THE INFLUENCE OF A MASS ON THE FREE FLEXURAL VIBRATIONS OF A CIRCULAR RING

by

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TABLES

FIGURES

LIST OF SYMBOLS

a, b	Real and imaginary parts, respectively, of mode parameters n_2 and n_3 in first mode
A_k, B_k	Mode constants for antisymmetrical and symmetrical branches, respectively
C, C_A, C_B	Frequency-point mass relationships where the subscripts A and B pertain to the antisymmetrical and symmetrical branches, respectively
D_A, D_B	Determinant of coefficients of mode constants A_k and B_k
E	Young's modulus
G	In-plane bending moment
H	Amplitude constant
I	Moment of inertia of ring cross section about the centroidal axis normal to the plane of the ring
i	$\sqrt{-1}$
k	Subscript relating the three mode constants A_k or B_k to the three mode parameters n_k
M	Point mass $\left(\frac{\text{lb - sec}^2}{\text{ft}}\right)$
m	Mass of ring per unit length $\left(\frac{\text{lb - sec}^2}{\text{ft}^2}\right)$
N	Shear on cross section of ring in radial direction
n	An integer greater than unity defining the mode of vibration for a ring without a point mass
n_k	Mode parameters
p	Circular frequency of vibration
Q	Frequency - mass relationship
r	Radius of centerline of ring
s	Arc length along ring centerline
T	Normal force on cross section of ring
t	Time variable

U, V	Radial and tangential displacement functions
u, v	Radial and tangential displacements of ring centerline
α_K	Circular argument $n_K \pi$ for coefficients of mode constants A_K and B_K
β, γ	Real and imaginary parts of symmetrical branch, first mode constants B_K
ϕ, λ	Real and imaginary parts of antisymmetrical branch, first mode constants A_K
Γ	Time function
ϵ	Tangential unit strain of ring centerline
$\Delta \rho$	Change in curvature of ring centerline from the undeformed to the deformed state
θ	Angular coordinate
δ	Phase angle
∞	Infinity

ABSTRACT

The general solution was obtained for the free flexural vibrations in the plane of a thin circular ring containing a point mass. As a degenerate case of the general solution, the solution for a uniform ring alone was derived from the general solution by taking the point mass to be zero. Numerical calculations of the frequencies and mode shapes of the first and second flexural modes were made for values of the point mass in the range from zero to infinity. The results are presented in graphical form.

The predominant feature of the investigation was the difference in frequency and mode shape found in the symmetrical and antisymmetrical modes, and the particular orientation of the nodes with respect to the point mass. It was noted that similar phenomena were observed experimentally for vibrations of imperfect bodies of revolution. In conclusion it was brought out that a ring with a point mass offers a convenient mathematical model for a preliminary theoretical investigation of the vibrations of imperfect bodies of revolution.

THE INFLUENCE OF A MASS ON THE FREE FLEXURAL VIBRATIONS OF A CIRCULAR RING

I. INTRODUCTION

The problem of the vibration of a ring containing a mass is of interest in many engineering situations (e.g., in the dynamic behavior of submarine pressure hulls supporting equipment, machinery and valves). It should be particularly suited also to the study of vibrations of imperfect bodies of revolution inasmuch as the effect of such imperfections can be quite distinct as has been shown by several experimental investigations [1, 2, 3, 4].* This matter is discussed more fully in Section VI.

The equation of motion and its general solution for the inextensional flexural vibrations of thin circular rings are given in the literature [5, 6, 7]. A comparison of theoretical and experimental results for an elastically supported ring is also available [8]. Theoretical consideration of the effect of centerline extension in flexural vibrations is discussed in [9]. Some theoretical work [10] has been done in deriving approximate fundamental frequencies by the Rayleigh-Ritz method for incomplete rings (arches and like structures) with attached masses.

Starting with the general solution, free vibration frequencies and mode shapes may be found by describing suitable continuity and equilibrium conditions at the mass. The influence of the mass may be ascertained by comparison with the vibrations of a ring without a point mass, the solution of which may be obtained from the general solution by taking the point mass to be zero.

*Numbers in brackets designate References listed on page 47.

II. OBJECTIVE AND SCOPE

The objective of this thesis is to determine the influence of a concentrated mass on the free flexural vibrations in the plane of a complete thin circular ring with particular application to the problem of imperfections in the vibrations of bodies of revolution.

The theoretical analysis is given in Part III which also contains a statement of the problem, the derivation of the differential equation of motion and its solution. Part III also contains a theoretical consideration of the point mass and the solution for a ring with and without a point mass. Part IV consists of numerical solutions for the first two modes with values of the point mass ranging from zero to infinity. The influence of the mass on the normal mode frequencies and shapes is then discussed in Part V. A summary of results and conclusions is offered in Part VI.

