1 Introduction

I wrote this thesis because I believe that a powerful solution technique known as the Riccati iteration, which is typically applied to complex (hundreds of variables), multiple-objective dynamic optimization problems and requires powerful computers and unique software[1] can also be applied successfully with a common personal computer and spreadsheet software to solve less complex problems that would otherwise be difficult or infeasible to solve. My objective in this thesis is to do just that: use the Riccati iteration with spreadsheet software to solve a difficult dynamic optimization problem.

In some ways, this approach is a little unusual: a solution in search of a problem. However, I believe the Riccati approach is so compelling that demonstrating its value when solving a difficult problem may encourage others to apply the Riccati technique to other problems. The problem that I will consider in this thesis is a capital replacement problem posed by Lotfi where multiple objectives have been identified.[2]

Lotfi introduced a multiple-objective dynamic optimization problem that asked: "What is the best way to replace capital equipment over time when the owners have identified multiple objectives to be optimized during the replacement?" The Lotfi problem as presented in his paper is too complex for my purposes, so it will be simplified and critiqued before it is used. My thesis will demonstrate how difficult even the simplified Lotfi problem is to solve using the popular Lagrange approach versus the

[1] The Riccati iteration has been employed in papers studying optimization problems as diverse as the macroeconomic performance of the U.S. economy, to the performance of a polyethylene chemical plant.
[2] Lotfi, V., "Implementing flexible automation: A multiple criteria decision making approach", *International Journal of Production Economics*, v38, pp. 255-268, (1995).

Riccati approach.

Before proceeding, some definitions should be presented: A **dynamic** optimization problem (as opposed to the more typical static optimization[3]) is a problem where resources are allocated over an interval of time from *initial time* to *terminal time*. Variables describing the system are called *state variables*. In mathematical terms, the problem is that of choosing time paths for certain variables, called *control variables*, from a given class of time paths, called the *control set*. The time paths for the control variables are chosen to maximize a given function, or functions depending on the time paths for the control and the state variables, called the *objective function(s)*. When presented in this form the problem is referred to as the *control problem*.[4]

1.1 Defense of a Multiple Objective Problem

The fact that a firm may attempt to optimize more than one objective function is a contentious proposition. It is usually assumed that firms choose a goal of profit maximization or cost minimization only. To do otherwise would imply that managers do not always work in the best interest of their shareholders. There is evidence, however, that firms do precisely this.

Cohen and Cyert state that firm managers may not make decisions purely for profit maximization, and that instead, decision-makers may be attempting to maximize

[3] A static optimization problem is one where resources are allocated at a given point in time. In mathematical terms, the problem is that of choosing values for certain variables, called *instruments*, from a given set, called the *opportunity set*, so as to maximize a given function, called the *objective function*. When expressed in this form, the problem is referred to as the *mathematical programming problem*. Optimization theory represents an effective way of solving this problem, and the Lagrangian maximization approach is one of the more sophisticated and popular optimization approaches employed in the solution of the mathematical programming problem in microeconomic theory.
[4] Intriligator, M.D., *Mathematical Optimization and Economic Theory*, Prentice-Hall, Inc., Englewood Cliffs, NJ (1971).

their own security or power.[5] Simon argues that managers may actually only be interested in satisficing goals, not maximizing them.[6] Studies of organizational objectives suggest that firms may actually be pursuing different goals at the same time, since goals are the result of a continuous bargaining-learning process.[7] A recent investigation asserts that many "visionary" firms (including Sony, 3M, Proctor & Gamble, Hewlett-Packard, Merck, Ford, etc.) were founded and maintain to this day principles and procedures that attempt to maximize more than profits.[8]

Even if one does not believe that firms attempt to optimize more than one objective (profits), a multiple objective problem may still be required for a firm that attempts to maximize profits only. Cyert and March contend that maximizing profits often entails several related goals, and identify six common profit-related targets used by firms: net earnings per share, net dollar sales, cash flow, return on investment, return on stockholders equity, and new orders received.[9] A dynamic optimization problem with these six goals would also be an appropriate implementation of a multiple objective problem formulation.

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[5] Cohen, K.J., R.M. Cyert, *Theory of the Firm, Resource Allocation in a Market Economy*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1975), pp 50-51.
[6] Simon, H.A., "A Behavioral Model of Rational Choice", *Quarterly Journal of Economics*, v69, 1955, pp. 99-118.
[7] Cyert, R.M., J.G. March, *A Behavioral Theory of the Firm*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1963).
[8] Collins, J.C., J.I. Porras, *Built to Last*, HarperCollins Publishers, Inc., New York, (1997).
[9] Cyert, R.M., J.G. March, *A Behavioral Theory of the Firm*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1963).

2 Review of Literature

2.1 Literature Review: Dynamic Optimization Problems

Textbooks typically emphasize dynamic optimization problems that have only one objective function.[10] Such problems are referred to as single-input, single-output problems. However, multiple-objective dynamic optimization problems often appear in the literature. Examples of such problems often surface in financial management and normative macroeconomic studies. A literature search reveals the following dynamic optimization papers:

Jorgensen and Kort studied a dynamic (multi-period) optimization problem in renewable resource harvesting, capital investment, and financing.[11] This work combined renewable resource harvesting with capital investments under borrowing and lending constraints. The problem was set up as a dynamic optimization of resource stock and stock of equity by controlling effort rate and dividend pay out rate. Jorgensen used the maximum principal to identify the solution.

Chow has investigated classic problems of portfolio selection and investment by modeling them as dynamic optimization problems involving stochastic differential equations.[12] Chow solves this problem by employing a method of Lagrange multipliers

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[10] All examples given in Intrilligator are single-variable.

[11] Jorgensen, S., P.M. Kort, "Optimal investment and finance in renewable resource harvesting", *Journal of Economic Dynamics and Control*, v20, pp. 1-18, (1996).

[12] Chow, G.C., "The Lagrangian method of optimization with applications to portfolio and investment decisions", *Journal of Economic Dynamics and Control*, v21, pp. 603-630, (1997).

that is a generalization of Pontryagin's maximum principal and avoids having to solve the Bellman equation for the value function. Chow suggests that this method has certain computational advantages.

Maranas investigates a multi-stage financial planning problem in which a specialized global optimization algorithm, which guarantees finite convergence, is employed.[13] The "dynamically balanced" decision rule requires the purchase and sale of assets at each time stage so as to keep constant asset proportions in the portfolio composition. This requirement results in a non-convex objective function. The computational results of the Maranas, study demonstrate the procedure's efficiency on a real world financial planning problem, and also demonstrate that the rule does not understate the efficient frontier.

Macroeconomic policy is another area inherently inclined to multi-objective dynamic optimization studies. It is generally agreed that the goals of macroeconomic policy are low unemployment, price stability, and economic growth. (There are differences of opinion about the relative weight assigned to each of these goals.) The overriding question for macroeconomic policy is one of how to set the policy instruments, (government spending, tax rates, etc.) to achieve the target unemployment, price, and growth levels.[14]

One recent paper in this area was by Juillard, which discusses alternative algorithms for solving nonlinear forward looking macroeconomic models.[15] The author

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[13] Maranas, C.D., "Solving Long-Term Financial Planning Problems via Global Optimization", *Journal of Economic Dynamics and Control*, v21, n 8-9, pp. 1405-1425, (1997).
[14] Froyen, R.T., *Macroeconomics Theories & Policies*, 5th ed., Prentice-Hall, Inc., Upper Saddle River, NJ (1996).
[15] Juillard, M., "An Algorithm Competition: First Order Iterations versus Newton-Based Techniques", *Journal of Economic Dynamics and Control*, v22, n 8-9, pp. 1291-1318, (1998).

suggests that Newton-based techniques are robust and efficient solution techniques for solving nonlinear forward looking models. Juillard suggests that the relative lack of such techniques may explain why development and use of forward looking macro models in policy-making institutions has proceeded at a much slower pace than what was predicted in the early 1980s.

Caccou and Lansing employed a simple growth model in order to derive an optimal transition path for the provision of public capital.[16] A normative comparison was conducted to quantify the conditions under which an increase in the stock of public capital is desirable. Caccou and Lansing then analyzed the ratio of public to private capital in the U.S. economy since 1925 in an attempt to investigate the degree to which non-optimal fiscal policies could account for U.S. productivity slowdown.

Tucci used dynamic optimization theory and rational expectations to study the impacts of employing uncertain hyperstructural parameters on the choice of control in an adaptive control framework.[17] He contended that under the rational expectation hypothesis, the standard results of ineffectiveness of monetary policy might not hold when some of the parameters of the system are uncertain. Tucci concluded that given the complexity of the typical solution procedures, it may be worth it, on some occasions, to use an approximate multivariate solution based on time varying parameters.

2.2 Problem Selection: The Lotfi Manufacturing Problem

As stated in the introduction, I conducted a literature search to find a particular

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[16] Caccou, S.P., K.J. Lansing, "Optimal Fiscal Policy, Public Capital, and the Productivity Slowdown", *Journal of Economic Dynamics and Control*, v22, n 6, pp. 911-935, (1998).
[17] Tucci, M.P., "Adaptive Control in the Presence of Time-Varying Parameters", *Journal of Economic Dynamics and Control*, v22, n1, pp. 39-47, (1997).

problem: a capital replacement problem where the decision maker has more than one objective to accomplish. I found something close in a paper published by Vahid Lotfi in 1995.[18]

Lotfi's paper dealt with the problem of replacing different machines in a parts manufacturing plant. Lotfi's goal was to replace six different types[19] of older numerical control machines with four new computer numerical control modules[20] over the span of five years. The machines and modules were flexible; each could be outfitted with different tool sets to produce different parts. The different machines formed a network, working together to produce a set of three different part types.

Lotfi's problem, however, was not simply a capital replacement problem. In addition to replacing the equipment, Lotfi wanted to control five different attributes of the solution. Lotfi translated these attributes into mathematical objective functions, including objectives on profitability, budget requirements, routing flexibility (the capability to process a given set of parts on alternate machines), firm disruption by module augmentation and finally, firm disruption by machine reduction. Unusually, Lotfi proposed solving this problem by optimizing on the different objectives simultaneously (a multiple objective optimization). He also defined constraints on his model that enforced various implementation restrictions, demand requirements, and setup charges.

Lotfi, using the stated objectives, constraints, and a phased, multi-period approach formulated a multi-objective, mixed integer linear programming (MOMILP) model to

[18] Lotfi, V., "Implementing flexible automation: A multiple criteria decision making approach", *International Journal of Production Economics*, v38, pp. 255-268, (1995).

[19] There were fifteen total machines: two of Type A, one of Type B, three of Type C, three of Type D, two of Type E, and four of Type F.

[20] There were eight new modules: two Type 1, three of Type 2, two of Type 3, and one of Type 4.

calculate a near-optimal schedule for replacement of the old machines and purchase of the new modules.[21]

2.3 Lotfi Model Modifications

One shortcoming of Lotfi's solution method (MOMILP) is that it only approximates an optimal solution. Since the Riccati iteration is a true optimal solution, (and the gist of my thesis is to demonstrate a practicable optimal solution method), I will forego a critique of Lotfi's solution method and instead focus on Lotfi's model. Optimal solution techniques will be applied to Lotfi's model.

In this thesis, I will review and modify Lotfi's model in three basic ways. First, I will reduce the problem size (variables). Second, I will critique Lotfi's model and then modify it based on the critique. Finally, I will modify the model so that it is amenable to the Riccati iteration.

The first modification will simply reduce the sheer size of Lotfi's problem. As mentioned in the introduction, the Riccati iteration has already been used to solve large, complex, multiple-objective dynamic optimization problems. In fact, Lotfi's model is a large, complex, multiple-objective dynamic optimization problem and would require a powerful computer and specialized software to solve. Since my objective is to implement the Riccati iteration using a common personal computer and spreadsheet software, I will reduce the scope of the Lotfi problem. Thankfully, the Lotfi model is amenable to simplifications, while still retaining its features of a real-word problem. I will reduce the

[21] Lotfi, V., "Implementing flexible automation: A multiple criteria decision making approach", *International Journal of Production Economics*, v38, pg 255, (1995).

number of modules, machines, and parts in the model, and will also make Lotfi's model a before-tax model. Doing so will result in a model that is well designed for this thesis.

The reduced problem I propose will consider only one type of old machine rather than six, one type of new machine modules rather than four, and will assume that the plant produces only one part type rather than three. I will also assume that each machine or module can produce the entire part by itself. Tax implications are very complicated, especially with regard to capital investments and salvage. In fact, although Lotfi included a tax rate, an investment credit factor, and capital gains or losses on selling material, he omitted several other tax implications, including depreciation and interest payments. Rather than including only select tax features, I will instead ignore tax implications by omitting all tax-related variables, and will produce a before-tax model.

The second set of modifications will be based on my critique of the resulting Lotfi model. Where I disagree with assumptions Lotfi made concerning the variables, the objectives, or the constraints, I will incorporate changes. I will also defend Lotfi's model when I believe it is correct.

The third modification will introduce changes to make Lotfi's model amenable to the Riccati iteration. One of these modifications will be to change the Lotfi objectives from maximizations to minimizations, as the Riccati is best solved as a minimization. Another will be put the objectives in linear quadratic form.

2.4 Comparative Optimizations: Riccati, Lagrange and Least-Squares

In this thesis, three alternative optimization methods, the *Riccati*, the *Lagrange*, and, (only to verify the Riccati solution) the *Least-Squares* will be investigated. The primary objective of this research is to demonstrate and compare the different optimization methods on a representative, multi-objective dynamic optimization problem.[22] The Riccati method[23] employs a series of matrix arithmetic problems to optimize the objective functions, as formally described in section five.

The *Lagrange* method, by contrast, calculates a gradient followed by a simultaneous solution of the resulting gradient equations. The size of the multi-objective, multi-period problem precludes the Lagrange solution from being practicable solution, as shown in sections 6.1, 6.2, and 6.3.

