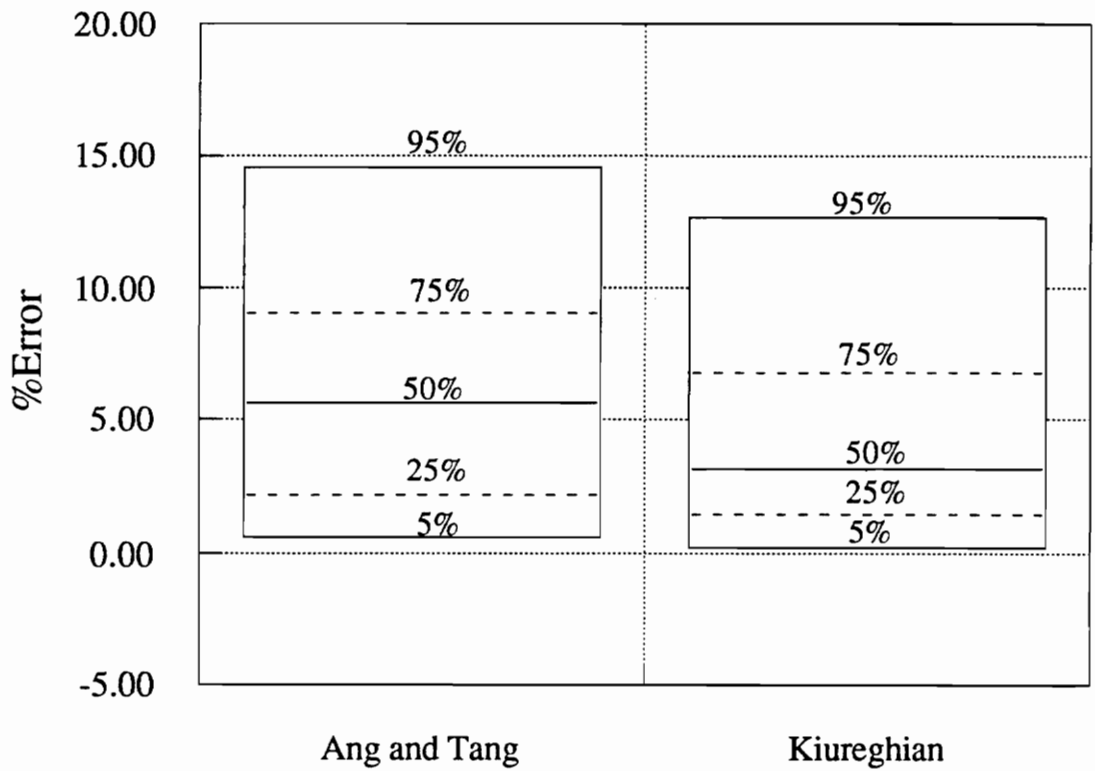


Figure 15. Percentile Absolute Error in Safety Index.