III. THEORETICAL ANALYSIS

A. STATEMENT OF THE PROBLEM

This investigation pertains to the case of a plane circular ring of constant and symmetrical cross section containing a point mass. One of the principal axes of the cross section is assumed to be situated in the plane of the ring with the vibrations occurring in that plane. The cross sectional dimensions are assumed to be small compared with the radius of the centerline, which implies a thin ring. The vibrations are assumed to occur in a form with no centerline extension. The mass is taken as rigid and integral with the ring, and for mathematical simplicity, is assumed to be concentrated at a point on the ring centerline. Damping and body forces are neglected in the analysis.

External forces are not considered; therefore, the motion will be free in nature.

B. THE DIFFERENTIAL EQUATION OF MOTION

The orientation of the polar coordinate system of the ring with respect to the point mass is shown in Fig. 1 as well as the positive directions of the radial and tangential displacements, u and v, of a point on the ring centerline.

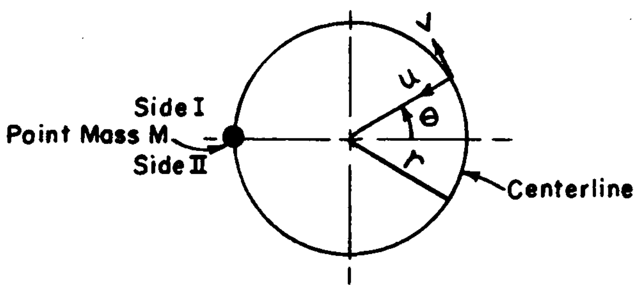

Fig. 1 Ring Coordinate System

To establish the equations of equilibrium [5, 6, 8] consider an element of the ring as shown in Fig. 2.

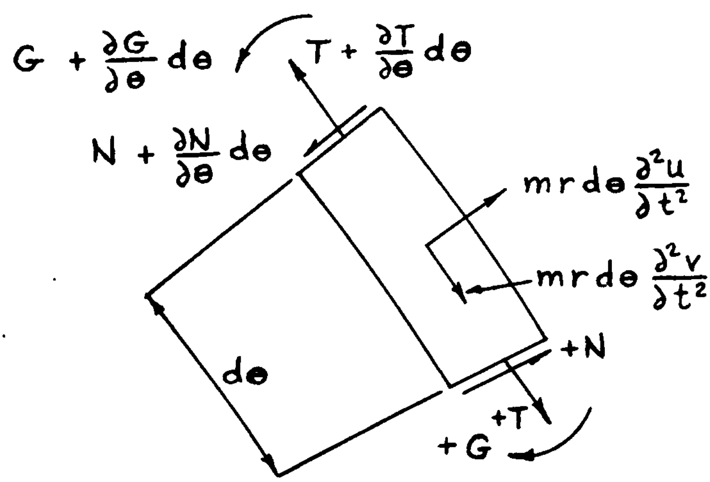

Fig. 2 Dynamical Equilibrium of a Ring Element

The variation of the internal shear and normal forces N and T, and flexural moment G along the element was obtained by Taylor's series expansions retaining only the first order term in each case. Shearing distortion is neglected and plane sections before flexure remain plane after flexure.

The sum of forces in the radial direction (positive inward) gives an equation of equilibrium,

$$\frac{\partial N}{\partial \theta} + T = mr\frac{\partial^2 u}{\partial t^2}, \tag{1}$$

and in the tangential direction (positive counterclockwise) an equilibrium equation,

$$\frac{\partial T}{\partial \theta} - N = mr\frac{\partial^2 v}{\partial t^2}. \tag{2}$$

Neglecting rotatory inertia, the sum of the moments gives an equation of equilibrium,

$$\frac{\partial G}{\partial \theta} + Nr = 0. \tag{3}$$

A single differential equation of motion is derived by eliminating the N, T and G terms of Eqs. (1), (2) and (3) with the aid of the relationship between bending moment and displacement [6, 7]. For thin circular bars the distribution of bending stresses approaches a linear one and the neutral axis is assumed to pass through the centroid of the cross section. The bending moment can then be related to the change in curvature of the bar as is done for straight bars. This can be expressed as

$$\Delta\rho = \frac{1}{r_1} - \frac{1}{r} = \frac{G}{EI}. \qquad (4)$$

The ring curvature before bending is $\frac{1}{r} = \frac{d\theta}{ds}$ and after bending, for small deflections, the curvature is

$$\frac{1}{r_1} = \frac{d\theta'}{ds'} \qquad (5)$$

where $d\theta'$ is the angle between normal cross sections of the deflected element and ds' is the length of the deflected element. Consider an element of the circular ring and the deflected element included in the same radii as shown in Fig. 3.