The *Least-Squares* method is a direct, brute-force approach.[24] The least-squares approach is the matrix equivalent of taking the derivative of an equation, setting it equal to zero, and solving for the minimum value. However, the least-squares approach involves not-so-simply deriving a matrix equation that is itself made up of other matrices (a super-matrix), and requires specialized software to solve, as demonstrated in Appendix D.

2.5 Game Plan: The Next Sections

Sections three, four, and five will follow the above game plan. Section three will introduce the Lotfi model—Lotfi's variables, Lotfi's objective equations, and Lotfi's constraints, and will reduce it based on the one old machine type, one new module type, one part type, and no tax (before tax model) assumptions. It should be stressed that in section three, Lotfi's model is not being changed in any way. I am simply selecting a

[22] Borrowed from Lotfi, V., "Implementing flexible automation: A multiple criteria decision making approach", *International Journal of Production Economics*, v38, pp. 255-268, (1995).
[23] Franklin, G.F., J.D. Powell and M.L. Workman, *Digital Control of Dynamic Systems*, 2nd ed., Addison-Wesley Publishing Co., New York, pp. 422-427 (1992).
[24] Professor J.B. Rawlings lecture notes, 1994, University of Texas.

particular instantiation of the Lotfi model. Section four, on the other hand, will change the Lotfi model. In this section, I will provide a critique of the resulting Lotfi model, and will incorporate changes meant to address those critiques. Finally, section five will introduce the Riccati solution method, show how the Lotfi model fits in to the Riccati iteration, and will modify the Lotfi model structure so that it conform to the Riccati iteration. This section, like section three will not fundamentally change the Lotfi model.

Section six will apply the Riccati iteration, will demonstrate why the Riccati solution is superior to the Lagrange solution, and will provide a numerical demonstration of the Riccati solution.

3 The Lotfi Model: Introduction and Reduction

In his paper, Lotfi presented a complex deterministic model, with the aim of deriving a Pareto optimal implementation and replacement schedule, as well as an aggregate production plan for the different parts.[25] This section will introduce the variables, objective functions, and constraints, as Lotfi introduced them. The model will then be pared-down, based on the assumption that the plant has only one old machine type, one new machine (module) type, one part type, and no taxes (a before tax model).

3.1 Lotfi's Variables

Lotfi's original model contained 39 variables. This section will help to reduce that number down to fourteen variables. The following sections will introduce all of Lotfi's original variables, and then will explain which shall be omitted.

3.1.1 Planning Horizon

The *planning* or *prediction horizon* is the length of the time considered when calculating the optimal decision. The prediction horizon is what allows a model-based control system to anticipate where the process is heading, and to make the proper calculations.[26] The prediction horizon can be one-step or greater, and more steps require

[25] Lotfi, V., "Implementing flexible automation: A multiple criteria decision making approach", *International Journal of Production Economics*, v38, p257, (1995).
[26] Seborg, P.E., et al, *Process Dynamics and Control*, John Wiley and Sons, Inc., New York, p. 654, (1989).

more calculations. The planning horizon in Lotfi's model, T, consisted of five periods where t denoted the time index.

Simplification

None. I will be using Lotfi's planning horizon.

Summary

Retain: T and t

3.1.2 Old Machine Types, Parts, and Operations

Lotfi denoted the three part types in his model by the index i, and the associated manufacturing steps for producing part type i were denoted by the index j. These manufacturing steps were performed on old machine types, m, for which the manufacturing capability of each machine was represented by an incidence matrix element, z'_{ijm}. The incidence matrix element assumed a value of one if operation j of part type i could be performed by machine type m, and a value of zero otherwise.[27] Lotfi defined ST'_{ijm} and p'_{ijm} to denote the setup and processing times of machine type m for step j and part i. Similarly, the setup cost (SC'_{ijm}), unit variable cost (d'_{ijm}) and a fixed cost element (e'_{ijm}) were specified for every combination of part, step, and machine type. The fixed cost element (e'_{ijm}) reflected the cost to retain the ability to create different products (i) using different operations (j) on different machines (m).

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[27] For example, if Machine A could perform the second manufacturing step on part type 3 then $z'_{32A} = 1$.

Simplification:

Since I am assuming only one part type and that each machine can produce that one part type by itself, it is not necessary to track either *i* part type or *j* operation, and therefore both indices *i* and *j* will be omitted from the model. Since there is only a single *i* part, no separate *j* operations, and only a single old machine, m and z'_{ijm} will also be omitted from the model.

Specifying the setup time, operating time, or setup cost is no longer necessary, either, since no tool changing will be required. Therefore, ST'_{ijm}, p'_{ijm}, and SC'_{ijm} will be omitted. The variable cost, d'_{ijm}, (now simply d') will be retained. Since there is only a single *i* part and no separate *j* operations, the fixed cost associated with retaining multiple capabilities can be omitted, and e'_{ijm} becomes zero.

Summary

Omit:	$i, j, m, z'_{ijm}, ST'_{ijm}, p'_{ijm}$, and SC'_{ijm}
Retain:	d'_{ijm} as d', and e'_{ijm} as 0

3.1.3 New Module Types, Parts, and Operations

For new modules, Lotfi also defined the three part types by the index *i*. The associated manufacturing steps for producing part type *i* are denoted by the index *j*, and the set of new module types by *n*. The manufacturing capabilities for new modules were defined as z''_{ijn}. z''_{ijn} assumed a value of one if operation *j* of part type *i* could be performed by module n, and a value of zero otherwise.[28] The notation for setup time, processing time, setup cost, unit variable cost and fixed cost elements for the new modules were updated to ST'', p''_{ijn}, SC''_n, d''_{ijn}, e''_{ijn}, respectively. Unlike the case for old machines, with the new modules Lotfi assumed that the setup time, ST'', was independent of the part being produced (i), manufacturing step (j), or module (n) and the setup cost, SC''_n, was independent of the part being produced (i) or the manufacturing step (j).

Simplification:

For the same reasons i, j, and m were omitted for the old machines, i, j, and n will be omitted for new modules and e''_{ijn} will again equal zero. Specifying the setup time, operating time, or setup cost is no longer necessary, either, since no tool changing will be required. Therefore, ST'', p''_{ijn}, and SC''_n will be omitted. The variable cost, d''_{ijn}, now d'' will be retained.

Summary

Omit: $i, j, m,$ z''_{ijn}, ST'', p''_{ijn}, and SC''_n

Retain: d''_{ijn} as d'', and e''_{ijn} as 0

[28] For Lotfi, the flexibility of the new modules was higher than the old machines, and therefore the number of z''_{ijn} elements with a value of one was greater than the number of z'_{ijm} elements with a value of one.

3.1.4 Number of Machines and Modules, Salvage Value and Cost

Lotfi denoted the number of type m current machines and type n new modules operating in a given period by $M_{m,t}$ and $N_{n,t}$, respectively. As the old machines were phased out during the transition periods, they were sold for the salvage value after capital gain or loss, $S_{m,t}$. Lotfi defined the capital costs for new modules, as $C_{n,t}$.

Simplification:

Since I am assuming only one machine and one module type, both m and n will be omitted from the model.

Summary:

Omit: m and n

Retain: $M_{m,t}$ as M_t, $N_{n,t}$ as N_t, $S_{m,t}$ as S_t, and $C_{n,t}$ as C_t

3.1.5 Manufacturing Steps

Lotfi assumed that as new modules were installed and current machines phased out, the three parts continued to be produced on all machines simultaneously. Lotfi optimized the assignment of manufacturing steps and production quantities between old machines and the new modules and denoted this segregation by the decision variables X_{it} (old machine production of part type i during period t) and s_{it} (the number of setups required on old machines for part type i during period t). Y_{it} (new module production of part type i during period t) and $y_{ijn,t}$ was the production quantity for operation (i,j) on new module type n during period t.

Simplification:

Again, index i, j, and n will be omitted, since there is only one new module type and it can produce the part by itself. X_{it} will become X_t, the quantity produced on old machines in time period t. Lotfi defines $\sum_n y_{ijn,t} = Y_{it}$ and since n and j are being omitted, $y_{ijn,t}$ is simply Y_{it}, which itself becomes Y_t, the quantity produced on new machines in time period t. Since all machines perform the same function, no new setups are required and therefore s_{it} will also be omitted.

Summary

Omit: i, j, n, s_{it}.

Retain: X_{it} as X_t, $\sum_n y_{ijn,t}$ or Y_{it} as Y_t

3.1.6 Flexibility

Lotfi was very concerned with manufacturing flexibility, and stated that flexibility was maintained by creating redundancy in each machine group. Lotfi went out of his way to define different types of flexibility in the manufacturing process: expansion flexibility, machine flexibility, routing flexibility, and process flexibility.

Lotfi defined expansion flexibility as the ability to add capacity through purchasing new equipment in a modular fashion.

Lotfi defined machine flexibility as the ability to process different part types on different machines. To help select and assign different manufacturing steps on machines and modules, Lotfi defined v_t to be equal to one if any manufacturing step was performed on a machine in period t, and w_n to be equal to one if a module type n was to be implemented. To help assign new modules, Lotfi defined K as the number of hours of available capacity per period. To help assign different existing machines, Lotfi defined the efficiency of machines and modules to be x_t, so the production function for old machines is $x_t'M$, and for new machines is $x_t''N$.

Lotfi ensured routing flexibility through planned redundancy and "slack" time on equipment. g was used to describe capacity slack, and separately, Lotfi proposed an objective function to facilitate redundancy within a machine group (the third objective, see below).

Lotfi accounted for process flexibility (ability to produce a given part type in several ways, specifically through loading a variety of parts in the tooling magazines) by assuming that this loading had a direct impact on minimizing changeover costs. The parameter L_t, the number of operating cycles in a period, was used with an associated cost function, $g(L_t)$ to provide a dollar value of process flexibility.[29]

Simplification:

Flexibility is not a concern of my model, since I am only proposing to produce one product and have only old machines and new modules. Therefore, variables relating directly to flexibility – g, $g(L_t)$, and L_t – will be omitted. Since it is assumed that all machines and modules can and will produce the same part independently, v_t, w_n, and K will also be omitted. x_t will be retained, but instead of efficiency, will be kept as the more standard marginal productivity.

[29] The associated cost function was not provided in the Lotfi paper.

Summary

Omit: g, L_t, $g(L_t)$, v_t, w_n and K

Retain: x_t

3.1.7 Taxes and Discount Factor

Lotfi accounted for the tax rate and for investment tax credits in his model. He defined the adjusted tax factor as $?_t$=(1-TR$_t$)r_t, where TR$_t$ is the tax rate and r_t is the discount factor. Lotfi also incorporated an investment credit factor, $IC_{n,t}$ in the capital cost calculations.

Simplification:

Tax implications are very complicated, especially with regard to capital investments and salvage. In fact, although Lotfi included an investment credit factor, he omitted many other tax features, including depreciation and interest payments deductions. Rather than including only select tax features, I will instead ignore tax implications by omitting the tax-related variables, TR$_t$ and $IC_{n,t}$, effectively rendering a before-tax model where $?_t$=r_t.

Summary

Omit: TR$_t$, $IC_{n,t}$

Retain: $?_t$ as r_t

3.1.8 Other Parameters

Lotfi also defined several other parameters. These parameters included the demand for part type *i* in period *t* (D_{it}), price (P_{it}), marginal cost (MC_{it}), and unit inventory carrying cost of part type *i* during time period *t* (h_{it}). t_m represented the final time by which machine type *m* needs to be phased out.

Simplification:

Since I am assuming only one part type, the index *i* will be omitted from the model. Since there is only one old machine type, *m* will also be omitted and t_m (now t) is simply the time by which the replacement is scheduled to be completed, now identical to the time horizon, *T*. All other parameters will be kept.

Summary

Retain:
- D_{it} as D_t for the part's demand
- P_{it} as P_t for the price
- MC_{it} as MC_t for the marginal cost
- h_{it} as h_t for the unit inventory carrying cost
- t_m as T for the time horizon

3.2 Next Step: Lotfi's Objective Functions

Lotfi's model also contained five objectives that he wished to accomplish. In this section, I will introduce the objectives and then will simplify them by omitting the variables discussed above.

The five objectives that Lotfi wished to accomplish were:

1. Maximize the net present value (NPV) of after-tax cash flows (ATCF).

2. Smooth the budgetary requirements over the planning horizon by minimizing the maximum budget requirement for any period.
3. Maximize routing flexibility by minimizing the number of machine groups selected from the candidate set of machines.
4. Minimize the affect of disruptions caused by removing old machines by minimizing the maximum number of current machines earmarked for salvage each period.
5. Minimize the affect of disruptions caused by installing new machines by minimizing the maximum number of new machines to be installed in any period.

3.2.1 Lotfi's First Objective Function: Maximize the NPV of ATCF

With the first objective, Lotfi attempted to maximize the total net present value of after tax cash flows rising from the sum of seven different sources. The sources were: (*a*) sales revenue less material cost, (*b*) capital cost of new modules, (*c*) salvage value of current machines, (*d*) variable costs, (*e*) fixed costs, (*f*) setup costs, and finally (*g*) inventory and process flexibility costs.

(a) Sales Revenue less Material Cost

$$+\sum_{t}\sum_{i}\left(P_{it}-MC_{it}\right)D_{it}h_{t} \tag{1.}$$

In the above equation, P_{it}, MC_{it}, and D_{it} are the parameters for price, marginal cost and demand, respectively, for part type i for period t. The adjusted tax factor is

$h_t = (1 - \text{TR}_t)r_t$, where TR_t is the tax rate and r_t is the discount factor.

Based on the above discussion of variables, *i* is omitted, D_{it} becomes D_t, P_{it} becomes P_t, h_t becomes r_t, and MC_{it} becomes MC_t, leaving:

$$+\sum_t (P_t - MC_t) D_t r_t \tag{2.}$$

(b) Capital Cost of New Modules

$$-\sum_t \sum_n (N_{n,t} - N_{n,t-1}) C''_{n,t} (1 - IC_{n,t}) r_t \tag{3.}$$

The term ($N_{n,t}$-$N_{n,t-1}$) is the number of new modules of type *n* installed in time *t*, and $C''_{n,t}$ is the capital cost applicable for that period. $IC_{n,t}$ is the investment credit factor, and r_t is the discount factor.