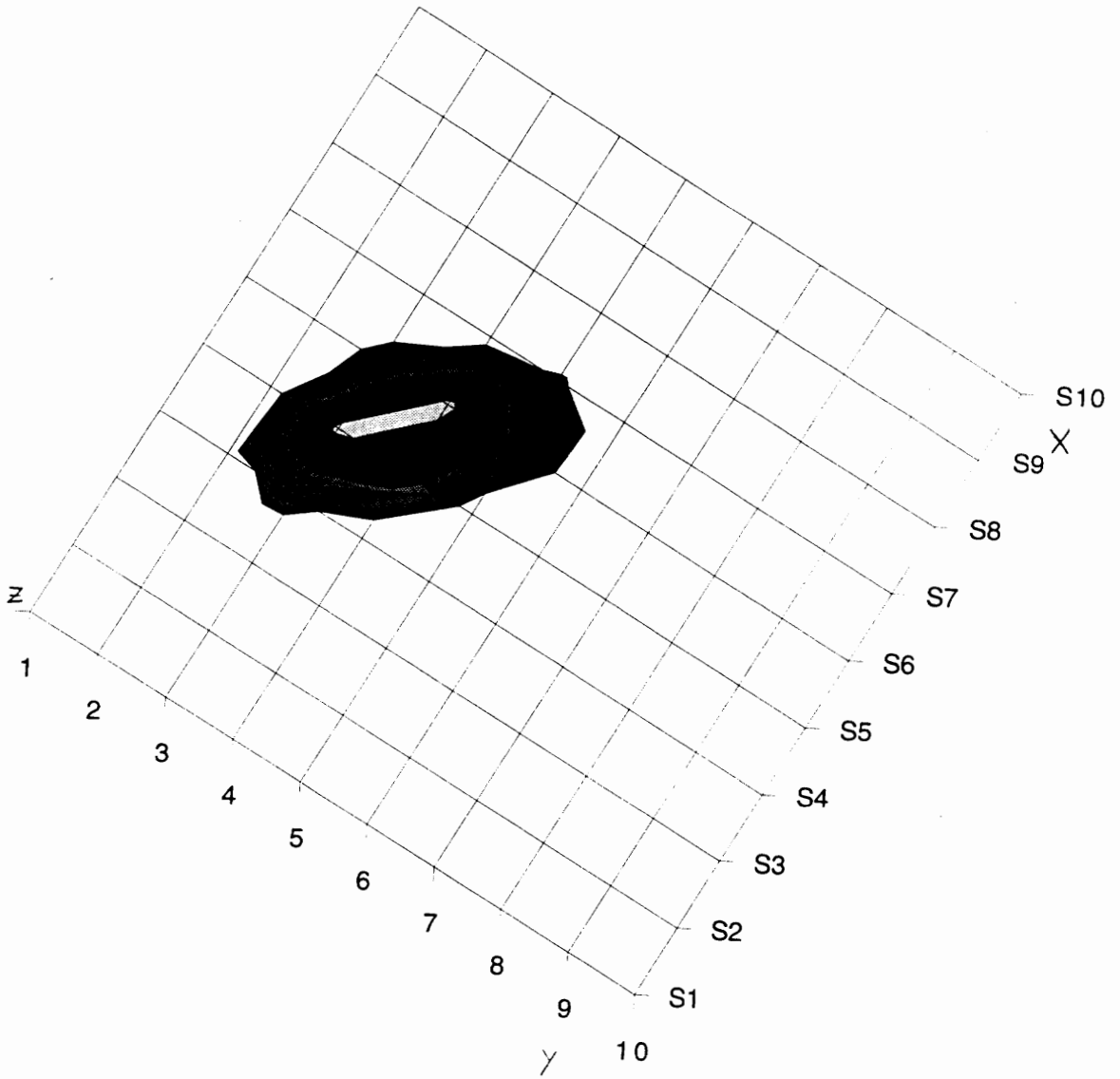


Figure 16. Joint PDF from Samples Generated M-C Approach 2. Top View.