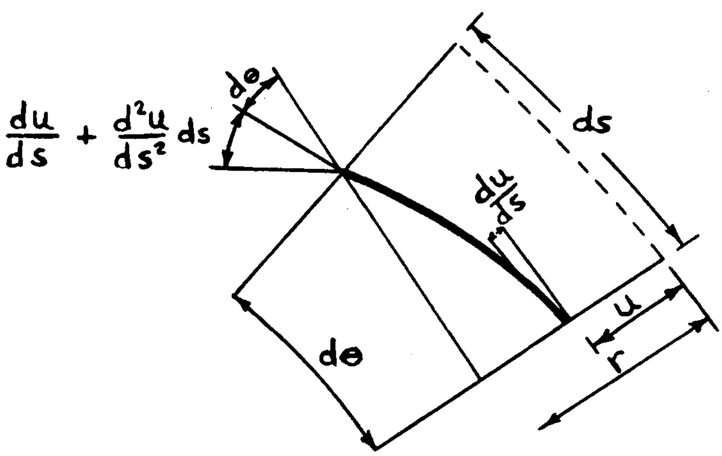

Fig. 3 Geometry of a Deflected Centerline Element

From Fig. 3 we have

$$d\theta' = d\theta + \frac{d^2u}{ds^2}\,ds \qquad (6)$$

and by neglecting the small angle $\frac{du}{ds}$ the length of the deflected element is

$$ds' = (r - u)\, d\theta. \tag{7}$$

Then from (6) and (7), (5) becomes

$$\frac{1}{r_1} = \frac{d\theta + \frac{d^2u}{ds^2}ds}{\left(1 - \frac{u}{r}\right)ds}, \tag{8}$$

which after neglecting small quantities of higher order reduces to

$$\frac{1}{r_1} = \frac{1}{r} + \frac{u}{r^2} + \frac{d^2u}{ds^2}. \tag{9}$$

Substituting (9) back in (4) we obtain

$$\frac{G}{EI} = \frac{1}{r^2}\left(\frac{d^2u}{d\theta^2} + u\right). \tag{10}$$

Since it is desired to employ (10) in a dynamic situation it may be written as

$$G = \frac{EI}{r^2}\left(\frac{\partial^2 u}{\partial\theta^2} + u\right). \tag{11}$$

For flexural vibrations without extension [5, 7, 8] the following geometrical condition must be satisfied

$$u = \frac{\partial v}{\partial \theta}. \tag{12}$$

This can be established in the following manner. Consider a centerline element deformed as a result of the displacements u and v, as shown in Fig. 4.

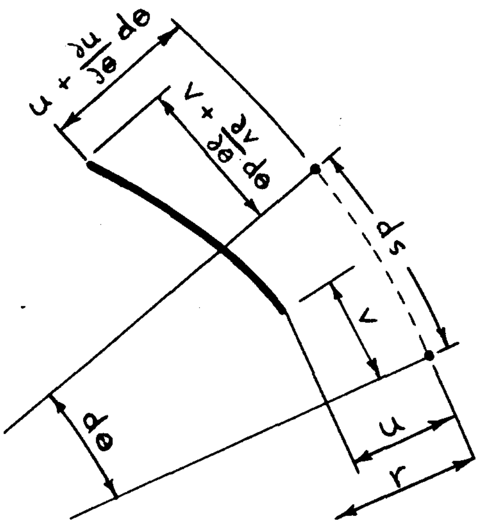

Fig. 4 Extension of a Centerline Element

The tangential unit strain due to the radial displacement is approximately

$$\epsilon = \frac{(r-u)\frac{ds}{r} - ds}{ds} = -\frac{u}{r} \qquad (13)$$

and due to the tangential displacement is

$$\epsilon = \frac{\left(v + \frac{\partial v}{\partial \theta} d\theta\right) - v}{ds} = \frac{\partial v}{r \partial \theta} \qquad (14)$$

or the total tangential unit strain is

$$\epsilon = \frac{\partial v}{r \partial \theta} - \frac{u}{r}. \qquad (15)$$

For no centerline extension, $\epsilon = 0$ and (15) becomes (12).

By use of the relationships (11) and (12), Eqs. (1), (2) and (3) may be reduced to obtain the equation of motion in terms of the tangential displacements:

$$\frac{EI}{r^4}\left(\frac{\partial^6 v}{\partial \theta^6} + 2\frac{\partial^4 v}{\partial \theta^4} + \frac{\partial^2 v}{\partial \theta^2}\right) = m\frac{\partial^2}{\partial t^2}\left(v - \frac{\partial^2 v}{\partial \theta^2}\right). \qquad (16)$$

C. SOLUTION OF THE DIFFERENTIAL EQUATION

Substituting an assumed product solution of the form

$$v(\theta, t) = V(\theta)\,\Gamma(t) \tag{17}$$

into (16), the usual process of separation of variables [12] leads to the equations

$$\frac{d^2\Gamma}{dt^2} + p^2\Gamma = 0 \tag{18}$$

and

$$\frac{d^6V}{d\theta^6} + 2\frac{d^4V}{d\theta^4} + \frac{d^2V}{d\theta^2}\left(1 - \frac{mr^4p^2}{EI}\right) + \frac{mr^4p^2}{EI}V = 0 \tag{19}$$

where p^2 is the separation constant.