Simplifying the above, based on the preceding discussion of variables, *n* is omitted because only one module is assumed, and $IC_{n,t}$ is omitted since calculations are assumed to be before-tax, resulting in:

$$-\sum_t (N_t - N_{t-1}) C''_t r_t \tag{4.}$$

(c) Salvage Value of Current Machines

$$+\sum_t \sum_m (M_{m,t-1} - M_{m,t}) S_{m,t} r_t \tag{5.}$$

The term ($M_{n,t-1}$ -$M_{n,t}$) is the number of old machines of type *m* disposed of in time *t*, $S_{m,t}$ is the salvage value after adjusting for tax on capital gain/loss, and r_t is the discount factor.

Based on the preceding discussion of variables, *m* is eliminated because the only one old machine is assumed and the salvage value S_t is no longer adjusted for capital gains or losses, resulting in:

$$+\sum_t (M_{t-1} - M_t) S_t r_t \tag{6.}$$

(d) Variable Costs

$$-\left[\sum_t \sum_i \sum_j \sum_n d''_{ijn} y_{ijn,t} h_t + \sum_t \sum_i \sum_j \sum_m d'_{ijm} X_{it} h_t\right] \tag{7.}$$

Where d'_{ijm} and d''_{ijn} are unit variable costs for operation (*i*,*j*) on machine types *m* and *n*. X_{it} is the production quantity for part type *i* on current machine types during time period t, $y_{ijn,t}$ is the production quantity for operation (*i*,*j*) on new machine type *n* in period *t*, and $h_t = (1 - \mathrm{TR}_t) r_t$ where TR_t is the tax rate and r_t is the discount factor.

Based on the above discussion of variables, *i*, *j*, *m*, and *n* are omitted, h_t is replaced by r_t, and $y_{ijn,t}$ is replaced by Y_t. Therefore, the above equation is greatly simplified, and becomes:

$$-\left[\sum_t d'' Y_t r_t + \sum_t d' X_t r_t\right] \tag{8.}$$

(e) Fixed Costs,

$$-\left[\sum_t \sum_i \sum_j \sum_n e''_{ijn} N_{n,t} h_t + \sum_t \sum_i \sum_j \sum_m e'_{ijm} M_{m,t} h_t\right] \tag{9.}$$

Where e'_{ijm} and e''_{ijn} are unit fixed costs for operation (*i*,*j*) on machine types *m* and *n*. $M_{m,t}$ in the number of current machines of type *m* in period *t*, $N_{m,t}$ is the number of current machines of type *n* in period *t*, and $h_t = (1 - \mathrm{TR}_t) r_t$ where TR_t is the tax rate and r_t

is the discount factor.

Based on the above discussion, *i*, *j*, *m*, and *n* are omitted and e'_{ijm} and e''_{ijn} are 0. Therefore, the above equation can be omitted as it becomes:

$$-\left[\sum_t 0 \cdot N_t h_t + \sum_t 0 \cdot M_t h_t\right] \tag{10.}$$

(f) Setup Costs

$$-\sum_t \left[L_t \sum_n SC''_n N_{nt} + \sum_i \sum_j \sum_m SC'_{ijm} s_{it}\right] h_t \tag{11.}$$

The first part of the above expression reflects setup costs for new modules. Setup costs depend on L_t, the number of system setups in a period, and the costs of setting up machines of various types in every system setup $\sum_n SC''_n N_{nt}$. The second part of the expression presents the setup costs for the current machine types based on traditional setups, $\sum_i \sum_j \sum_m SC'_{ijm} s_{it}$ and $h_t = (1 - \mathrm{TR}_t) r_t$ where TR_t is the tax rate and r_t is the discount factor.

Based on the above discussion of variables, *i*, *j*, *m*, SC'_{ijm}, and L_t, should be omitted and h_t should be replaced by r_t, resulting in the following equation:

$$-\sum_t (0 \cdot N_t) r_t \tag{12.}$$

The whole equation shall be omitted, since there are no longer setup costs for switching lines between products.

(g) Inventory and Process Flexibility Costs

$$-\sum_t \left[\sum_i \left(\frac{\left(D_{it} \middle/ L_t \right)}{2} \right) h_{it} + g(L_t) \right] h_t \tag{13.}$$

D_{it} is the demand for a given part, *i*, during a given time, *t*, and L_t is the number of operating cycles in a period. (Lotfi states that L_t values are typically either 52 and 12, corresponding to weekly or monthly setups.) D_{it}/L_t provides the average number of products produced during a cycle, which Lotfi assumes to be the average lot size. The function $g(L_t)$ is the cost of providing process flexibility, h_{it} is the unit inventory carrying cost of part type *i* during time period *t*, and $?_t$=(1-TR_t)r_t where TR_t is the tax rate and r_t is the discount factor.

Based on the above discussion of variables, $g(L_t)$, *i*, and L_t are omitted, and $?_t$ becomes r_t, resulting in[30]:

$$-\sum_t \frac{D_t}{2} h_t r_t \tag{14.}$$

3.2.2 Lotfi's Simplified First Objective

Lotfi presented the first objective:

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[30] Lotfi does not explain the term $\frac{D_t}{2}$. Presumably, either the average unit inventory in a given time period is half the demand for that period, or the inventories are drawn down to zero at a constant rate over period *t*.

$$\max\ f_1 = -\sum_t \sum_n \left(N_{n,t} - N_{n,t-1}\right) C''_{n,t} \left(1 - IC_{n,t}\right) r_t + \sum_t \sum_m \left(M_{m,t} - M_{m,t-1}\right) S_{m,t} r_t$$
$$- \left[\sum_t \sum_i \sum_j \sum_n d''_{ijn} y_{ijn,t} h_t + \sum_t \sum_i \sum_j \sum_m d'_{ijm} X_{it} h_t \right]$$
$$- \left[\sum_t \sum_i \sum_j \sum_n e''_{ijn} N_{n,t} h_t + \sum_t \sum_i \sum_j \sum_m e'_{ijm} M_{m,t} h_t \right] \quad (15.)$$
$$- \sum_t \left[L_t \sum_n SC''_n N_{nt} + \sum_i \sum_j \sum_m SC'_{ijm} s_{it} \right] h_t - \sum_t \left[\sum_i \left(\frac{\left(D_{it} / L_t \right)}{2} \right) h_{it} + g\left(L_t\right) \right] h_t$$

Substituting for (*a*), (*b*),…(*g*) with the new, simplified variables produces:

$$\max\ f_1 = \sum_t \left(P_t - MC_t\right) D_t r_t - \sum_t \left(N_t - N_{t-1}\right) C''_t r_t + \sum_t \left(M_t - M_{t-1}\right) S_t r_t$$
$$- \left[\sum_t d'' Y_t r_t + \sum_t d' X_t r_t \right] - \sum_t \frac{D_t}{2} h_t r_t \quad (16.)$$

3.2.3 Lotfi's Second Objective Function: Budgetary Considerations

Lotfi states that his second objective function attempts to smooth the budgetary requirements over the planning horizon by minimizing the maximum (minimax) budget requirement for any period, and informs us that this is a major preference in practice. Lotfi explains that the equation is set up as a maximization of a negative maximization, which is mathematically equivalent to a minimax. Unfortunately, Lotfi does a poor job explaining these equations. See section 4.2 for a critique of these equations:

$$\text{Max}\ \ f_2 = -\max \left[\sum_n \left(N_{nt} - N_{n,t-1}\right) C''_{nt} - \sum_m \left(M_{m,t-1} - M_{mt}\right) S_{mt} \right] \quad (17.)$$

subject to

$$\sum_n \left(N_{nt} - N_{n,t-1}\right)C''_{nt} - \sum_m \left(M_{m,t-1} - M_{mt}\right)S_{mt} \leq$$
$$\max\left[\sum_n \left(N_{nt} - N_{n,t-1}\right)C''_{nt} - \sum_m \left(M_{m,t-1} - M_{mt}\right)S_{mt}\right] \quad \forall t \tag{18.}$$

Based on the above discussion of variables, n and m should omitted, resulting in:

$$\text{Max} \quad f_2 = -\max\left[\left(N_t - N_{t-1}\right)C''_t - \left(M_{t-1} - M_t\right)S_t\right] \tag{19.}$$

subject to

$$\left(N_t - N_{t-1}\right)C''_t - \left(M_{t-1} - M_t\right)S_t \leq \max\left[\left(N_t - N_{t-1}\right)C''_t - \left(M_{t-1} - M_t\right)S_t\right] \quad \forall t \tag{20.}$$

3.2.4 Lotfi's Third Objective Function: Expanding Routing Flexibility

Lotfi introduces his third objective as an attempt to expand the routing flexibility.

$$f_3 = -\sum_n w_n \tag{21.}$$

subject to:

$$\sum_t N_{nt} \leq V w_n \quad \forall t \tag{22.}$$

Based on the above discussion of variables, w_n should be omitted. Since this problem is only dealing with one part, routing flexibility is no longer an issue, and therefore this objective shall be omitted altogether.

3.2.5 Lotfi's Fourth Objective Function: Firm Disruption due to Equipment Removal

Lotfi stated that objective four facilitated controlling disruptions caused by removal and salvaging of current equipment. This objective attempted to minimize the maximum number of current machines earmarked for salvage in any period.

$$\text{Max} \quad f_4 = -f'_4 \tag{23.}$$

subject to:

$$\sum_{m}\left(M_{m,t-1}-M_{m,t}\right)\leq f_4' \quad \forall t \tag{24.}$$

Based on the above variable discussion, m should be omitted, resulting in:

$$\text{Max} \quad f_4 = -f_4' \tag{25.}$$

subject to:

$$\left(M_{t-1}-M_{t}\right)\leq f_4' \quad \forall t \tag{26.}$$

3.2.6 Lotfi's Fifth Objective Function: Firm Disruption due to Equipment Installation

Lotfi stated that objective five facilitated controlling disruptions caused by installation of new modules. This objective minimized the maximum number of new modules to be installed in any period.

$$\text{Max} \quad f_5 = -f_5' \tag{27.}$$

subject to:

$$\sum_{n}\left(N_{n,t}-N_{n,t-1}\right)\leq f_5' \quad \forall t \tag{28.}$$

Based on the above discussion of variables, n should be omitted, since there is only a single module type, resulting in:

$$\text{Max} \quad f_5 = -f_5' \tag{29.}$$

subject to:

$$\left(N_{t}-N_{t-1}\right)\leq f_5' \quad \forall t \tag{30.}$$

3.3 Final Part of Lotfi's Model: Constraints

Lotfi's model also introduced three constraint sets. The constraints will be presented and then will be simplified based on the omission of variables discussed above. The first constraint set ensured that current machines would be phased out as new modules were installed. The second constraint set provided equations concerned with output and capacity of new modules. The third set of constraints placed restrictions on the output and production on old machines.

3.3.1 Lotfi's First Constraints: Implementation Constraints

Lotfi stated that the following constraints ensured current machines would be phased out as new modules were installed. Lotfi also defined the last period by which an old machine had to be phased out (useful life of equipment). It was assumed that new modules were not replaced during the planning horizon.

$$M_{m,t} \leq M_{m,t-1} \quad \forall m,t \tag{31.}$$

$$N_{n,t} \geq N_{n,t-1} \quad \forall n,t \tag{32.}$$

$$M_{m,t} = 0 \quad \forall m, t = t_m \tag{33.}$$

Where *M* and *N* denote old and new equipment, and t_m is the last period by which a current machine of type *m* has to be phased out.

Based on the above discussion of variables, both *n* and *m* should be omitted, since there is only a single module type, and t_m will be replaced by *T*, resulting in:

$$M_t \leq M_{t-1} \quad \forall t \tag{34.}$$

$$N_t \geq N_{t-1} \quad \forall t \tag{35.}$$

$$M_T = 0 \qquad (36.)$$

3.3.2 Lotfi's Second Constraints: Demand Requirements and Assignment of New Modules

With the second set of constraints, Lotfi stated that the annual demand for a part type i (D_{it}) should be split into the quantity produced completely on old machine types (X_{it}) and the quantity produced on new machine types (Y_{it}). This division stemmed mainly from processing requirement differences and assumed low interchangeability of work between traditional (old) and computer numerical control equipment (new modules). Thus, the following constraint was introduced:

$$Y_{it} + X_{it} = D_{it} \quad \forall i,t \qquad (37.)$$

Lotfi next placed a constraint on production with new modules ($y_{ijn,t}$). Lotfi stated that this constraint restricts an operation (i,j) in period t to be assigned to a machine group only if that group was capable of performing the operation.[31] Further, a given operation (i,j) could be assigned to different new machine types, but the total quantity produced, $\sum_n y_{ijn,t}$, should equal Y_{it}:

$$y_{ijn,t} \le z''_{ijn} D_{it} \quad \forall i, j, n, t \qquad (38.)$$

$$\sum_n y_{ijn,t} = Y_{it} \quad \forall i, j, t \qquad (39.)$$

Finally, Lotfi provided a capacity constraint for a new module of type n:

$$\sum_i \sum_j p''_{ijn} y_{ijn,t} + L_t ST'' N_{n,t} \le KgN_{n,t} \quad \forall n, t \qquad (40.)$$

[31] Since z''_{ijn} is binary, z''_{ijn} cannot equal 0 (and hence $z''_{ijn} D_{it}$ cannot equal 0) to make the assignment.