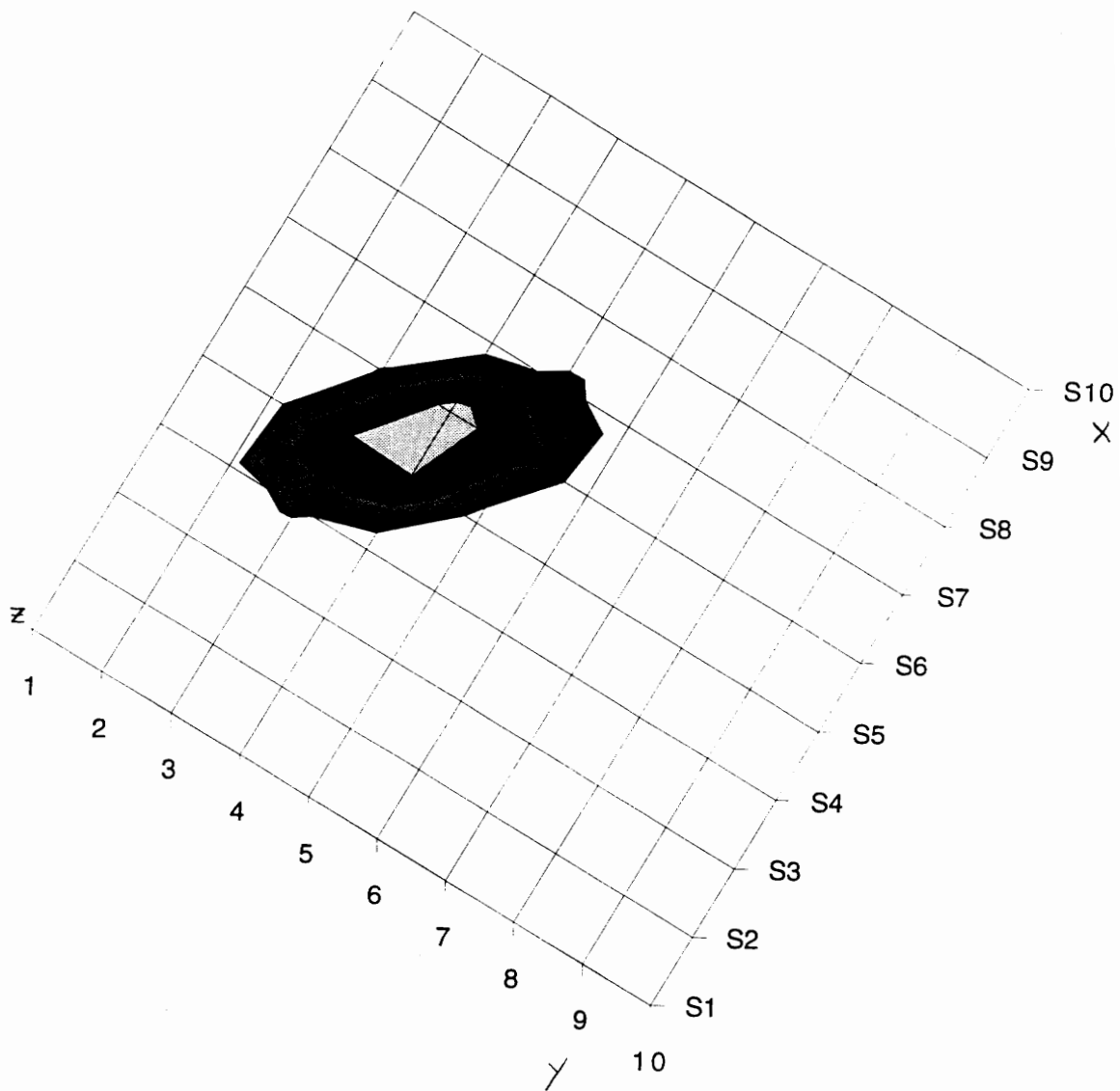


Figure 17. Joint PDF from Theory. Top View.

Tables

Table 1. Extrm 2L/ Extrm 2L.

$$g = X_1 - X_2$$

Variable	Type	M	D
X1	Extreme 2 Largest	200.0	100.0
X2	Extreme 2 Largest	150.0	75.0

ρ	-0.5	-0.3	0.0	0.3	0.5
F	1.956	1.782	1.000	1.360	1.250
Mnt Carlo					
β	0.456	0.517	0.640	0.811	1.005
P.F	0.324	0.302	0.261	0.209	0.157
Ang Tang					
β	0.512	0.549	0.623	0.737	0.862
P.F	0.304	0.291	0.267	0.230	0.194
% Error(β)	12.28	6.19	-2.66	-9.12	-14.23
%Error(Pf)	-6.17	-3.64	2.30	10.05	23.57
b(β)	0.891	0.942	1.027	1.100	1.166
b(Pf)	1.066	1.038	0.977	0.909	0.809
Kiuregh.					
β	0.448	0.507	0.623	0.797	0.982
P.F	0.327	0.306	0.267	0.213	0.163
% Error(β)	-1.75	-1.93	-2.66	-1.73	-2.29
%Error(Pf)	0.92	1.32	2.30	1.91	3.82
b(β)	1.018	1.020	1.027	1.018	1.023
b(Pf)	0.991	0.987	0.977	0.981	0.963
Independ.					
β	0.623	0.623	0.623	0.623	0.623
P.F	0.267	0.267	0.267	0.267	0.267
% Error(β)	36.60	20.50	-2.66	-23.20	-38.00
%Error(Pf)	-17.60	-11.59	2.30	27.75	70.06

Table 2. Extrm 2L/ Extrm 2L.

$$g = X_1 - X_2$$

Variable	Type	M	D
X1	Extreme 2 Largest	500.0	200.0
X2	Extreme 2 Largest	300.0	80.0

ρ	-0.5	-0.3	0.0	0.3	0.5
F	1.559	1.456	1.000	1.208	1.143
Mnt Carlo					
β	1.042	1.163	1.375	1.726	2.064
P.F	0.149	0.122	0.845 E-1	0.422 E-1	0.195 E-1
Ang Tang					
β	1.133	1.212	1.372	1.618	1.884
P.F	0.129	0.113	0.850 E-1	0.528 E-1	0.298 E-1
% Error(β)	8.73	4.21	-0.22	-6.26	-8.72
% Error(Pf)	-13.42	-7.38	0.59	25.12	52.82
b(β)	0.920	0.960	1.002	1.067	1.095
b(Pf)	1.155	1.080	0.994	0.779	0.654
Kiuregh.					
β	1.043	1.156	1.327	1.689	2.017
P.F	0.148	0.124	0.850 E-1	0.456 E-1	0.218 E-1
% Error(β)	9.60 E-2	-0.60	-0.22	-2.14	-2.28
% Error(Pf)	-0.67	1.64	0.59	8.06	11.79
b(β)	0.999	1.006	1.002	1.022	1.023
b(Pf)	1.007	0.984	0.994	0.925	0.894
Independ.					
β	1.372	1.372	1.372	1.372	1.372
P.F	0.850 E-1	0.850 E-1	0.850 E-1	0.850 E-1	0.850 E-1
% Error(β)	31.67	17.97	-0.22	-20.51	-33.53
% Error(Pf)	-42.95	-30.33	0.59	101.42	335.90