Solving (18) gives the time function

$$\Gamma = H\cos(pt + \delta) \tag{20}$$

where p is the circular frequency, and H (amplitude) and δ (phase angle) are constants that can be determined from the initial conditions. The amplitude factor H is associated with the space function V so that (20) is usually expressed [5, 8] as

$$\Gamma = \cos(pt + \delta). \tag{21}$$

For frequencies and mode shapes of free vibration the solution of (19) for the space function is of primary interest. The complete solution of (19) is of the form [5]

$$V = \sum_{k=1}^{k=3} \left(A_k \cos n_k \theta + B_k \sin n_k \theta \right) \tag{22}$$

where n_1 , n_2 and n_3 are roots of the equation

$$n_k^2 \left(n_k^2 - 1 \right)^2 = \left(n_k^2 + 1 \right) \frac{m r^4 p^2}{EI} = \left(n_k^2 + 1 \right) Q \cdot \tag{23}$$

Equation (23) results from the substitution of (22) in (19). Notice that (23) is a sixth order algebraic equation in n_k but third order in n_k^2 ; therefore the six roots are $\pm n_1$, $\pm n_2$, and $\pm n_3$. Since (22) is in terms of circular functions either the positive or the negative set of three roots may be chosen. In this investigation the positive set is taken.

D. CONSIDERATION OF THE POINT MASS

As the solution of (19) contains six constants of integration, A_k and B_k (in general at least one of these always remains arbitrary as (19) is homogeneous), and since the frequency p in (19) must also be determined, then six conditions are needed at the mass. Continuity conditions at sides $I\ (\theta = \pi)$, and $II\ (\theta = -\pi)$ (see Figs. 1 and 5) of the mass require that the radial and tangential deflections, u and V , and the slope of the radial deflection, $\frac{du}{r d\theta}$, be equal. Equilibrium conditions at the mass require that the sum of the forces in the radial and tangential directions and the sum of the moments be zero.

From the continuity conditions the following equations can be written

$$(u)_I = (u)_{II}, \tag{24}$$

$$(v)_I = (v)_{II}, \tag{25}$$

and

$$\left(\frac{du}{d\theta}\right)_I = \left(\frac{du}{d\theta}\right)_{II} \tag{26}$$

where from (1 2) it follows that

$$U(\theta) = \sum_{K=1}^{K=3} \left(-A_K n_K \sin n_K\theta + B_K n_K \cos n_K \theta\right). \tag{27}$$

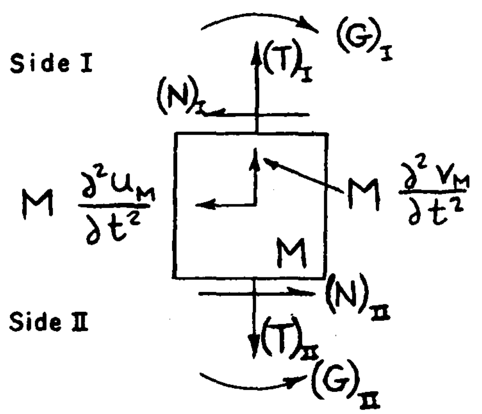

Fig. 5 Dynamical Equilibrium of the Point Mass

For steady-state harmonic motion the inertia terms may be expressed as

$$M\frac{\partial^2 v_M}{\partial t^2} = -Mp^2 v_M \tag{28}$$

and $$M \frac{\partial^2 u_M}{\partial t^2} = - M p^2 u_M \tag{29}$$

where the displacements of the mass, u_M and v_M, are

$$v_M = (v)_I = (v)_{II} \tag{30}$$

and $$u_M = (u)_I = (u)_{II} . \tag{31}$$

Summing forces in the radial direction gives

$$M p^2 u_M - (N)_I + (N)_{II} = 0, \tag{32}$$

in the tangential direction gives

$$M p^2 v_M - (T)_I + (T)_{II} = 0, \tag{33}$$

and summing the moments (disregarding rotatory inertia since the mass is concentrated at a point) gives

$$(G)_I - (G)_{II} = 0. \tag{34}$$

Evaluating (24), (25), (26), (32), (33) and (34) with expressions (1), (2), (3), (11), (12), (28), (29), (30) and (31) with $(\theta)_I = \pi$ and $(\theta)_{II} = -\pi$ gives six linear homogeneous algebraic equations arranged for convenience as follows:

From (24)

$$\sum_{K=1}^{K=3} A_K n_K \sin \alpha_K = 0 \tag{35}$$

From (34)

$$\sum_{K=1}^{K=3} A_K n_K^3 \sin \alpha_K = 0 \tag{36}$$

From (33)

$$\sum_{K=1}^{K=3} A_K \left(\cos\alpha_K + C n_K^5 \sin\alpha_K \right) = 0 \tag{37}$$

From (25)

$$\sum_{K=1}^{K=3} B_K \sin\alpha_K = 0 \tag{38}$$

From (26)

$$\sum_{K=1}^{K=3} B_K n_K^2 \sin\alpha_K = 0 \tag{39}$$

From (32)

$$\sum_{K=1}^{K=3} B_K n_K \left(\cos\alpha_K + C n_K^3 \sin\alpha_K \right) = 0 \tag{40}$$

where $\alpha_K = n_K \pi \quad (K = 1, 2 \text{ or } 3)$

and

$$C = \frac{2EI}{M r^3 p^2}. \tag{41}$$

For nontrivial values of the constants, A_K and B_K, the determinant of their coefficients must vanish and the frequency equation may be established in this manner. Since (35), (36) and (37) contain only A_K terms and (38), (39) and (40) only B_K terms, the general sixth order determinant of all the coefficients may be expressed as a product of two third-order determinants, one independent of the A_K coefficients and the other independent of the B_K coefficients.

The determinant from (35), (36) and (37) is

$$D_A = \begin{vmatrix} n_1 \sin\alpha_1 & n_2 \sin\alpha_2 & n_3 \sin\alpha_3 \\ n_1^3 \sin\alpha_1 & n_2^3 \sin\alpha_2 & n_3^3 \sin\alpha_3 \\ \cos\alpha_1 + C n_1^5 \sin\alpha_1 & \cos\alpha_2 + C n_2^5 \sin\alpha_2 & \cos\alpha_3 + C n_3^5 \sin\alpha_3 \end{vmatrix} \tag{42}$$

and the determinant from (38), (39) and (40) is

$$D_B = \begin{vmatrix} \sin\alpha_1 & \sin\alpha_2 & \sin\alpha_3 \\ n_1^2 \sin\alpha_1 & n_2^2 \sin\alpha_2 & n_3^2 \sin\alpha_3 \\ n_1\cos\alpha_1 + Cn_1^4\sin\alpha_1 & n_2\cos\alpha_2 + Cn_2^4\sin\alpha_2 & n_3\cos\alpha_3 + Cn_3^4\sin\alpha_3 \end{vmatrix} \qquad (43)$$

and the general determinant D can be written

$$D = D_A \times D_B . \qquad (44)$$

Now, nontrivial values of the constants are possible if either D_A or D_B is zero. If D_A is zero, then, in general, D_B will not be; therefore, at least one of the A_K constants will be nonzero and all of the B_K constants will be zero. This situation is reversed if D_B is zero. By setting $D_A = 0$ and evaluating (42) the frequency equation associated with the constants A_K is obtained as follows:

$$\begin{aligned} D_A = {} & n_2 n_3 (n_2^2 - n_3^2)\cos\alpha_1 \sin\alpha_2 \sin\alpha_3 \\ & + n_1 n_3 (n_3^2 - n_1^2)\sin\alpha_1 \cos\alpha_2 \sin\alpha_3 \\ & + n_1 n_2 (n_1^2 - n_2^2)\sin\alpha_1 \sin\alpha_2 \cos\alpha_3 \\ & - C n_1 n_2 n_3 (n_1^2 - n_2^2)(n_2^2 - n_3^2)(n_3^2 - n_1^2)\sin\alpha_1 \sin\alpha_2 \sin\alpha_3 . \end{aligned} \qquad (45)$$

Solving (45) for C (let $C = C_A$ for $D_A = 0$) the frequency equation is

$$\begin{aligned} C_A = {} & \frac{\cos\alpha_1}{n_1(n_1^2 - n_2^2)(n_3^2 - n_1^2)\sin\alpha_1} \\ & + \frac{\cos\alpha_2}{n_2(n_1^2 - n_2^2)(n_2^2 - n_3^2)\sin\alpha_2} \\ & + \frac{\cos\alpha_3}{n_3(n_2^2 - n_3^2)(n_3^2 - n_1^2)\sin\alpha_3} . \end{aligned} \qquad (46)$$