Where p''_{ijn} is the processing time for operation (*i*,*j*) on machine type *n*, $y_{ijn,t}$ is the production quantity for operation (*i*,*j*) on new machine type *n* in period *t*, and L_t is the number of operating cycles in a period. ST'' is the setup time, *K* is the number of available hours of capacity per period, and g is slack for routing flexibility. The first term $\left(\sum_i \sum_j p''_{ijn} y_{ijn,t}\right)$ represents the total processing time, and the second $\left(L_t ST'' N_{n,t}\right)$, the total time lost due to system setups on a given machine type. $KgN_{n,t}$ is the total number of hours available.

Based on the above discussion of variables, *m*, *n*, *i*, *j*, *g*, ST'', L_t, p''_{ijn}, and *K* should be omitted, and Y_t should be substituted for $\sum_n y_{ijn,t}$, resulting in:

$$Y_{it} + X_{it} = D_{it} \quad \forall i,t$$

$$y_{ijn,t} \leq z''_{ijn} D_{it} \quad \forall i, j, n, t$$

$$\sum_n y_{ijn,t} = Y_{it} \quad \forall i, j, t$$

$$\sum_i \sum_j p''_{ijn} y_{ijn,t} + L_t ST'' N_{n,t} \leq KgN_{n,t} \quad \forall n,t$$

becoming:

$$Y_t + X_t = D_t \quad \forall t \tag{41.}$$

$$Y_t \leq D_t \quad \forall t \text{, which is trivial, and can be omitted based on eq. 41} \tag{42.}$$

$$Y_t = Y_t \text{, which is trivial and can be omitted} \tag{43.}$$

$$N_t \leq N_t \quad \forall t \text{ , which is trivial and can be omitted} \tag{44.}$$

3.3.3 Lotfi's Third Constraints: Operation Assignment for Existing Machines

For the third constraint set, Lotfi defined the following equations for production assignments on current machine types:

$$\sum_i \sum_j p'_{ijm} X_{it} + \sum_i \sum_j ST'_{ijm} s_{it} \leq K x_t M_{m,t} \qquad \forall m, t \tag{45.}$$

$$X_{it} \leq V v_t \qquad \forall i, t \tag{46.}$$

$$s_{it} = L_t v_t \qquad \forall i, t \tag{47.}$$

Where p'_{ijm} is the processing time for operation (i,j) on machine type m, and X_{it} is the production quantity of part i in time period t from old machines. ST'_{ijm} is the setup time for operation (i,j) on machine type m, and s_{it} is the number of setups for part type i on old machines in period t. K is the number of available hours of capacity per period, and x_t is efficiency of the old machines. The first term $\left(\sum_i \sum_j p'_{ijm} X_{it}\right)$ represents the total processing time, and the second term is the total setup time, $\left(\sum_i \sum_j ST'_{ijm} s_{it}\right)$. $K x_t M_{m,t}$ is the total number of hours available on old machines.

In the second equation, since V is a large positive number, this constraint looks to simply ensure that if no work is done on an old machine ($v_t = 0$) then there can be no old-machine production ($X_{it} = 0$). Lotfi does explain the third constraint: After the transition period (when $M_{m,t} = 0$), both the production quantities and setups are forced to zero by the first equation, while during the transition, the number of setups (s_{it}) for an operation (i,j) is assumed to be equal to the number of operating cycles in a period (L_t). Thus, setups are either 0 or L_{it}.

Based on the above discussion of variables, m, i, j, g, p'_{ijm}, K, ST'_{ijm}, s_{it}, L_t, and v_t should be omitted, resulting in:

$$\sum_i \sum_j p'_{ijm} X_{it} + \sum_i \sum_j ST'_{ijm} s_{it} \le K x_t M_{m,t} \qquad \forall m, t$$

$$X_{it} \le V v_t \qquad \forall i, t$$

$$s_{it} = L_t v_t \qquad \forall i, t$$

becoming

$$X_t \le x_t M_t \qquad \forall t \tag{48.}$$

$X_t \le V \qquad \forall t$ which can be omitted since it is always true (*V* defined as large) (49.)

$$0 = 0 \tag{50.}$$

3.4 Overall Summary: One Part, One Module, One Machine, Before Taxes

For an overall summary of the revised Lotfi model and its simplification based on one machine type, one module type, one part type, and a before-tax model, please see Appendix A.

4 The Lotfi Model: Critique and Modifications

The previous section simplified the Lotfi model, this section will critique and modify each of Lotfi's objective functions and constraints.

4.1 Lotfi's First Objective Function: Variable Costs and Time Independence of Material Cost, Capital Cost, and Salvage Value

Lotfi's first objective function from section 3.2.2 is:

$$\max\ f_1 = \sum_t (P_t - MC_t)D_t r_t - \sum_t (N_t - N_{t-1})C_t'' r_t + \sum_t (M_{t-1} - M_t)S_t r_t$$
$$-\left[\sum_t d'' Y_t r_t + \sum_t d' X_t r_t\right] - \sum_t \frac{D_t}{2} h_t r_t$$

This objective reflects the fact that Lotfi is trying to maximize profits by choosing the number of new machines and old machines to employ at each time step, and the equation simply calculates revenue less expense. There are two revenue terms, the third term in the above equation, $(M_{t-1} - M_t)S_t$, is the revenue derived from selling old machines and is calculated by multiplying the number of machines sold in any given time period $(M_{t-1} - M_t)$ by the salvage price, S_t. The first term contains the other source of revenue, sales revenue ($P_t D_t$), which Lotfi has awkwardly combined with material cost ($MC_t D_t$) in $(P_t - MC_t)D_t$.

Lotfi has included four cost terms in addition to material cost. The second term $(N_t - N_{t-1})C_t''$ describes the cash outlay for purchasing new modules and is calculated by multiplying the number of modules purchased in any given time period $(N_t - N_{t-1})$ by the module cost, C_t. The third term, $d''Y_t$ describes the variable costs of production on a new module, as d'' is the variable cost for production on new modules and Y_t is the quantity of parts produced on new modules. Likewise, $d'X_t$ describes the variable costs of production on an old machine. Finally, $\frac{D_t}{2}h_t$ describes the inventory carrying cost, as h_t is the unit inventory carrying cost and $\frac{D_t}{2}$ is the inventory quantity. It should be noted that all of the terms are summed and are discounted by r_t.

In general, I agree with Lotfi's equations, however, I disagree with several of the details. First of all, Lotfi's equations used nominal costs, nominal prices, and nominal interest rates, and assumed that the marginal cost, salvage value, and capital cost were all time dependant[32]. Instead, I will use real prices and costs (indexed back to the first year of the study), and the real interest rate.[33] This simplifying change requires no modifications to any of the equations, just a redefinition of the existing variables. It will also allow marginal cost, capital cost, and salvage value to be constant. A constant marginal cost and capital cost is a conservative assumption, and merely assumes that the there are no supply or demand shocks to the either of these markets.

My assumption of a constant salvage value deserves a little more discussion, since machinery typically loses real value over time. It is reasonable, however, to assume that the relative value of machinery declines at a smaller rate as machinery gets older. We can assume that for very old machinery, the relative drop from one year to the next is small enough to be negligible, and can be omitted from the model without impact.

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[32] Interestingly, Lotfi did not assume that the variable costs (d' or d'') were time dependant.

[33] The real interest rate is equal to $\frac{R-i}{1+i}$ where R is the nominal interest rate, and i is the inflation rate.

I also disagree with how Lotfi handles variable costs. A variable cost, by definition, changes as the quantity produced changes.[34] The material cost ($MC_t D_t$) and inventory carrying cost ($\frac{D_t}{2} h_t$) are both variable costs, since they change with quantity (D_t). There is no reason that variable costs should be separated into different components, and therefore the material and inventory-carrying costs will be combined along with the variable cost (d_t' or d_t'') into two new marginal cost terms, MC' for new modules and MC'' for old machines.

To help clean up the equations, I will also define two new variables to describe the change in modules and machines from one time period to the next: $\Delta N_{t-1} = N_t - N_{t-1}$ and $\Delta M_{t-1} = M_t - M_{t-1}$. Now, removing time dependencies for marginal cost, salvage value, capital cost, and using Lotfi's second constraint, $Y_t + X_t = D_t$, and the ΔN and ΔM definitions, I will substitute and rearrange the first objective function to yield:

$$\max\ f_1 = \sum_t (P_t - MC_t)(Y_t + X_t) r_t - \sum_t (N_t - N_{t-1}) C_t'' r_t + \sum_t (M_{t-1} - M_t) S_t r_t$$
$$- \left[\sum_t d'' Y_t r_t + \sum_t d' X_t r_t \right] - \sum_t (Y_t + X_t) \frac{h_t}{2} r_t$$

$$\vdots$$

$$\max\ f_1 = \sum_t \left[P_t Y_t r_t - \left(MC + d'' + \frac{h}{2} \right) Y_t r_t \right] + \sum_t \left[P_t X_t r_t - \left(MC + d' + \frac{h}{2} \right) X_t r_t \right]$$
$$- \sum_t \Delta N_{t-1} C'' r_t - \sum_t \Delta M_{t-1} S r_t$$

$$\vdots$$

[34] Mansfield, E., *Managerial Economics*, 3rd Ed., W.W. Norton & Co., pp300-302, (1990).

$$\max\ f_1 = \sum_t \left(P_t Y_t r_t - MC'' Y_t r_t\right) + \sum_t \left(P_t X_t r_t - MC' X_t r_t\right)$$
$$- \sum_t \Delta N_{t-1} C'' r_t - \sum_t \Delta M_{t-1} S r_t$$

Summary: Lotfi's First Objective Function

$$\max\ f_1 = \sum_t \left(P_t Y_t r_t - MC'' Y_t r_t\right) + \sum_t \left(P_t X_t r_t - MC' X_t r_t\right) - \sum_t \Delta N_{t-1} C'' r_t - \sum_t \Delta M_{t-1} S r_t \tag{51.}$$

And:

$$\Delta N_{t-1} = N_t - N_{t-1} \tag{52.}$$

$$\Delta M_{t-1} = M_t - M_{t-1} \tag{53.}$$

4.2 Lotfi's Second Objective Function: A Poor Constraint and Time Independence of Capital Cost and Salvage Value

Lotfi's second objective function in section 3.2.3 is:

$$\text{Max}\ \ f_2 = -\max\left[(N_t - N_{t-1})C_t'' - (M_{t-1} - M_t)S_t\right]$$

subject to

$$(N_t - N_{t-1})C_t'' - (M_{t-1} - M_t)S_t \le \max\left[(N_t - N_{t-1})C_t'' - (M_{t-1} - M_t)S_t\right] \quad \forall t$$

With this objective, Lotfi is attempting to minimize the maximum budget requirement for any given time period. Lotfi states that the equation is written in maximization form to be compatible with the proposed solution procedure, but does not explain the equation. An explanation does not appear to be necessary, however, as the formula simply reflects the budgetary outlay: $(N_t - N_{t-1})C_t''$ is cash outlay for purchasing new modules, less the value cash gained by salvaging old equipment, $(M_{t-1} - M_t)S_t$.

To quickly summarize, I disagree with this objective altogether. This objective makes no economic sense and may only serve to double count this amount. Note these very terms appear in the first objective function. Further, the discounting term, r_t, takes into account financial cost of capital, which is the only rational basis for making a decision on whether to replace equipment or not. Therefore, it is my contention that Lotfi's desire to minimize the outlay of cash in any time period is properly accounted for in the first objective function and therefore this objective will be completely omitted.

4.3 Lotfi's Third Objective Function

Lotfi's third objective function was completely omitted in section 3.2.4.

4.4 Lotfi's Fourth and Fifth Objective Functions: A Real Minimax

Lotfi's fourth and fifth objective functions in sections 3.2.5 and 3.2.6 were:

$$\text{Max} \quad f_4 = -f_4'$$

subject to:

$$\left(M_{t-1} - M_t\right) \leq f_4' \quad \forall t$$

and

$$\text{Max} \quad f_5 = -f_5'$$

subject to:

$$\left(N_t - N_{t-1}\right) \leq f_5' \quad \forall t$$

Lotfi stated that the above objectives were an attempt to control disruptions caused by removal or salvaging of current equipment. Both of these equations are minimax formulations. The fourth objective is minimizing the maximum number of old machines earmarked for salvage in any period, and the fifth objective is minimizing the maximum number new modules purchased in any period.

Again, I agree with Lotfi's intent – since the stated goal is to minimize the disruption caused by removal or salvaging of equipment, as Lotfi states, then a minimax construction is a fine way to go about it. However, I disagree with Lotfi's construction of these objectives—nether are a minimax construction. In fact, both of these equations are simply minimizations. Therefore, these equations will be re-written in the correct minimax formulation, and will be incorporated into the constraints introduced in objective one:

Summary: Lotfi's Fourth and Fifth Objective Functions

$$\text{Min} \quad f_4 = \max\left(-\Delta M_{t-1}\right) \quad \forall t \tag{54.}$$

and

$$\text{Min} \quad f_5 = \max\left(\Delta N_{t-1}\right) \quad \forall t \tag{55.}$$

4.5 Lotfi's Constraints: Elimination, Addition and Time Independence of Efficiency

In sections 3.3.1, 3.3.2, and 3.3.3, the Constraints were reduced to the following:

Constraint Set 1:

$$M_t \le M_{t-1} \quad \forall t$$
$$N_t \ge N_{t-1} \quad \forall t$$
$$M_T = 0$$

Constraint Set 2:

$$Y_t + X_t = D_t \quad \forall t$$

Constraint Set 3:

$$X_t \leq x_t M_t \quad \forall t$$

Lotfi states that the first set of constraints are needed to implement the solution; namely to ensure that old machines are sold and new machines are purchased. The first equation ($M_t \leq M_{t-1}$) states that the number of old machines owned in any year must be less than or equal to the number of machines owned the previous year—guaranteeing that old machines will be sold, not purchased. The second equation ($N_t \geq N_{t-1}$) likewise ensures that new modules will be purchased, not sold. The third equation ($M_T = 0$) states that in the final period (*T*) all old machines will be sold.