Table 3. Shifted Expo./ Extrm 2L.

$$g = X_1 - X_2$$

Variable	Type	M	D
X1	Shifted Exponential	200.0	100.0
X2	Extreme 2 Largest	125.0	60.0

ρ	-0.5	-0.3	0.0	0.3	0.5
F	1.670	1.549	1.000	1.248	1.169
Mnt Carlo					
β	0.597	0.689	0.854	1.094	1.344
P.F	0.275	0.245	0.196	0.137	0.089
Ang Tang					
β	0.651	0.699	0.797	0.952	1.126
P.F	0.258	0.242	0.213	0.171	0.130
% Error(β)	9.04	1.45	-6.67	-12.98	-16.22
% Error(Pf)	-6.18	-1.22	8.67	24.82	46.07
b(β)	0.917	0.986	1.075	1.149	1.194
b(Pf)	1.066	1.012	0.920	0.801	0.685
Kiuregh.					
β	0.589	0.659	0.797	1.007	1.234
P.F	0.278	0.255	0.213	0.157	0.109
% Error(β)	-1.34	-4.35	-6.67	-7.95	-8.18
% Error(Pf)	1.09	4.08	8.67	14.60	22.47
b(β)	1.013	1.045	1.075	1.086	1.089
b(Pf)	0.989	0.961	0.920	0.873	0.816
Independ.					
β	0.797	0.797	0.797	0.797	0.797
P.F	0.213	0.213	0.213	0.213	0.213
% Error(β)	33.50	15.67	-6.67	-27.15	-40.70
% Error(Pf)	-22.54	-13.06	8.67	55.47	139.33

Table 4. Lognormal/ Shifted Expo.

$$g = X_1 - X_2$$

Variable	Type	M	D
X1	Lognormal	500.0	250.0
X2	Shifted Exponential	200.0	100.0

ρ	-0.5	-0.3	0.0	0.3	0.5
F	1.297	1.250	1.000	1.121	1.082
Mnt Carlo					
β	1.095	1.190	1.377	1.715	2.056
P.F	0.137	0.117	0.084	0.043	0.020
Ang Tang					
β	1.133	1.216	1.385	1.653	1.951
P.F	0.129	0.112	0.083	0.049	0.025
%Error(β)	3.47	2.18	0.58	-3.61	-5.11
%Error (Pf)	-5.84	-4.27	-1.19	13.95	25.00
b(β)	0.966	0.979	0.994	1.037	1.054
b(Pf)	1.060	1.045	1.012	0.877	0.800
Kiuregh.					
β	1.081	1.183	1.385	1.697	2.035
P.F	0.140	0.118	0.083	0.045	0.021
%Error(β)	-4.59	-0.59	0.58	-1.05	-1.02
%Error(Pf)	2.19	0.85	-1.19	4.65	5.00
b(β)	1.013	1.006	0.994	1.011	1.010
b(Pf)	0.979	0.991	1.012	0.955	0.952
Independ.					
β	1.385	1.385	1.385	1.385	1.385
P.F	0.083	0.083	0.083	0.083	0.083
%Error(β)	26.48	16.39	0.58	-19.24	-32.64
%Error(Pf)	-39.42	-29.05	-1.19	93.02	315.00

Table 5. Lognormal/ Lognormal.

$$g = X_1 - X_2$$

Variable	Type	M	D
X1	Lognormal	500.0	200.0
X2	Lognormal	300.0	80.0

ρ	-0.5	-0.3	0.0	0.3	0.5
F	1.086	1.074	1.000	1.040	1.029
Mnt Carlo					
β	0.830	0.885	1.022	1.209	1.392
P.F	0.203	0.188	0.153	0.113	0.819 E-1
Ang Tang					
β	0.835	0.894	1.011	1.190	1.382
P.F	0.202	0.186	0.156	0.117	0.835 E-1
% Error(β)	0.60	1.02	-1.08	-1.57	-0.72
% Error(Pf)	-0.49	-1.06	1.96	3.54	1.95
b(β)	0.994	0.990	1.011	1.016	1.007
b(Pf)	1.005	1.011	0.981	0.966	0.981
Kiuregh.					
β	0.824	0.887	1.011	1.200	1.400
P.F	0.205	0.188	0.156	0.115	0.808 E-1
% Error(β)	-0.72	0.23	-1.08	-0.74	0.57
% Error(Pf)	0.98	0.00	1.96	1.77	-1.34
b(β)	1.007	0.998	1.011	1.007	0.994
b(Pf)	0.990	1.00	0.981	0.983	1.014
Independ.					
β	1.011	1.011	1.011	1.011	1.011
P.F	0.156	0.156	0.156	0.156	0.156
% Error(β)	21.81	14.24	-1.08	-16.38	-27.37
% Error(Pf)	-23.15	-17.02	1.96	38.05	90.48