By the same procedure we find from (43)

$$
\begin{aligned}
D_B = \; & n_1(n_2^2 - n_3^2)\cos\alpha_1 \sin\alpha_2 \sin\alpha_3 \\
& + n_2(n_3^2 - n_1^2)\sin\alpha_1 \cos\alpha_2 \sin\alpha_3 \\
& + n_3(n_1^2 - n_2^2)\sin\alpha_1 \sin\alpha_2 \cos\alpha_3 \\
& - C(n_1^2 - n_2^2)(n_2^2 - n_3^2)(n_3^2 - n_1^2)\sin\alpha_1 \sin\alpha_2 \sin\alpha_3 \qquad (47)
\end{aligned}
$$

from which the frequency equation is (let $C = C_B$ for $D_B = 0$)

$$
\begin{aligned}
C_B = \; & \frac{n_1 \cos\alpha_1}{(n_1^2 - n_2^2)(n_3^2 - n_1^2)\sin\alpha_1} \\
& + \frac{n_2 \cos\alpha_2}{(n_1^2 - n_2^2)(n_2^2 - n_3^2)\sin\alpha_2} \\
& + \frac{n_3 \cos\alpha_3}{(n_2^2 - n_3^2)(n_3^2 - n_1^2)\sin\alpha_3}. \qquad (48)
\end{aligned}
$$

Through a numerical trial and error process (23), (46) and (48) could be used to determine the two frequencies of a selected mode for a particular ring and mass. The mode shapes could then be found by solving for the constants A_K and B_K in (35) through (40). But to aid in interpretation of the results and for identification of modes, the solution for a ring without a point mass will be determined first.

E. RING WITHOUT A POINT MASS

Setting (45) and (47) equal to zero, dividing each by C (from (41) $C = \frac{EI}{Mr^3p^2}$) and then setting $M = 0$, we obtain the expressions; from (45)

$$
n_1 n_2 n_3 (n_1^2 - n_2^2)(n_2^2 - n_3^2)(n_3^2 - n_1^2)\sin\alpha_1 \sin\alpha_2 \sin\alpha_3 = 0 \qquad (49)
$$

and from (47)

$$(n_1^2-n_2^2)(n_2^2-n_3^2)(n_3^2-n_1^2)\sin\alpha_1\sin\alpha_2\sin\alpha_3 = 0. \tag{50}$$

Both (49) and (50) are identically zero if one of the n_K's is an integer $(\alpha_K = n_K \pi)$ greater than unity (see (23)). Solving the six simultaneous equations (35) through (40) it is found that two of the constants, an A_K and a B_K, are arbitrary and the others are zero. From (27) the expression for the mode shape is then

$$U(\theta) = A_n \sin n\theta + B_n \cos n\theta. \tag{51}$$

Note that (51) is composed of an antisymmetrical branch, $\sin n\theta$, and a symmetrical branch, $\cos n\theta$. For a ring without a point mass it is possible to express the solution as a superposition of the two branches in the form of (51) because the frequency in each branch is the same. From (23) the frequency equation is

$$n^2(n^2-1)^2 = (n^2+1)Q. \tag{52}$$

Since n is an integer in (51) and (52) the subscript K is usually omitted. However, as will be shown in Section IV, there are two other solutions of (52) that will give the same value of the frequency p, but they are meaningless since the four associated mode constants, A_K and B_K in (27) are zero.

It is interesting to note from (51) that the axis of symmetry of the antisymmetrical branch is rotated $\frac{\pi}{2n}$ radians or $\frac{90}{n}$ degrees from the axis of symmetry of the symmetrical branch. Also that the expression (51) relates the mode form to the integer n. For a given value of n greater than unity (see (52)) the mode shapes will be such that there are n wave-lengths and $2n$ nodes to the circumference of the ring.

From the standpoint of this study the most interesting aspect of the solution of a ring without a point mass is the fact that the node positions are completely arbitrary. Any point of the ring can be a nodal point depending only upon the way in which the vibrations were initiated.

F. RING WITH A POINT MASS

Equations (23), (48), (23) and (46) constitute the frequency expressions for the symmetrical and antisymmetrical branches, respectively. Since the frequency is different for each branch it is not possible to superimpose the mode shapes of the two branches as was done in (51) for a ring without a point mass. The normal mode frequencies and shapes must be determined separately for each branch.