I strongly disagree with Lotfi's first constraints. Lotfi is gaming the system to force replacement of the old machines with new modules. If the model says I should consider not replacing old machines, I would like to know that. More interestingly, the model may tell me that instead of purchasing brand new modules, it may be more economical to purchase older machines that other companies are scrapping, rather than purchase brand new modules. Since I do not agree with any of these constraints all of them will be removed from my model.

The second constraint ($Y_t + X_t = D_t$) is an assumption that the quantity of parts produced on old machines (X_t) and new modules (Y_t) will be equal to the annual demand (D_t). And, finally, the third constraint ($X_t \leq x_t M_t$) states that the quantity of parts produced on old machines (X_t) must be less than or equal to the number of old machines (M_t) times the efficiency of those machines (x_t).

In Lotfi's second constraint set, I am uncomfortable with his treatment of demand[35] and do not agree with this constraint, either. Lotfi simply introduces demand (D_t) without explanation and forces the plant to produce that amount. With this constraint, Lotfi overlooks the fact that it may be impossible or uneconomical for the factory to meet the demand number during the changeover period. I agree that an objective should be to meet the quantity demanded during this time period, but will not treat this as a firm constraint. Instead, I will separate the demand (D_t) from the quantity produced (Q_t), and will introduce a sixth objective to minimize the difference between the quantity produced and the quantity demanded over time $\sum_t (Q_t - D_t)$, and will reformulate the constraint to $Y_t + X_t = Q_t$.

The third constraint, $X_t \le x_t M_t$, where $x_t M_t$ is the production function, is a little strange. The original intent of this constraint was to take into account the productivity lost through setup time for different parts. Since no setup times are now required, this equation no longer makes sense as an inequality. Instead, I will modify this equation to an equality statement: that the number of parts produced on an old machines is equal to the number of old machines times the efficiency of those machines ($X_t = x_t M_t$). I disagree with Lotfi's assumption that efficiency is time dependant (x_t). If kept in good condition, a machine should be able to produce the same quantity of parts over a given time period regardless of age. Therefore, efficiency will be treated as a time-independent quantity (x). The third constraint brings up another interesting quirk of the Lotfi equations—unlike tying production (X_t) to machine quantity (M_t) as he did with old

[35] Please see section 5.3.1 for further discussion of Lotfi's demand.

machines, Lotfi instead tied the quantity of parts produced on new modules (Y_t) to the time spent processing those modules and his concept of slack time. When I eliminated slack time, I also eliminated the tie between modules (N_t) and quantity produced on those modules (Y_t). To recover this relationship, I propose a new constraint on new modules similar to that for old machines: $Y_t = x''N_t$, where x'' is efficiency of new modules. Keeping with Lotfi's convention for apostrophes, I will also modify the original constraint to become: $X_t = x'M_t$.

Summary: Lotfi's Constraints and New Objective

$$M_t \le M_{t-1} \quad \forall t$$
$$N_t \ge N_{t-1} \quad \forall t$$
$$M_t = 0 \quad \forall t = t$$
$$Y_t + X_t = D_t \quad \forall t$$
$$X_t \le x_t M_t \quad \forall t$$

becomes:

$$Y_t + X_t = Q_t \quad \forall t \tag{56.}$$
$$X_t = x'M_t \quad \forall t \tag{57.}$$
$$Y_t = x''N_t \quad \forall t \tag{58.}$$

and objective six:

$$\text{Min } f_6 = \sum_t (Q_t - D_t) \tag{59.}$$

4.6 A Simple Substitution

It should be noted that the constraints $X_t = x'M_t$ and $Y_t = x''N_t$ can be and will be substituted into the first constraint and the first objective function to yield:

$$\max\ f_1 = \sum_t \left(P_t x'' N_t r_t - MC'' x'' N_t r_t\right) + \sum_t \left(P_t x' M_t r_t - MC' x' M_t r_t\right) - \sum_t \Delta N_{t-1} C'' r_t - \sum_t \Delta M_{t-1} S r_t \tag{60.}$$

and

$$x'' N_t + x' M_t = Q_t \quad \forall t \tag{61.}$$

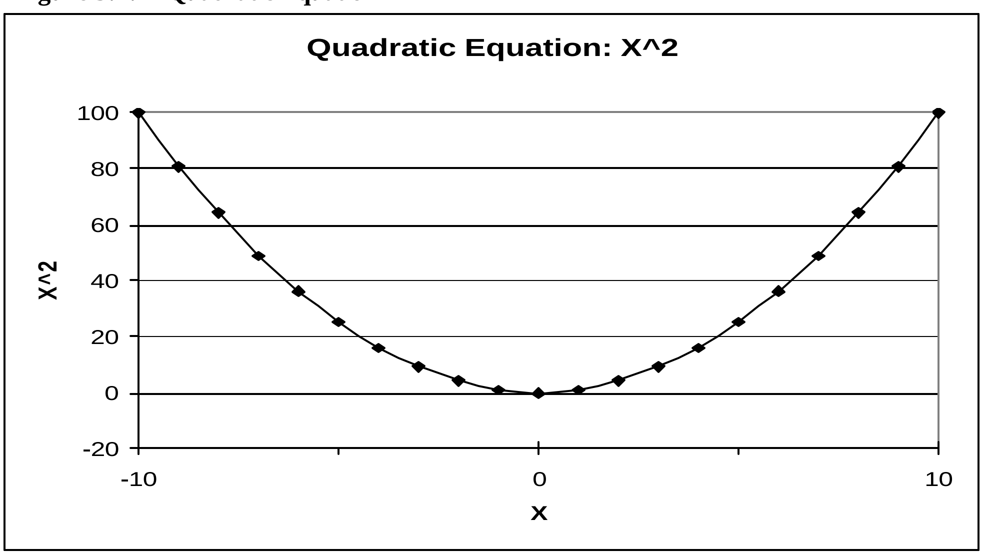
4.7 Overall Summary: Variable Costs, Time Independence, Real Minimax, and Constraints

For an overall summary of the revised Lotfi model and its simplification based on combining variable costs, making several variables time-independent, introducing true minimax objectives, and an examination of constraints, please see Appendix B.

5 The Riccati Iteration: Solution Technique for the Lotfi Model

In many ways this is the most important and most difficult section of the thesis. This section ties the Lotfi model to the Riccati iteration. Let me start by saying what the Riccati iteration does: Very simply, the Riccati iteration attempts to minimize a matrix Linear Quadratic equation to zero. (It is useful to keep the picture of a quadratic – or squared – equation in mind, see figure 5.1 below). The following section will present the transformation of the Lotfi problem to a problem amenable to the Riccati iteration by breaking the problem into the above requirements: Linear Quadratic problem (sections 5.1 and 5.2), a minimization (sections 5.2 and 5.3), and variables that go to zero (section 5.4).

Figure 5.1: A Quadratic Equation



5.1 The Riccati Problem: A Linear Quadratic

As mentioned above, the Riccati iteration minimizes Linear Quadratic problems.

The matrix form of the Linear Quadratic problem is:

$$\min_{\{\mathbf{u}_t\}} f = \sum_t \left(\mathbf{y}_t^{\mathbf{T}} \mathbf{Q} \mathbf{y}_t + \mathbf{u}_t^{\mathbf{T}} \mathbf{R} \mathbf{u}_t \right) \tag{62.}$$

subject to the constraints:

$$\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1} + \mathbf{B}\mathbf{u}_{t-1} \tag{63.}$$

$$\mathbf{y}_t = \mathbf{C}\mathbf{x}_t + \mathbf{D}\mathbf{u}_t \tag{64.}$$

Where **y** is a vector of outputs, **u** is a vector of inputs, and **Q**, **R**, **A**, **B, C** and **D** are matrices. Admittedly, this notation is arcane. To simplify, for a two-by-two matrix, the matrix linear quadratic problem can be written in non-linear form as:

$$\min_{\{u_{1,t}, u_{2,t}\}} f = \sum_t q_1 y_{1,t}^2 + \sum_t q_2 y_{2,t}^2 + \sum_t r_1 u_{1,t}^2 + \sum_t r_1 u_{2,t}^2 \tag{65.}$$

Written in this form, it is clear why this is called a linear-quadratic equation. The equation is a linear addition of quadratic terms. The "$\min_{\{u_{1,t}, u_{2,t}\}} f$" simply means pick $u_{1,t}$ and $u_{2,t}$ that will minimize f.[36] Note from the constraints that since $\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1} + \mathbf{B}\mathbf{u}_{t-1}$ and $\mathbf{y}_t = \mathbf{C}\mathbf{x}_t + \mathbf{D}\mathbf{u}_t$ that $u_{1,t}$, $u_{2,t}$, and $\mathbf{x}_0$ completely determine $y_{1,t}$ and $y_{2,t}$.

The Lotfi optimizations presented so far are not linear-quadratic; in fact, the optimizations are a linear maximization, two minimax objectives, and a linear minimization. Putting these different objectives into Linear Quadratic form will be presented in the next sections.

5.2 Linear Quadratic from Minimax

Objectives four and five are constructions to minimize a maximum (minimax).

[36] In our case, $u_{1,t}$ is the number of new modules bought and $u_{2,t}$ is the number of old machines scrapped.

$$\text{Min } f_4 = \max \sum_t \left(-\Delta M_{t-1}\right)$$

$$\text{Min } f_5 = \max \sum_t \Delta N_{t-1}$$

The Linear Quadratic problem shares many minimax qualities. In the Linear Quadratic formulation, the largest numbers are weighted exponentially (squared) rather than linearly. This feature provides a preferential reduction of the largest numbers, effectively minimizing the maximum values. Since the minimization of a Linear Quadratic problem effectively serves the same function as a minimax, these objectives can simply be rewritten as Linear Quadratic minimizations:

$$\text{Min } \quad f_4 = \sum_t \left(-\Delta M_{t-1}\right)^2 \text{ or simply, } f_4 = \sum_t \Delta M_{t-1}^2 \tag{66.}$$

$$\text{Min } \quad f_5 = \sum_t \Delta N_{t-1}^2 \tag{67.}$$

5.3 Linear Quadratic Minimizations: The Connection Between Objectives One and Six

Lotfi states that objective one is supposed to be profit maximization. As mentioned in section 4.5 above, Lotfi introduced demand (D_t) without much explanation. Likewise, he introduced price (P_t) in a similar fashion. Unfortunately, in neither case did Lotfi state that the quantity or price provided were profit maximizing, although profit maximization is a stated objective. In fact, upon analysis, the demand and price provided turn out not to be profit maximizing. (Please see section 5.3.1 for a treatment of profit maximization.)

I will modify objective one and six to overcome this problem, while transforming the two objectives into Linear Quadratic minimizations. Instead of maximizing profit in objective one, I will calculate the profit maximizing quantity (Q_{PMax}) and price (P_{PMax}) in a separate calculation. Q_{PMax} is the profit maximizing output quantity in the final time period, when all old machines have been replaced and output is produced entirely on new machines. The profit maximizing quantity (Q_{PMax}) will then be driven with objective six by minimizing the difference between the quantity produced in each period and the profit maximizing quantity produced in the last period $\left(Q_t - Q_{PMax}\right)$. Finally, the first objective will be changed to a cost minimization (the mirror image of a profit maximization).

To smoothly incorporate the quadratic nature of this minimization, I will incorporate two further modifications. I will define a constraint, $TC_t = MC''x''N_t r_t + MC'x'M_t r_t + \Delta N_{t-1}C''r_t + \Delta M_{t-1}Sr_t$, to represent the total costs in objective one, and will also square the objective function terms to make the equations quadratic. Therefore, objectives One and Six

$$\max\ f_1 = \sum_t \left(P_t x''N_t r_t - MC''x''N_t r_t\right) + \sum_t \left(P_t x'M_t r_t - MC'x'M_t r_t\right) - \sum_t \Delta N_{t-1}C''r_t + \sum_t \Delta M_{t-1}Sr_t$$

$$\text{Min}\ f_6 = \sum_t \left(Q_t - D_t\right)$$

Will become:

$$\text{Min}\ f_1 = \sum_t TC_t^2 \qquad (68.)$$

$$\text{Min}\ f_6 = \sum_t \left(Q_t - Q_{PMax}\right)^2 \qquad (69.)$$

subject to:

$$TC_t = MC''x''N_t r_t + MC'x'M_t r_t + \Delta N_{t-1}C''r_t + \Delta M_{t-1}Sr_t \qquad (70.)$$

5.3.1 Determination of Q_{PMax}

Q_{PMax}, is an important term in the minimization, as it ensures that the solution is driven to the profit maximizing quantity. Lotfi assumed that output was exogenous, and the quantity demand was simply provided. However, a demand function may be constructed based on data in the Lotfi paper, and this function, along with the production function allows the profit maximizing quantity and price to be determined.

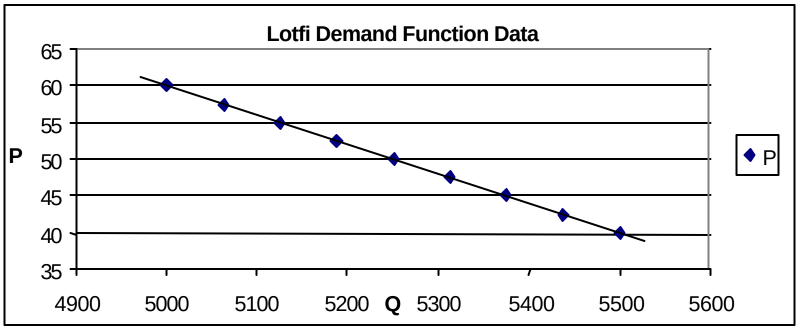
The following calculations will illustrate a solution where the profit is maximized. Profit maximization will be calculated where $MR=MC$, and the profit maximizing price and quantity will define the beginning and end points for the Riccati solution. The two profit maximizing cases considered here:

1. The initial case, for period $t = 0$, where N_0 (the number of new machines) = 0, will determine the initial profit maximizing conditions.
2. The final case, for period $t = t_f$ where M_f (the number of old machines) = 0, will determine the final profit maximizing conditions.