Table 6. Extrm 2L/ Shifted Expo.

$$g = X_1 - X_2$$

Variable	Type	M	D
X1	Extreme 2 Largest	500.0	200.0
X2	Shifted Exponential	300.0	80.0

ρ	-0.5	-0.3	0.0	0.3	0.5
F	1.443	1.353	1.000	1.145	1.097
Mnt Carlo					
β	1.021	1.136	1.337	1.659	1.962
P.F	0.154	0.128	0.906 E-1	0.486 E-1	0.249 E-1
Ang Tang					
β	1.080	1.154	1.301	1.528	1.774
P.F	0.140	0.124	0.966 E-1	0.632 E-1	0.380 E-1
% Error(β)	5.79	1.584	-2.69	-7.90	-9.58
%Error(Pf)	-9.09	-3.12	6.62	30.04	52.61
b(β)	0.945	0.984	1.028	1.086	1.106
b(Pf)	1.100	1.032	0.938	0.769	0.655
Kiuregh.					
β	1.012	1.113	1.301	1.573	1.855
P.F	0.156	0.133	0.966 E-1	0.579 E-1	0.318 E-1
% Error(β)	-0.88	-2.02	-2.69	-5.18	-5.45
% Error(Pf)	1.30	3.91	6.62	19.13	27.71
b(β)	1.009	1.021	1.028	1.055	1.058
b(Pf)	0.987	0.962	0.938	0.839	0.783
Independ.					
β	1.301	1.301	1.301	1.301	1.301
P.F	0.966 E-1	0.966 E-1	0.966 E-1	0.966 E-1	0.966 E-1
% Error(β)	27.42	14.52	-2.69	-21.58	-33.69
%Error(Pf)	-37.27	-24.53	6.62	98.76	287.95

Table 7. Extrim 2L/ Extrim 1L.

$$g = X_1 - X_2$$

Variable	Type	M	D
X1	Extreme 2 Largest	500.0	200.0
X2	Extreme 1 Largest	300.0	80.0

ρ	-0.5	-0.3	0.0	0.3	0.5
F	1.233	1.200	1.000	1.111	1.084
Mnt Carlo					
β	1.016	1.117	1.304	1.620	1.951
P.F	0.155	0.132	0.962 E-1	0.526 E-1	0.255 E-1
Ang Tang					
β	1.038	1.113	1.263	1.500	1.762
P.F	0.150	0.133	0.103	0.668 E-1	0.390 E-1
% Error(β)	2.16	-0.36	-3.14	-7.41	-9.68
% Error(Pf)	-3.22	0.76	7.07	27.00	52.94
b(β)	0.979	1.003	1.032	1.08	1.107
b(Pf)	1.033	0.992	0.934	0.787	0.654
Kiuregh.					
β	1.001	1.089	1.263	1.535	1.837
P.F	0.158	0.138	0.103	0.623 E-1	0.331 E-1
% Error(β)	-1.48	-2.51	-3.14	-5.25	-5.84
% Error(Pf)	1.93	4.54	7.07	18.44	29.80
b(β)	1.015	1.026	1.032	1.055	1.062
b(Pf)	0.981	0.956	0.934	0.844	0.770
Independ.					
β	1.263	1.263	1.263	1.263	1.263
P.F	0.103	0.103	0.103	0.103	0.103
% Error(β)	24.31	13.07	-3.14	-22.04	-35.26
% Error(Pf)	-33.55	-21.97	7.07	95.82	303.92

Table 8. Lognormal/ Normal.

$$g = X_1 - X_2$$

Variable	Type	M	D
X1	Lognormal	500.0	200.0
X2	Normal	350.0	150.0

ρ	-0.5	-0.3	0.0	0.3	0.5
F	1.044	1.044	1.000	1.044	1.044
Mnt Carlo					
β	0.443	0.495	0.580	0.718	0.861
P.F	0.329	0.310	0.281	0.236	0.195
Ang Tang					
β	0.416	0.447	0.509	0.609	0.720
P.F	0.339	0.327	0.305	0.271	0.236
% Error(β)	-6.09	-9.70	-12.24	-15.18	-16.38
% Error(Pf)	3.04	5.48	8.54	14.83	21.02
b(β)	1.065	1.107	1.139	1.179	1.196
b(Pf)	0.970	0.948	0.921	0.871	0.826
Kiuregh.					
β	0.413	0.445	0.509	0.615	0.737
P.F	0.340	0.328	0.305	0.269	0.231
% Error(β)	-6.77	-10.10	-12.24	-14.34	-14.40
% Error(Pf)	3.34	5.81	8.54	13.98	18.46
b(β)	1.073	1.112	1.139	1.167	1.168
b(Pf)	0.968	0.945	0.921	0.877	0.844
Independ.					
β	0.509	0.509	0.509	0.509	0.509
P.F	0.305	0.305	0.305	0.305	0.305
% Error(β)	14.90	2.83	-12.24	-29.11	-40.88
% Error(Pf)	-7.29	-1.61	8.54	29.24	56.41