The symmetrical branch mode constants B_K may be found by solving (38) and (39)

$$\frac{B_2}{B_1} = \frac{(n_3^2 - n_1^2)\sin\alpha_1}{(n_2^2 - n_3^2)\sin\alpha_2} \tag{53}$$

$$\frac{B_3}{B_1} = \frac{(n_1^2 - n_2^2)\sin\alpha_1}{(n_2^2 - n_3^2)\sin\alpha_3} \tag{54}$$

and the antisymmetrical constants A_K, may be found from (35) and (36)

$$\frac{A_2}{A_1} = \frac{n_1(n_3^2 - n_1^2)\sin\alpha_1}{n_2(n_2^2 - n_3^2)\sin\alpha_2} \tag{55}$$

$$\frac{A_3}{A_1} = \frac{n_1(n_1^2 - n_2^2)\sin\alpha_1}{n_3(n_2^2 - n_3^2)\sin\alpha_3}. \tag{56}$$

The expressions (22) and (27) for the mode shapes for symmetrical vibrations are

$$\frac{V}{B_1} = \sin n_1\theta + \frac{B_2}{B_1}\sin n_2\theta + \frac{B_3}{B_1}\sin n_3\theta \tag{57}$$

and

$$\frac{U}{B_1} = n_1\cos n_1\theta + \frac{B_2}{B_1} n_2\cos n_2\theta + \frac{B_3}{B_1} n_3\cos n_3\theta \tag{58}$$

and for antisymmetrical vibrations are

$$\frac{V}{A_1} = \cos n_1\theta + \frac{A_2}{A_1}\cos n_2\theta + \frac{A_3}{A_1}\cos n_3\theta \tag{59}$$

and

$$\frac{U}{A_1} = -n_1\sin n_1\theta - \frac{A_2}{A_1} n_2\sin n_2\theta - \frac{A_3}{A_1} n_3\sin n_3\theta. \tag{60}$$

IV. NUMERICAL SOLUTIONS

To illustrate the influence of the point mass on the free vibrations of the ring, frequencies and several mode shapes were calculated for the first and second flexural modes by the following procedure:

A. VARIATION OF FREQUENCY WITH MASS

1. Several values of n_1 (positive and real) were chosen and (23) was solved for corresponding values of $Q = \frac{m r^4 p^2}{EI}$ (see Table 1).

2. This reduced (23) to a quadratic equation in n_K^2 which was then solved for n_2 and n_3 (listed in Table 1).

TABLE 1 Mode Parameters

First Mode

$n_2 = a + bi$ $n_3 = a - bi$

n_1	Q	a	b
2.00	7.20000	0.41331	1.08205
1.99	6.99566	0.41777	1.07452
1.975	6.69678	0.42425	1.06316
1.95	6.21861	0.43450	1.04405
1.90	5.33440	0.45310	1.00514
1.80	3.83420	0.48366	0.92408
1.70	2.65382	0.50659	0.83764

Second Mode

n_1	Q	$\frac{n_2}{i}$	$\frac{n_3}{i}$
3.00	57.6000	1.03987	2.43283
2.99	56.7027	1.04060	2.42018
2.975	55.3757	1.04174	2.40113
2.95	53.2137	1.04373	2.36920
2.80	41.4931	1.05910	2.17217
2.70	34.7916	1.07418	2.03375

3. The n_k values were then inserted into the frequency expressions (46) and (48) to obtain values for C_A and C_B . These are listed in Table 2.

TABLE 2 Frequency Functions

	Antisymmetrical Branch	Symmetrical Branch
n_1	C_A	C_B
First Mode		
2.00	∞	∞
1.99	0.53341	2.46279
1.975	0.15947	0.98112
1.95	0.02890	0.48424
1.94	0	---
1.90		0.22959
1.80		0.08430
1.70		0.00779
1.68		0
Second Mode		
3.00	∞	∞
2.99	0.05744	0.62880
2.975	0.01479	0.24764
2.95	0	0.11991
2.80		0.01909
2.70		0.00341
2.675		0

From Eq. (41), $C = \dfrac{2EI}{Mr^3p^2}$, then $C = 0$ represents an infinite mass M and, as noted in Table 2, $C = \infty$ represents zero mass and gives the solution for the ring alone where n_1 is an integer. For the limiting case of an infinite mass approximate n_1 values were obtained by plotting C versus n_1 (not shown) and were found, as listed in Table 2, to be 1.94 $(C_A = 0)$ and 1.68 $(C_B = 0)$ for the first mode and 2.95 $(C_A = 0)$ and 2.675 $(C_B = 0)$ for the second mode.

4. Frequency ratios of the ring with mass M to that of the ring alone were obtained by Eq. (23):

$$p_M^2 = \frac{EI}{mr^4} Q_M$$

and

$$p_m^2 = \frac{EI}{m r^4} Q_m .$$

Then using these expressions the following is obtained

$$\frac{p_M}{p_m} = \left(\frac{Q_M}{Q_m}\right)^{\frac{1}{2}} \tag{61}$$

where the subscripts M and m pertain to the ring with mass and the ring alone, respectively.