The following linear demand data was provided in the Lotfi paper. Lotfi did not assume any stochastic processes, and neither does this analysis.

Table 5.1: Lotfi Demand Function

Q	P
5000	60
5063	57.5
5125	55
5188	52.5
5250	50
5313	47.5
5375	45
5438	42.5
5500	40



A linear regression of the data results in the following demand function:

$$Q = f(P) = 13000 - 50P$$

Rearranging:

$$P = 260 - \frac{Q}{50}$$

We also know that pre-tax profits will be maximized when *MC*=*MR*. In the initial case, $N_0 = 0$, therefore all production is conducted with old machines, M_0. Also, the marginal cost is MC'.

$$MC' = 28$$

We know that:

$$TR_t = P \cdot Q_t$$

Therefore,

$$TR_0 = P \cdot Q_0$$

$$TR_0 = \left(260 - \frac{Q_0}{50}\right) Q_0$$

$$TR_0 = 260Q_0 - \frac{Q_0^2}{50}$$

And, then

$$MR_0 = \frac{\partial TR_0}{\partial Q_0} = 260 - \frac{Q_0}{25}$$

Finally, Setting $MR_0 = MC'$ we can solve for Q_0:

$$Q_0 = 5800$$

In the final case, $M_f = 0$, so all production is conducted with new machines, *N*. Also, the marginal cost is MC''.

$$MC'' = 19.8$$

Finally, setting $MR_f = MC''$ we can solve for the optimal Q_f and P_f.

$$MR_f = 260 - \frac{Q_f}{25} = 19.8$$

$$\vdots$$

$$Q_f = 6005$$
$$P_f = 139.9$$

5.4 Variables to Zero: Transversality Conditions

As stated above, the Riccati iteration will drive all objectives to zero. If the final values are not zero, a proxy variable must be assigned that will allow the Riccati iteration to go to zero. For instance, the final value of the Total Cost, TC_f, will not be zero, nor will the final number of modules in use (*N*), but proxy variables defined as $(TC_t - TC_f)$ and $(N_t - N_f)$ will, indeed, go to zero. Other expressions like $(Q_t - Q_{PMax})$, ΔM_{t-1}, and ΔN_{t-1}, should be zero in the final time period, and therefore need no further modification.

The Riccati assumption that variables go to zero melds nicely with the idea of *transversality*. Transversality is a condition defined when a dynamic optimization is complete. Transversality is met when the *solution path* is normal to the *terminal surface,* a situation that only occurs at the termination of the optimal control path (i.e. the final control and state variable values). This result is true for both linear and non-linear systems, and is often difficult to calculate. However, the transversality conditions for Riccati's Linear Quadratic problem are actually incorporated directly into the formulation. Remember, the Riccati problem assumes that in the final state, all variables are zero, and the Riccati iteration drives all objectives to zero. This unique feature of

Riccati's Linear Quadratic equations guarantees that the transversality conditions are met and easily identified: when all proxy variables are equal to 0, the transversality conditions are satisfied.[37] The proxy variables introduced above already contain the transversality conditions, namely TC_f, N_f, and that ΔN_{t-1} and ΔM_{t-1} are both zero. Q_{PMax} is slightly different, since it was not introduced as a transversality condition, however, if the profit maximizing quantity is met at the final period, this value will also meet the transversality condition for quantity.

5.5 The Final Lotfi Model: Normal and Transversal

Putting together the equations from above, the final Lotfi model is:

$$\text{Min } f_1 = \sum_t TC_t^2$$

$$\text{Min } f_4 = \sum_t \Delta M_{t-1}^2$$

$$\text{Min } f_5 = \sum_t \Delta N_{t-1}^2$$

$$\text{Min } f_6 = \sum_t \left(Q_t - Q_{PMax}\right)^2$$

subject to:

$$TC_t = MC''x''N_t r_t + MC'x'M_t r_t + \Delta N_{t-1}C''r_t + \Delta M_{t-1}Sr_t$$

$$Q_t = x''N_t + x'M_t$$

$$\Delta N_{t-1} = N_t - N_{t-1}$$

$$\Delta M_{t-1} = M_t - M_{t-1}$$

[37] All numerical solutions in this thesis were calculated with proxy variables. The numbers and results reported in this paper will be the nominal rather than the proxy variables.

It should be noted that the four objectives can be combined into a simplified Linear Quadratic problem[38],

$$\text{Min } F = \sum_t \left(\frac{TC_t}{a}\right)^2 + \sum_t \left(\frac{Q_t - Q_{PMax}}{b}\right)^2 + \sum_t \left(\frac{\Delta N_{t-1}}{c}\right)^2 + \sum_t \left(\frac{\Delta M_{t-1}}{d}\right)^2 \qquad (71.)$$

And the final Transversal model is:

$$\text{Min } F = \sum_t \left(\frac{TC_t - TC_f}{a}\right)^2 + \sum_t \left(\frac{Q_t - Q_{PMax}}{b}\right)^2 + \sum_t \left(\frac{\Delta N_{t-1}}{c}\right)^2 + \sum_t \left(\frac{\Delta M_{t-1}}{d}\right)^2 \qquad (72.)$$

Subject to:

$$\left(TC_t - TC_f\right) = MC''x''\left(N_t - N_f\right)r_t + MC'x'\left(M_t - M_f\right)r_t + \Delta N_{t-1}C''r_t + \Delta M_{t-1}Sr_t \qquad (73.)$$

$$\left(Q_t - Q_{PMax}\right) = x''\left(N_t - N_f\right) + x'\left(M_t - M_f\right) \qquad (74.)$$

$$\Delta N_{t-1} = N_t - N_{t-1} \qquad (75.)$$

$$\Delta M_{t-1} = M_t - M_{t-1} \qquad (76.)$$

Where *a*, *b*, *c*, and *d* are normalization factors: *a* = \$1000, *b*= 100 units, *c* = 1 new machine, *d* = 1 old machine. (See section 5.5.3 for explanation of *a*, *b*, *c*, *d*.)

5.5.1 Economic Interpretation of Equations

The economic interpretation of each term in equation (72) is akin to the economic interpretation of the squared terms in a linear regression. The linear regression is trying to minimize the error of a set of points to a line, just as the Riccati is trying to minimize the error of the optimal trajectory to the transversal values. Equation 72 is a least-squares

[38] This form should look familiar. It is the form of Linear Quadratic equation presented in Section 5.1.

minimization of the four Lotfi objectives. Observe the terms:

$$\text{Min}\, F = \sum_t \left(\frac{TC_t - TC_f}{a} \right)^2 + \sum_t \left(\frac{Q_t - Q_{PMax}}{b} \right)^2 + \sum_t \left(\frac{\Delta N_{t-1}}{c} \right)^2 + \sum_t \left(\frac{\Delta M_{t-1}}{d} \right)^2$$

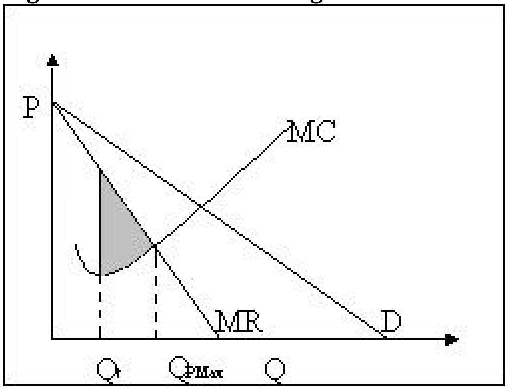
Each term could be re-written as:

$$\text{Min}\, F = \sum_t \left(\frac{TC_t - TC_f}{a} \right)^2 + \sum_t \left(\frac{Q_t - Q_{PMax}}{b} \right)^2 + \sum_t \left(\frac{\Delta N_{t-1} - 0}{c} \right)^2 + \sum_t \left(\frac{\Delta M_{t-1} - 0}{d} \right)^2$$

With the zeros placed in the equation, it is clear that the quantity subtracted in each term is the transversal value (TC_f, Q_{PMax}, 0 and 0) and that the equation is minimizing the error of each of the individual optimal trajectories to the transversal values. TC_f is the transversal total cost when all machines are in place in the final period of adjustment. Likewise, Q_{PMax} is defined to be the profit maximizing output (transversal output), and the term $\sum_t (Q_t - Q_{PMax})^2$ ensures that the problem will go to the profit maximizing quantity. (Q_{PMax} is derived in section 5.3.1.)

Minimizing the error of each of the individual optimal trajectories to the transversal induces a tension between the different terms. For example, in period *t*, a tension exists between $\sum_t (Q_t - Q_{PMax})^2$ and $\sum_t (TC_t - TC_f)^2$. Figure 5.2 shows that while the firm is producing at point Q_t, it is foregoing profit (the shaded area displays the foregone profit.) The further Q_t is toward the origin, the bigger is foregone profit in period *t*.

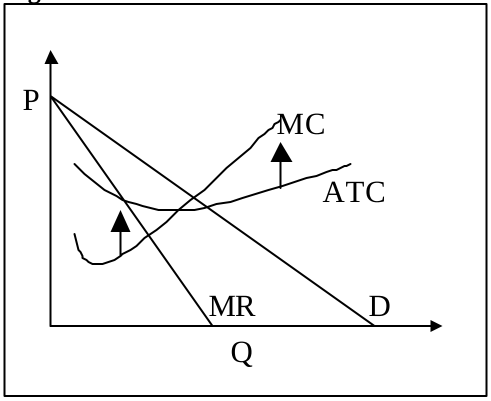
Figure 5.2: Demand vs. Marginal Cost Curves



In trying to reduce the shaded area in figure 5.2 by driving Q_t to Q_{PMax} (where MC_t = MR_t) and reducing $\sum_t (Q_t - Q_{PMax})^2$, the firm must also expend money to switch out old machines and switch in new machines. This expenditure pushes up MC_t and ATC_t, drives up total cost, $\sum_t (TC_t - TC_f)^2$, and this tension prevents all machines from being switched out in one period.

MC and *ATC* are driven up due to disruption of the production process in period *t* caused by switching in new machines and switching out old machines. This disruption drives up the average variable cost (*AVC*) and the *MC* (the variable cost directly attributable to producing output in period *t*). These disruption costs are represented mathematically by the cross product terms containing ΔM_{t-1} and ΔN_{t-1}. The bigger the ΔM_{t-1} and ΔN_{t-1} in period *t*, the larger the disruption costs, the larger $\sum_t (TC_t - TC_f)^2$, and the more MC_t and ATC_t are pushed up.

Figure 5.3: MC and ATC - DM and DN Cross-Product Impact



Additionally, ATC is driven up by the average fixed cost (AFC), since $ATC_t = AFC_t + AVC_t$. Money spent to purchase new machines increases AFC, which in turn drives up the overall ATC. Purchasing new machines in and of itself does not affect the AVC, since it does not affect the variable costs of producing output in period t. It is only through disruption of the production process that AVC is affected.

Recall that for a quadratic equation of the form $(a+b+c+d)^2$, the cross product terms are: $2(ab+ac+ad+bc+bd+cd)$. The cost equations yield the following cross-product terms: $(MC''x''N_t r_t \cdot MC'x'M_t r_t)$, $(MC''x''N_t r_t \cdot \Delta N_{t-1}C''r_t)$, $(MC''x''N_t r_t \cdot \Delta M_{t-1}Sr_t)$, $(MC'x'M_t r_t \cdot \Delta N_{t-1}C''r_t)$, $(MC'x'M_t r_t \cdot \Delta M_{t-1}Sr_t)$, $(\Delta N_{t-1}C''r_t \cdot \Delta M_{t-1}Sr_t)$, and $(x''N_t \cdot x'M_t)$. $(MC''x''N_t r_t \cdot \Delta N_{t-1}C''r_t)$, $(MC''x''N_t r_t \cdot \Delta N_{t-1}C''r_t)$, $(MC''x''N_t r_t \cdot \Delta M_{t-1}Sr_t)$, $(MC'x'M_t r_t \cdot \Delta N_{t-1}C''r_t)$, $(MC'x'M_t r_t \cdot \Delta M_{t-1}Sr_t)$, and $(\Delta N_{t-1}C''r_t \cdot \Delta M_{t-1}Sr_t)$ clearly demonstrate the disruption costs of switching machines, as all of these cost contributors are equal to zero when ΔM_{t-1} and ΔN_{t-1} are zero. These terms also demonstrate relationships playing out in the minimization. For instance, $(MC'x'M_t r_t \cdot \Delta M_{t-1}Sr_t)$ exhibits the relationship between producing with old machines $(MC'x'M_t r_t)$ and the salvage value of those machines $(\Delta M_{t-1} S r_t)$.[39]

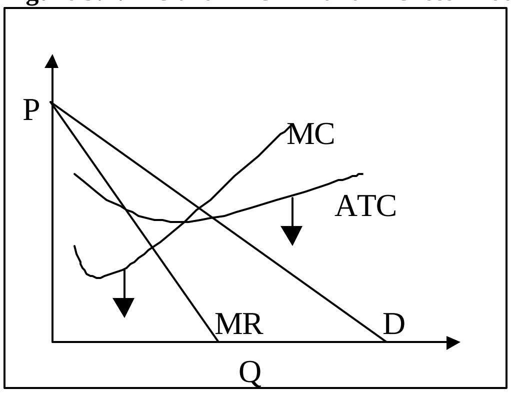


Figure 5.4: MC and ATC - M and N Cross-Product Impact

The cross-product terms containing only *M* or *N*, $(MC'' x'' N_t r_t \cdot MC' x' M_t r_t)$ and $(x'' N_t \cdot x' M_t)$, also affect the *MC* and *ATC* curves. The mix of new and old machines changes every period, being weighted more heavily with new machines each successive period. From period to period, the *M* and *N* cross-product terms continually lower the "baseline" of the *MC* and *ATC* curves, while the ΔM_{t-1} and ΔN_{t-1} cross-product terms drive up the *MC* and *ATC* curves from that baseline. With this graphical interpretation, we can appreciate the deeper tension playing out in the minimization of Equation 72.