Table 9. Extrm 1L/ Extrm 1L.

$$g = X_1 - X_2$$

Variable	Type	M	D
X1	Extreme 1 Largest	500.0	200.0
X2	Extreme 1 Largest	300.0	80.0

ρ	-0.5	-0.3	0.0	0.3	0.5
F	1.100	1.085	1.000	1.044	1.031
Mnt Carlo					
β	0.833	0.882	1.004	1.167	1.335
P.F	0.202	0.189	0.158	0.122	0.909 E-1
Ang Tang					
β	0.859	0.919	1.038	1.218	1.403
P.F	0.195	0.179	0.150	0.112	0.803 E-1
% Error(β)	3.12	4.19	3.39	4.37	5.09
% Error(Pf)	-3.46	-5.29	-5.06	-8.20	-11.66
b(β)	0.970	0.960	0.967	0.958	0.951
b(Pf)	1.036	1.056	1.053	1.089	1.132
Kiuregh.					
β	0.846	0.910	1.038	1.228	1.421
P.F	0.199	0.181	0.150	0.110	0.777 E-1
% Error(β)	1.56	3.17	3.39	5.23	6.44
% Error(Pf)	-1.48	-4.23	-5.06	-9.84	-14.52
b(β)	0.985	0.970	0.967	0.950	0.939
b(Pf)	1.015	1.044	1.053	1.109	1.170
Independ.					
β	1.038	1.038	1.038	1.038	1.038
P.F	0.150	0.150	0.150	0.150	0.150
% Error(β)	24.61	17.69	3.39	-11.05	-22.25
% Error(Pf)	-25.74	-20.63	-5.06	22.95	65.02

Table 10. Lognormal/ Extrm I L.

$$g = X_1 - X_2$$

Variable	Type	M	D
X1	Lognormal	500.0	200.0
X2	Extreme I Largest	100.0	45.0

ρ	-0.5	-0.3	0.0	0.3	0.5
F	1.127	1.109	1.000	1.057	1.040
Mnt Carlo					
β	2.290	2.478	2.894	3.450	4.108
P.F	0.110 E-1	0.660 E-2	0.190 E-2	0.280 E-3	0.200 E-4
Ang Tang					
β	2.328	2.501	2.851	3.408	4.032
P.F	0.995 E-2	0.619 E-2	0.218 E-2	0.327 E-3	0.277 E-4
% Error(β)	1.66	0.93	-1.49	-1.22	-1.85
% Error(Pf)	-9.54	-6.21	14.74	16.79	38.50
b(β)	0.984	0.991	1.015	1.012	1.019
b(Pf)	1.105	1.066	0.871	0.856	0.722
Kiuregh.					
β	2.281	2.470	2.851	3.450	4.114
P.F	0.113 E-1	0.675 E-2	0.218 E-2	0.280 E-3	0.194 E-4
% Error(β)	-0.39	-0.32	-1.49	0.00	0.15
% Error(Pf)	2.73	2.27	14.74	0.00	-3.00
b(β)	1.004	1.003	1.015	1.00	0.998
b(Pf)	0.973	0.978	0.871	1.00	1.031
Independ.					
β	2.851	2.851	2.851	2.851	2.851
P.F	0.218 E-2	0.218 E-2	0.218 E-2	0.218 E-2	0.218 E-2
% Error(β)	24.50	15.05	-1.49	-17.36	-30.60
% Error(Pf)	-80.18	-66.97	14.74		

Table 11. Extrm 1L/ Extrm 2L.

$$g = X_1 - X_2$$

Variable	Type	M	D
X1	Extreme 1 Largest	500.0	200.0
X2	Extreme 2 Largest	100.0	45.0

ρ	-0.5	-0.3	0.0	0.3	0.5
F	1.362	1.317	1.000	1.191	1.152
Mnt Carlo					
β	2.081	2.237	2.583	3.090	3.568
P.F	0.187 E-1	0.115 E-1	0.490 E-2	0.100 E-2	0.180 E-3
Ang Tang					
β	2.238	2.398	2.710	3.142	3.759
P.F	0.126 E-1	0.824 E-2	0.336 E-2	0.840 E-3	0.854 E-4
% Error(β)	7.54	7.197	4.92	1.68	5.35
%Error(Pf)	-32.62	-28.35	-31.43	-16.00	-52.56
b(β)	0.930	0.933	0.953	0.983	0.949
b(Pf)	1.484	1.396	1.458	1.190	2.108
Kiuregh.					
β	2.117	2.318	2.710	3.239	3.818
P.F	0.171 E-1	0.100 E-1	0.336 E-2	0.599 E-3	0.672 E-4
% Error(β)	1.73	3.62	4.92	4.82	7.01
%Error(Pf)	-8.56	-13.04	-31.43	-40.1	-62.67
b(β)	0.983	0.965	0.953	0.954	0.934
b(Pf)	1.094	0.150	1.458	1.669	2.679
Independ.					
β	2.710	2.710	2.710	2.710	2.710
P.F	0.336 E-2	0.336 E-2	0.336 E-2	0.336 E-2	0.336 E-2
% Error(β)	30.20	21.14	4.92	-12.30	-24.05
%Error(Pf)	79.68	-70.78	-31.43	236.00	1766.67

Table 12. Extrm 1L/ Extrm 2L.

$$g = X_1 - X_2$$

Variable	Type	M	D
X1	Extreme 1 Largest	500.0	380.0
X2	Extreme 2 Largest	100.0	80.0

ρ	-0.5	-0.3	0.0	0.3	0.5
F	1.679	1.611	1.000	1.416	1.354
Mnt Carlo					
β	1.055	1.101	1.185	1.287	1.351
P.F	0.146	0.135	0.118	0.099	0.088
Ang Tang					
β	1.152	1.200	1.268	1.328	1.361
P.F	0.125	0.115	0.102	0.092	0.087
% Error(β)	9.19	8.99	7.00	3.19	0.74
%Error(Pf)	-14.38	-14.81	-13.56	-7.07	-12.12
b(β)	0.916	0.917	0.934	0.969	0.993
b(Pf)	1.168	1.174	1.157	1.076	1.011
Kiuregh.					
β	1.076	1.156	1.268	1.349	1.386
P.F	0.141	0.124	0.102	0.089	0.083
% Error(β)	1.99	4.99	7.00	4.82	2.59
%Error(Pf)	-3.42	-8.15	-13.56	-10.10	-16.16
b(β)	0.980	0.952	0.934	0.954	0.975
b(Pf)	1.035	1.089	1.157	1.112	1.060
Independ.					
β	1.268	1.268	1.268	1.268	1.268
P.F	0.102	0.102	0.102	0.102	0.102
% Error(β)	20.19	15.17	7.00	-1.48	-6.14
%Error(Pf)	-30.14	-24.44	-13.56	3.03	15.91

Table 13. Extrm 1L/ Extrm 2L.

$$g = X_1 - X_2$$

Variable	Type	M	D
X1	Extreme 1 Largest	500.0	200.0
X2	Extreme 2 Largest	250.0	112.5

ρ	-0.5	-0.3	0.0	0.3	0.5
F	1.362	1.317	1.000	1.191	1.152
Mnt Carlo					
β	1.059	1.130	1.286	1.566	1.861
P.F	0.145	0.129	0.992 E-1	0.587 E-1	0.314 E-1
Ang Tang					
β	1.138	1.223	1.394	1.665	1.966
P.F	0.127	0.111	0.817 E-1	0.480 E-1	0.246 E-1
% Error(β)	7.46	8.23	8.40	6.32	5.64
%Error(Pf)	-12.41	-13.95	-17.64	-18.23	-21.66
b(β)	0.931	0.924	0.922	0.940	0.947
b(Pf)	1.142	1.162	1.214	1.223	1.276
Kiuregh.					
β	1.075	1.180	1.394	1.737	2.129
P.F	0.141	0.119	0.817 E-1	0.412 E-1	0.166 E-1
% Error(β)	1.51	4.42	8.40	10.92	14.40
%Error(Pf)	-2.76	-7.75	-17.64	-29.81	-47.13
b(β)	0.985	0.958	0.922	0.902	0.874
b(Pf)	1.028	1.084	1.214	1.425	1.892
Independ.					
β	1.394	1.394	1.394	1.394	1.394
P.F	0.817 E-1	0.817 E-1	0.817 E-1	0.817 E-1	0.817 E-1
% Error(β)	31.63	23.36	8.40	-10.98	14.09
%Error(Pf)	-43.65	-36.67	-17.64	39.18	160.19

Table 14. Extrm 2L/ Lognormal.