5.5.2 Economic Interpretation of Equations: Numerical Example

The following tables provide a numerical example to illustrate the tension implicit between terms in equation 72. Equation 72 was solved under two different scenarios: the first is the baseline Lotfi situation, where the Marginal Cost for producing with new machines is \$19.80, MC'' = \$19.80. The second is where MC'' = \$9.00.

Tables 5.2 and 5.3 present the optimal trajectories when equation 72 is

[39] MC is marginal cost, x is efficiency, C'' is the cost of a new machine, S is salvage value.

minimized. The tables present the number of new and old machines required, the number of machines to be changed, the total cost, quantity, price, marginal revenue, marginal costs, and profit for each time period (from 0 to 6). The optimal trajectories are also displayed graphically in figure 5.5. During the transition, $MC \neq MR$, however, this should not be a surprise. It is during this time that the tension between terms in equation 72 is playing out, so, although we are not at a profit maximizing point during this time, equation 72 *is* at its minimal value. Note that with a lower MC'', the transition is faster, more equipment is changed out in each time period, final total cost is lower, and profits are higher. Also note that the optimization ends up at a different final quantity and price (which it should, based on the downward-sloping Demand curve).

Table 5.2: MC?=$19.8

Time	DN	N	DM	M	TC	Q	P	MR	MC'	MC"	Profit
						5800	144	28	28	19.8	
0	5	0	-11	28	147,000	5880	142.4	24.8	28	19.8	690,312
1	5	5	-8	17	108,786	5220	155.6	51.2	28	19.8	703,446
2	3	10	-5	9	114,411	5190	156.2	52.4	28	19.8	696,267
3	2	13	-3	4	100,904	5130	157.4	54.8	28	19.8	706,558
4	2	15	-1	1	97,518	5160	156.8	53.6	28	19.8	711,570
5	1	17	0	0	116,239	5610	147.8	35.6	28	19.8	712,919
6	0	18	0	0	116,618	5940	141.2	22.4	28	19.8	722,110
						6005	139.9	19.8	28	19.8	

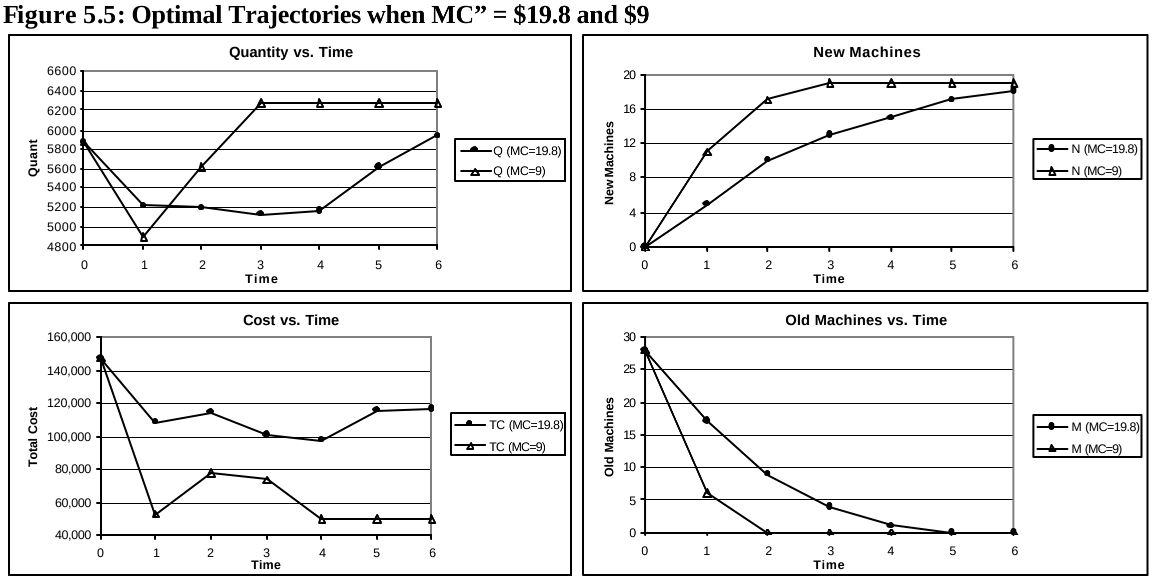
Table 5.3: MC?=$9

Time	DN	N	DM	M	TC	Q	P	MR	MC'	MC"	Profit
						5800	144	28	28	9	
0	11	0	-22	28	147,000	5880	142.4	24.8	28	9	690,312
1	6	11	-6	6	53,009	4890	162.2	64.4	28	9	740,149
2	2	17	0	0	77,813	5610	147.8	35.6	28	9	751,346
3	0	19	0	0	73,598	6270	134.6	9.2	28	9	770,344
4	0	19	0	0	50,384	6270	134.6	9.2	28	9	793,558
5	0	19	0	0	50,384	6270	134.6	9.2	28	9	793,558
6	0	19	0	0	50,384	6270	134.6	9.2	28	9	793,558
						6275	134.5	9	28	9	

For comparison, the first and last rows of the tables (without time designations) present the theoretically optimal (transversal) solutions where $MC' = MR$ and $MC'' = MR$. Why doesn't *MC=MR* in the rows with time designations? In this study, it was assumed that machines could not be "partially" employed. Therefore, every new machine produces 330 parts per year, and every old machine produces 210 parts per year – no more and no less. Therefore, although the transversal point for old machines is 5800, Lotfi's plant cannot produce 5800 parts with old machines. 27 old machines produce 5670 parts, and 28 old machines produce 5880 parts. Therefore, 5880 was chosen as the beginning production quantity.

Likewise, the true profit maximizing point for new machines (when MC'' = \$19.8) is 6005. 6005 cannot be produced with new machines. 18 new machines produce 5940 parts, and 19 new machines produce 6270 parts. Therefore, 5940 was chosen as the final production quantity for this study.

Figure 5.5: Optimal Trajectories when MC" = \$19.8 and \$9



5.5.3 Interpreting the Normalization

Lotfi wanted to minimize the four objectives given in eqn 72, but failed to provide any discussion or guidance on how to relate dollars to quantity to number of machines. This is a classic case of comparing apples to oranges. Just as is done elsewhere, relative prices were employed to compare these dissimilar objects one against the other. Those relative prices were then used to normalize the different terms.

The following relative prices were employed in the Lotfi analysis: $1000 ~ 100 units ~ 1 new machine ~ 1 old machine. (*a* = $1000, *b*= 100 units, *c* = 1 new machine, *d* = 1 old machine in eqns. 71 and 72.)

Just as relative prices of 3 apples ~ 2 orange implies that one would trade 3 apples for 2 oranges, or that one views 3 apples and 2 oranges as equally desirable, the above pricing scheme implies the same thing for Lotfi: that Lotfi would be willing to trade, for example, being $1000 away from the transversal total cost for being 100 units away from the transversal quantity for the disruption of one machine change.

To compare quantities one to another, one would employ a ratio of the relative prices. For example, how many oranges is 12 apples equivalent to?

$$12 \text{ Apples} \times \left(\frac{2 \text{ Oranges}}{3 \text{ Apples}} \right) = 8 \text{ Oranges} \qquad (77.)$$

For Lotfi: the disruption of 3 new machine is equivalent to being how many dollars away from the transversal TC?:

$$3\text{ New Machines} \times \left(\frac{\$1000}{1\text{ New Machine}} \right) = \$3000 \text{ away from the transversal TC} \tag{78.}$$

3 new machine purchases is equivalent to being $3000 away from the transversal TC. As shown above, ratios are used to calculate equivalent quantities, however, to **normalize**, one would instead divide by the relative price. For example, what has a higher price, 11 apples or 7 oranges?

For Apples:

$$\frac{\text{Number}}{\text{Rel Price}} = \frac{11\text{ Apples}}{3\text{ Apples}} = 3\tfrac{2}{3} \tag{79.}$$

For Oranges:

$$\frac{\text{Number}}{\text{Rel Price}} = \frac{7\text{ Oranges}}{2\text{ Oranges}} = 3\tfrac{1}{2} \tag{80.}$$

11 apples has a higher price than 7oranges, since $3\tfrac{2}{3} > 3\tfrac{1}{2}$. Note that the calculation is absolutely correct, but the final values are unitless. To calculate the price in terms of apples, one needs simply to multiply the unitless value by the price of apples. Likewise, to find out the price in terms of oranges, one would multiply the unitless value by price of oranges. Any one of the three methods provides the correct answer.

As an example for Lotfi: what has a higher relative price, being $3249 away from the transversal total cost, or being 350 units away from the transversal quantity?

For Total Cost:

$$\frac{\text{Number}}{\text{Rel Price}} = \frac{\$3249}{\$1000} = 3.249 \qquad (81.)$$

For Quantity:

$$\frac{\text{Number}}{\text{Rel Price}} = \frac{350 \text{ units}}{100 \text{ units}} = 3.5 \qquad (82.)$$

Being 350 units away from the transversal quantity has a higher relative price than being $3249 away from the transversal total cost, since 3.5 > 3.249. Again, to determine the price in terms of dollars from the transversal total cost, or units from the transversal quantity, or in disruption from machines, etc. one would multiply the unitless number by the relative price.

These relative pricing schemes are arbitrary, however, so are relative pricing for apples and oranges based on individual taste. Unfortunately, Lotfi did not price his objectives. Lotfi did, however, believe that each of these objectives was important. I chose relative prices so that the resulting normalized quantities had values of roughly equal size. This is important, since the terms are squared, large numbers for one of the objectives could dominate all the others. Again, this is akin to having numerically large outliers dominate a linear regression calculation.

5.6 Riccati Iteration: Final Presentation with the Linear Quadratic Problem

In this section, the equations of the Riccati iteration and the final matrix formulation of the Linear Quadratic problem will be introduced. The Riccati iteration is rather unimpressive when compared to the exertions required to modify the Lotfi model. However that is the beauty of the Riccati iteration. It is simply expressed, but very powerful. Remember, the purpose of the Riccati iteration is to calculate the optimal inputs, $\mathbf{u}_0$, that will minimize $f = \sum_t \left(\mathbf{y}_t^{\mathbf{T}} \mathbf{Q} \mathbf{y}_t + \mathbf{u}_t^{\mathbf{T}} \mathbf{R} \mathbf{u}_t \right)$.

The Riccati Iteration:

$$y_T = \overline{\mathbf{Q}} \tag{83.}$$

$$y_{t-1} = \overline{\mathbf{Q}} + \mathbf{A}^{\mathbf{T}} y_t \mathbf{A} - \left(\mathbf{A}^{\mathbf{T}} y_t \mathbf{B} + \overline{\mathbf{N}} \right) \left(\mathbf{B}^{\mathbf{T}} y_t \mathbf{B} + \overline{\mathbf{R}} \right)^{-1} \left(\mathbf{B}^{\mathbf{T}} y_t \mathbf{A} + \overline{\mathbf{N}}^{\mathbf{T}} \right) \tag{84.}$$

$$\mathbf{J}_t = -(\mathbf{B}^{\mathbf{T}} y_{t+1} \mathbf{B} + \overline{\mathbf{R}})^{-1} \left(\mathbf{B}^{\mathbf{T}} y_{t+1} \mathbf{A} + \overline{\mathbf{N}}^{\mathbf{T}} \right) \tag{85.}$$

$$\mathbf{u}_t = \mathbf{J}_t \mathbf{x}_t \tag{86.}$$

where:

$$\begin{bmatrix} \overline{\mathbf{Q}} & \overline{\mathbf{N}} \\ \overline{\mathbf{N}}^{\mathbf{T}} & \overline{\mathbf{R}} \end{bmatrix} = \begin{bmatrix} \mathbf{C}^{\mathbf{T}} & \mathbf{0} \\ \mathbf{D}^{\mathbf{T}} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{87.}$$

That's it. The Riccati iteration is simply a series of multiplications, additions and subtractions. In the above equations, y is called the Riccati matrix, and $\mathbf{u}_t$ is the optimal vector of inputs. The Riccati iteration is not difficult to use. For instance, for our prediction horizon ($K = 5$), the present input vector ($\mathbf{u}_0$) would be calculated by iteratively calculating y_1 (e.g. we are given $y_5 = \mathbf{Q}$, and we can then calculate y_4, y_3, …, y_1). y_1 will allow us to calculate $\mathbf{J}_0$ and $\mathbf{u}_0$. Forecasting inputs is straightforward. For instance, forecasting inputs for the next two time periods (k=1 and k=2) would simply require using y_2 and y_3, which were already calculated, to determine $\mathbf{J}_1$, $\mathbf{J}_2$, $\mathbf{u}_1$, and $\mathbf{u}_2$.

5.6.1 The Lotfi Problem: Matrix Format

The Linear Quadratic problem presented in section 5.5 is properly presented as

$$\min_{\{\mathbf{u}_t\}} f = \sum_t \left(\mathbf{y}_t^{\mathbf{T}}\mathbf{Q}\mathbf{y}_t + \mathbf{u}_t^{\mathbf{T}}\mathbf{R}\mathbf{u}_t\right)$$

subject to:

$$\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1} + \mathbf{B}\mathbf{u}_{t-1}$$

$$\mathbf{y}_t = \mathbf{C}\mathbf{x}_t + \mathbf{D}\mathbf{u}_t$$

when:

$$\mathbf{y}_t = \begin{bmatrix} TC_t \\ Q_t - Q_{PMax} \\ \Delta N_{t-1} \\ \Delta M_{t-1} \end{bmatrix} \qquad \mathbf{u}_t = \begin{bmatrix} \Delta N_t \\ \Delta M_t \end{bmatrix} \qquad \mathbf{x}_t = \begin{bmatrix} N_t \\ \Delta N_{t-1} \\ M_t \\ \Delta M_{t-1} \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} MC''x''r_t & C''r_t & MC'x'r_t & Sr_t \\ x'' & 0 & x' & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Or with Transversality variables[40]:

[40] Please see Appendix E for proof of the veracity of the above equations.

$$\bar{\mathbf{y}}_t = \begin{bmatrix} TC_t - TC_f \\ Q_t - Q_{PMax} \\ \Delta N_{t-1} \\ \Delta M_{t-1} \end{bmatrix} \qquad \bar{\mathbf{x}}_t = \begin{bmatrix} N_t - N_f \\ \Delta N_{t-1} \\ M_t - M_f \\ \Delta M_{t-1} \end{bmatrix}$$

6 Results: Riccati Iteration Applied to Lotfi's Model and Proof of Riccati Value vs. Lagrange

This section demonstrates the usefulness of the Riccati iteration by highlighting the differences in solving the Lotfi problem using the Lagrange and the Riccati approaches.

6.1 The General Lagrange Solution:

The Lagrange optimization is one of the best-known, and most-used optimization techniques, typically employed as a method to solve constrained optimization problems. This method involves the construction of an equation, the Lagrange function, which combines the function to be minimized or maximized and any possible constraints. The Lagrange function is then solved for the optimal input variables. For instance, if the equation to be minimized is of the general form $f(x_1, \ldots, x_n)$ and the constraints (if any) are of the general form $g(x_1, \ldots, x_n)$, then the Lagrange equation is written as:

$$\Lambda = f(x_1,...,x_n) + l\,g(x_1,...,x_n) \qquad (88.)$$

where λ is the Lagrange multiplier and represents the shadow price of the constraint resource.[41]

Derivatives are then taken, set equal to zero, and solved simultaneously:

$$\frac{\partial \Lambda}{\partial x_1} = 0$$

[41] The Lagrange multiplier is also the rate of change of the objective function's maximum or minimum value with respect to parametric changes in the constraint value.

$$\vdots$$

$$\frac{\partial \Lambda}{\partial l_n} = 0$$

or, simply:

$$\nabla \Lambda = 0 \qquad (89.)$$

To ensure that the solution is indeed a maximum or minimum, second-order conditions must also be met. Recall that a maximum will have a positive definite bordered Hessian matrix (det(H)>0), and a minimum has a negative definite bordered Hessian matrix (det(H)<0), where the bordered Hessian matrix is defined as:

$$\mathbf{H} = \begin{bmatrix} \Lambda_{11} & \cdots & \Lambda_{1n} & g_1 \\ \vdots & \ddots & \vdots & \vdots \\ \Lambda_{n1} & \cdots & \Lambda_{nn} & g_n \\ g_1 & \cdots & g_n & 0 \end{bmatrix} \qquad (90.)$$

6.2 The Particular Lagrange Solution:

In this case, we will assume that the second order requirements are met (i.e. the bordered Hessian matrix is negative), and will use the Lagrange method attempt to solve for the optimal selection of input variables, ΔM_{t-1} and ΔN_{t-1}. In non-matrix form:

$$\text{Min } f = \sum_t \left(TC_t - TC_f\right)^2 + \sum_t \left(Q_t - Q_{PMax}\right)^2 + \sum_t \Delta N_{t-1}^2 + \sum_t \Delta M_{t-1}^2$$

Subject to:

$$\begin{aligned} &\left(TC_t - TC_f\right) = MC''x''\left(N_t - N_f\right)r_t + MC'x'\left(M_t - M_f\right)r_t + \Delta N_{t-1}C''r_t + \Delta M_{t-1}Sr_t \\ &\left(Q_t - Q_{PMax}\right) = x''\left(N_t - N_f\right) - x'\left(M_t - M_f\right) \\ &\Delta N_{t-1} = N_t - N_{t-1} \\ &\Delta M_{t-1} = M_t - M_{t-1} \end{aligned}$$

Now, forming the Lagrange equation:

$$\Lambda = \sum_t \left(TC_t - TC_f\right)^2 + \sum_t \left(Q_t - Q_{PMax}\right)^2 + \sum_t \Delta N_{t-1}^2 + \sum_t \Delta M_{t-1}^2 +$$
$$+ \sum_t l_1\left(MC''x''\left(N_t - N_f\right)r_t + MC'x'\left(M_t - M_f\right)r_t + \Delta N_{t-1}C''r_t + \Delta M_{t-1}Sr_t - TC_t\right)$$
$$+ \sum_t l_2\left(x''\left(N_t - N_f\right) + x'\left(M_t - M_f\right) - Q_t\right) + \sum_t l_3\left(\left(N_t - N_f\right) - N_{t-1} - \Delta N_{t-1}\right)$$
$$+ \sum_t l_4\left(\left(M_t - M_f\right) - M_{t-1} - \Delta M_{t-1}\right)$$

The kickers in the above equation are the summations. The above equation, as written, is actually an abbreviated form. If written in full to eliminate the summation signs, this equation would actually have 90 terms. The second step to solve the Lagrange equation would require deriving fourteen partial differential equations from the above 90 term equation, setting each of those fourteen equations equal to zero, and solving them simultaneously:

$$\frac{\partial \Lambda}{\partial \Delta N_0} = 0\,,\ \ldots,\ \frac{\partial \Lambda}{\partial \Delta N_4} = 0\,,$$

$$\frac{\partial \Lambda}{\partial \Delta M_0} = 0\,,\ \ldots,\ \frac{\partial \Lambda}{\partial \Delta M_4} = 0\,,$$

$$\frac{\partial \Lambda}{\partial \Delta l_1} = 0\,,\ \ldots,\ \frac{\partial \Lambda}{\partial \Delta l_4} = 0$$

The fact that TC_t, Q_t, N_t, and M_t are all functions of ΔN_{t-1}, and ΔM_{t-1} make these equations both nonlinear and complex. Solving such a set of fourteen equations simultaneously would be extraordinarily difficult and time consuming.

6.3 The Riccati Solution:

The Riccati Solution, as compared to the Lagrange, is simply a set of matrix arithmetic problems. The matrices were introduced in the previous section, and the solution of the Riccati iteration for a 5-period horizon simply requires calculating the following eight equations in order:

$$\begin{bmatrix} \overline{\mathbf{Q}} & \overline{\mathbf{N}} \\ \overline{\mathbf{N}}^{\mathbf{T}} & \overline{\mathbf{R}} \end{bmatrix} = \begin{bmatrix} \mathbf{C}^{\mathbf{T}} & \mathbf{0} \\ \mathbf{D}^{\mathbf{T}} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{C} & \mathbf{D} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

$$y_5 = \overline{\mathbf{Q}}$$

$$y_4 = \overline{\mathbf{Q}} + \mathbf{A}^{\mathbf{T}} y_5 \mathbf{A} - \left(\mathbf{A}^{\mathbf{T}} y_5 \mathbf{B} + \overline{\mathbf{N}}\right)\left(\mathbf{B}^{\mathbf{T}} y_5 \mathbf{B} + \overline{\mathbf{R}}\right)^{-1}\left(\mathbf{B}^{\mathbf{T}} y_5 \mathbf{A} + \overline{\mathbf{N}}^{\mathbf{T}}\right)$$

$$y_3 = \overline{\mathbf{Q}} + \mathbf{A}^{\mathbf{T}} y_4 \mathbf{A} - \left(\mathbf{A}^{\mathbf{T}} y_4 \mathbf{B} + \overline{\mathbf{N}}\right)\left(\mathbf{B}^{\mathbf{T}} y_4 \mathbf{B} + \overline{\mathbf{R}}\right)^{-1}\left(\mathbf{B}^{\mathbf{T}} y_4 \mathbf{A} + \overline{\mathbf{N}}^{\mathbf{T}}\right)$$

$$y_2 = \overline{\mathbf{Q}} + \mathbf{A}^{\mathbf{T}} y_3 \mathbf{A} - \left(\mathbf{A}^{\mathbf{T}} y_3 \mathbf{B} + \overline{\mathbf{N}}\right)\left(\mathbf{B}^{\mathbf{T}} y_3 \mathbf{B} + \overline{\mathbf{R}}\right)^{-1}\left(\mathbf{B}^{\mathbf{T}} y_3 \mathbf{A} + \overline{\mathbf{N}}^{\mathbf{T}}\right)$$

$$y_1 = \overline{\mathbf{Q}} + \mathbf{A}^{\mathbf{T}} y_2 \mathbf{A} - \left(\mathbf{A}^{\mathbf{T}} y_2 \mathbf{B} + \overline{\mathbf{N}}\right)\left(\mathbf{B}^{\mathbf{T}} y_2 \mathbf{B} + \overline{\mathbf{R}}\right)^{-1}\left(\mathbf{B}^{\mathbf{T}} y_2 \mathbf{A} + \overline{\mathbf{N}}^{\mathbf{T}}\right)$$

$$\mathbf{J}_0 = -(\mathbf{B}^{\mathbf{T}} y_1 \mathbf{B} + \overline{\mathbf{R}})^{-1}\left(\mathbf{B}^{\mathbf{T}} y_1 \mathbf{A} + \overline{\mathbf{N}}^{\mathbf{T}}\right)$$

$$\mathbf{u}_0 = \mathbf{J}_0 \mathbf{x}_0$$

This is where the true power of the Riccati iteration is shown: eight matrix arithmetic problems versus the simultaneous solution of fourteen separate non-linear partial differential equations. This simplicity is why the Riccati iteration can be calculated with spreadsheet software, whereas other solution techniques require specialized software and often a powerful computer. The next section will demonstrate the power of the Riccati iteration, by providing a numerical example of it as applied to the Lotfi problem.

6.4 Riccati Solution: A Numerical Example

This section will provide a numerical example of the Riccati iteration. The values were derived from the Lotfi paper, and the calculations were conducted on an Excel spreadsheet.[42] This section will first introduce the values that Lotfi provided for his variables, then will present the final (transversal) values for the state variables, and ultimately will present the results of the Riccati iteration.

6.4.1 The Lotfi Variables: Lotfi's Constants

The following values for these important Lotfi variables were derived from the Lotfi paper:

$T = 5$
$MC'' = \$19.80$ /part
$MC' = \$28.00$ / part
$x_t'' = 330$ parts / year
$x' = 210$ parts / year
$r_t = 1/(1+.12)$
$C'' = \$13,000$
$S = \$6890$
$P = \$140$
$M_0 = 28$

6.4.2 Transversality: The Final Values

The following values were calculated from the profit-maximizing quantity derived from information in the Lotfi paper. The demand curve is presented and discussed in section 5.3.1.

$TC_f = \$117,767$
$N_f = 18$
$M_f = 0$
$Q_{PMax} = 6005$

[42] The Excel spreadsheet was provided on a disk to the Thesis committee.

6.4.3 Results: Riccati Works!

Table 6.1 provides the decision for each time step calculated by the Riccati iteration, along with the total cost, the quantity produced, and the profit all discounted and in year zero dollars.

Table 6.1: Riccati Optimal Solution

Time	ΔN	N	ΔM	M	TC	Q	P	MR	MC'	MC"	Profit
						5800	144	28	28	19.8	
0	5	0	-11	28	147,000	5880	142.4	24.8	28	19.8	690,312
1	5	5	-8	17	108,786	5220	155.6	51.2	28	19.8	703,446
2	3	10	-5	9	114,411	5190	156.2	52.4	28	19.8	696,267
3	2	13	-3	4	100,904	5130	157.4	54.8	28	19.8	706,558
4	2	15	-1	1	97,518	5160	156.8	53.6	28	19.8	711,570
5	1	17	0	0	116,239	5610	147.8	35.6	28	19.8	712,919
6	0	18	0	0	116,618	5940	141.2	22.4	28	19.8	722,110
						6005	139.9	19.8	28	19.8	

As we can see, it actually takes into the sixth year to get everything settled, rather than the fifth year, as we were originally targeting. The numbers here are a little off of the calculated optimal (e.g. 6005 units produced) because of the restrictions placed on output. Without slack or oversubscrption (overtime), the old machines can only produce in increments of 210, and the new modules can only produce in increments of 330, so, although the profit-maximizing output is 6005, the amount that I can realize with this model is 5940, since 6005 does not divide evenly into 330. As further verification of the Riccati calculations, the NPV of manufacturing with the new modules turns out to be \$3,087,222, whereas the NPV of manufacturing without a switchover is only \$3,002,544, a difference of about \$85,000. As should be expected, it makes economic sense to do the switchover.

6.4.4 Verification: Least Squares

As a verification of the Riccati solution, a different solution method called the least-squares method was implemented to solve the Linear Quadratic problem independently. The least-squares approach is a brute-force method to solve the Linear Quadratic problem. This approach is cumbersome, requires specialized software and knowledge of Matrix calculus, and also demands ample familiarity with an arcane programming language.[43]

The good news is that the least-squares approach results in an identical solution to the Riccati iteration, therefore validating my Riccati solution. (See tables 6.2 and 6.3 below.) Appendix D presents the least-squares solution, and the program used to calculate the least-squares solution.

Table 6.2: Least Squares Optimal Sol'n

Time	DN	DM
0	5	-11
1	5	-8
2	3	-5
3	2	-3
4	2	-1
5	1	0

Table 6.3: Riccati Optimal Sol'n

Time	DN	DM
0	5	-11
1	5	-8
2	3	-5
3	2	-3
4	2	-1
5	1	0

[43] The specialized computer software is MATLAB, produced by The Math Works, Inc. MATLAB employs its own, unique programming language which is powerful, but somewhat abstruse.

7 Conclusions

This thesis has presented the Riccati iteration as the most practicable method for solving some otherwise difficult-to-solve dynamic optimization problems. It was shown that the Riccati iteration succeeded in solving a modified capital equipment problem proposed by Lotfi. The popular Lagrange solution method proved to be prohibitively difficult in solving the Lotfi problem, and the only other method found to work was a matrix least-squares method that required specialized software. It is the true hope of this author that with the demonstration in this thesis, others may decide to use the Riccati iteration as a potential problem solving technique.