$$g = X_1 - X_2$$

Variable	Type	M	D
X1	Extreme 2 Largest	500.0	250.0
X2	Lognormal	250.0	125.0

ρ	-0.5	-0.3	0.0	0.3	0.5
F	1.469	1.410	1.000	1.244	1.192
Mnt Carlo					
β	0.968	1.059	1.269	1.580	1.893
P.F	0.166	0.145	0.102	0.570 E-1	0.292 E-1
Ang Tang					
β	1.017	1.083	1.211	1.407	1.617
P.F	0.155	0.139	0.113	0.797 E-1	0.530 E-1
% Error(β)	5.06	2.27	-4.57	-10.95	-14.58
%Error(Pf)	-6.63	-4.14	10.78	39.82	81.51
b(β)	0.952	0.978	1.048	1.123	1.171
b(Pf)	1.071	1.043	0.903	0.715	0.551
Kiuregh.					
β	0.954	1.041	1.211	1.473	1.763
P.F	0.170	0.149	0.113	0.703 E-1	0.389 E-1
% Error(β)	-1.45	-1.70	-4.57	-6.77	-6.87
%Error(Pf)	2.41	2.76	10.78	23.33	33.22
b(β)	1.015	1.017	1.048	1.073	1.074
b(Pf)	0.976	0.973	0.903	0.811	0.751
Independ.					
β	1.211	1.211	1.211	1.211	1.211
P.F	0.113	0.113	0.113	0.113	0.113
% Error(β)	25.10	14.35	-4.57	-23.35	-36.03
%Error(Pf)	-31.93	-22.07	10.78	98.25	286.99

Table 15. Lognormal/ Uniform.

$$g = X_1 - X_2$$

Variable	Type	M	D
X1	Lognormal	500.0	200.0
X2	Uniform	300.0	80.0

ρ	-0.5	-0.3	0.0	0.3	0.5
F	1.043	1.041	1.000	1.041	1.043
Mnt Carlo					
β	0.785	0.858	0.995	1.186	1.385
P.F	0.216	0.196	0.160	0.118	0.831 E-1
Ang Tang					
β	0.717	0.767	0.869	1.028	1.205
P.F	0.237	0.221	0.192	0.152	0.114
% Error(β)	-8.66	-10.61	-12.66	-13.32	-13.00
%Error(Pf)	9.72	12.75	20.00	28.81	37.18
b(β)	1.095	1.119	1.145	1.154	1.149
b(Pf)	0.911	0.887	0.833	0.776	0.729
Kiuregh.					
β	0.712	0.764	0.869	1.037	1.231
P.F	0.238	0.222	0.192	0.150	0.109
% Error(β)	-9.30	-10.96	-12.66	-12.56	-11.12
%Error(Pf)	10.18	13.26	20.00	27.12	31.17
b(β)	1.102	1.123	1.145	1.144	1.125
b(Pf)	0.908	0.883	0.833	0.787	0.762
Independ.					
β	0.869	0.869	0.869	0.869	0.869
P.F	0.192	0.192	0.192	0.192	0.192
% Error(β)	10.70	1.28	-12.66	-26.73	-37.26
%Error(Pf)	-11.11	-2.04	20.00	62.71	131.05

Table 16. Highest Error found for Each One of the Methods.

$$G=X1-X2$$

Variable	Type	M	D
X 1	Lognormal	500.0	200.0
X 2	Normal	350.0	150.0

Correlation Coefficient = 0.5

	%E β	%E Pf	bias (β)	bias (Pf)
Ang and Tang	-16.38	21.02	1.196	0.826
Kiureghian	-14.40	18.46	1.168	0.844
Independent	-40.88	56.41	1.692	0.639

Table 17. Mean, Standard Deviation, and Coefficient of Variation: a) For % Error. b) For Bias.

	μ %E	σ %E	δ %E	μ %E	σ %E	δ %E
Ang & T	6.18	4.37	0.71	-1.35	7.48	-5.54
Kiureghian	4.41	3.95	0.90	-1.62	5.72	-3.51
Independ.	18.85	11.23	0.60	-1.64	21.98	-13.40

	μ b	σ b	δ b
Ang & T.	1.0197	0.080	0.078
Kiureghian	1.0194	0.060	0.059

Table 18. Example of Safety Index Obtained by different Approaches.

$$G=X1-X2$$

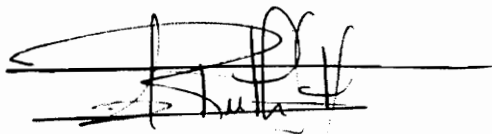
Variable	Type	M	D
X 1	Extreme 2 L	199.88	79.523
X 2	Extreme 2 L	154.78	125.480

Correlation Coefficient = 0.421

	β
M-C 1	0.951
M-C 2	1.229
Ang & T.	0.829
Kiureghian	0.926

Vita

I was born on the 24th of October, 1969 in Madrid, Spain. I received my Bachelor's Degree in Aerospace Engineering in Parks College of Saint Louis University in December, 1991. In January 1992, I joined the Master of Science program in the Aerospace and Ocean Engineering department at Virginia Polytechnic Institute and State University.

A handwritten signature in black ink, appearing to be 'R. P. ...', written over a horizontal line.