Mass and frequency ratios were related as follows:

From Eq. (23) $$p^2 = \frac{EI}{m r^4} Q$$

From Eq. (41) $$p^2 = \frac{2EI}{M r^3 C} .$$

Then, eliminating p^2 from the above

$$\frac{M}{2rm} = \frac{1}{CQ} \tag{62}$$

and dividing both sides,(62), by π it becomes

$$\frac{M}{2\pi rm} = \frac{1}{\pi CQ} . \tag{63}$$

Graphs of frequency ratios $\left(\frac{p_M}{p_m}\right)$ versus mass ratios $\left(\frac{M}{2\pi rm}\right)$ for the first and second modes are shown in Figs. 6 and 7.

B. VARIATION OF MODE SHAPES WITH MASS

1. For the n_K values listed in Table 1 the mode constants A_K and B_K were calculated by Eqs. (53) through (56) (Eqs. (37) and (40) were used as a check) and are listed below in Table 3. A_1 and B_1 are taken as the arbitrary constants in the antisymmetrical and symmetrical branches, respectively.

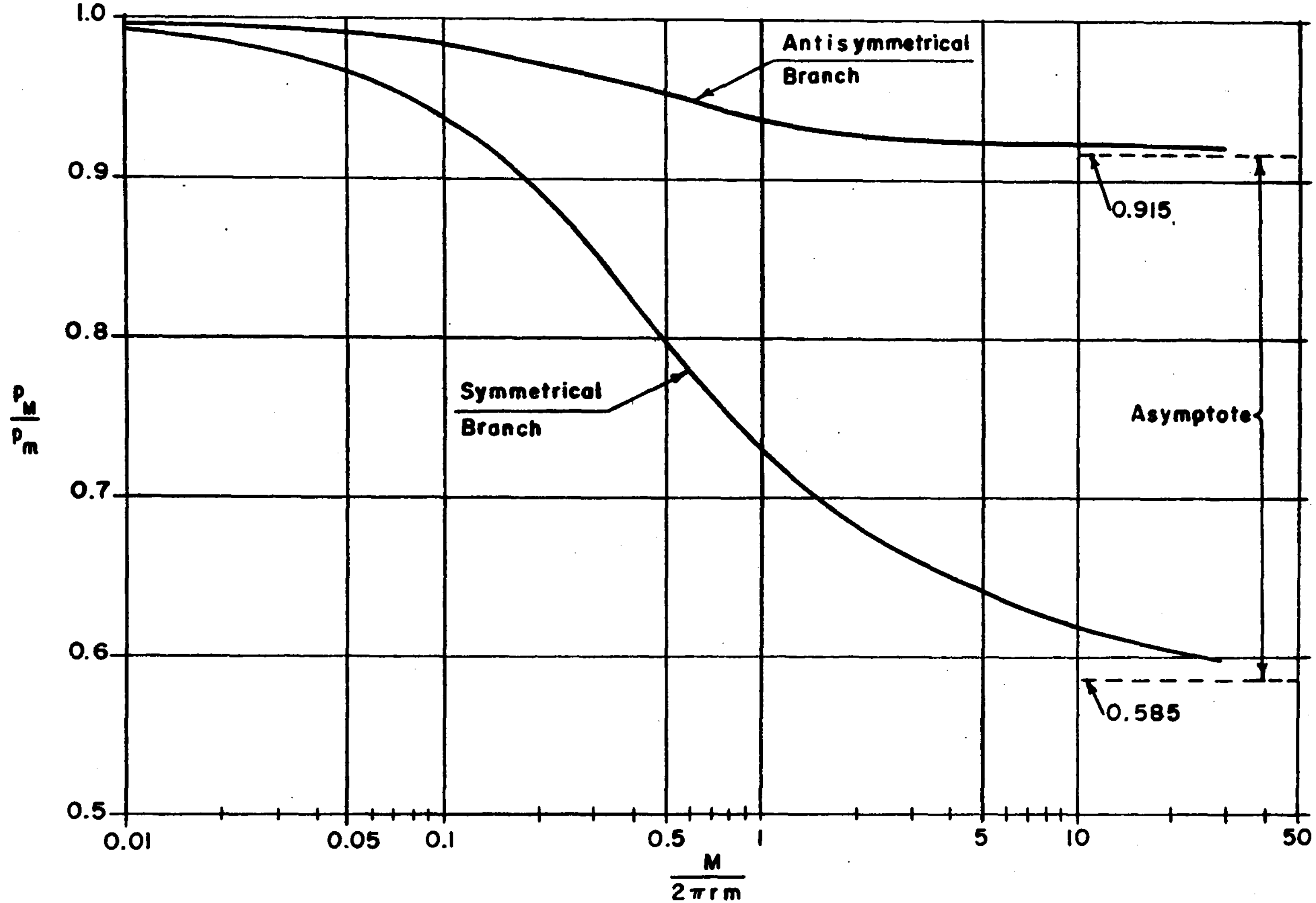

Fig. 6 Variation of Frequency with Mass, First Mode

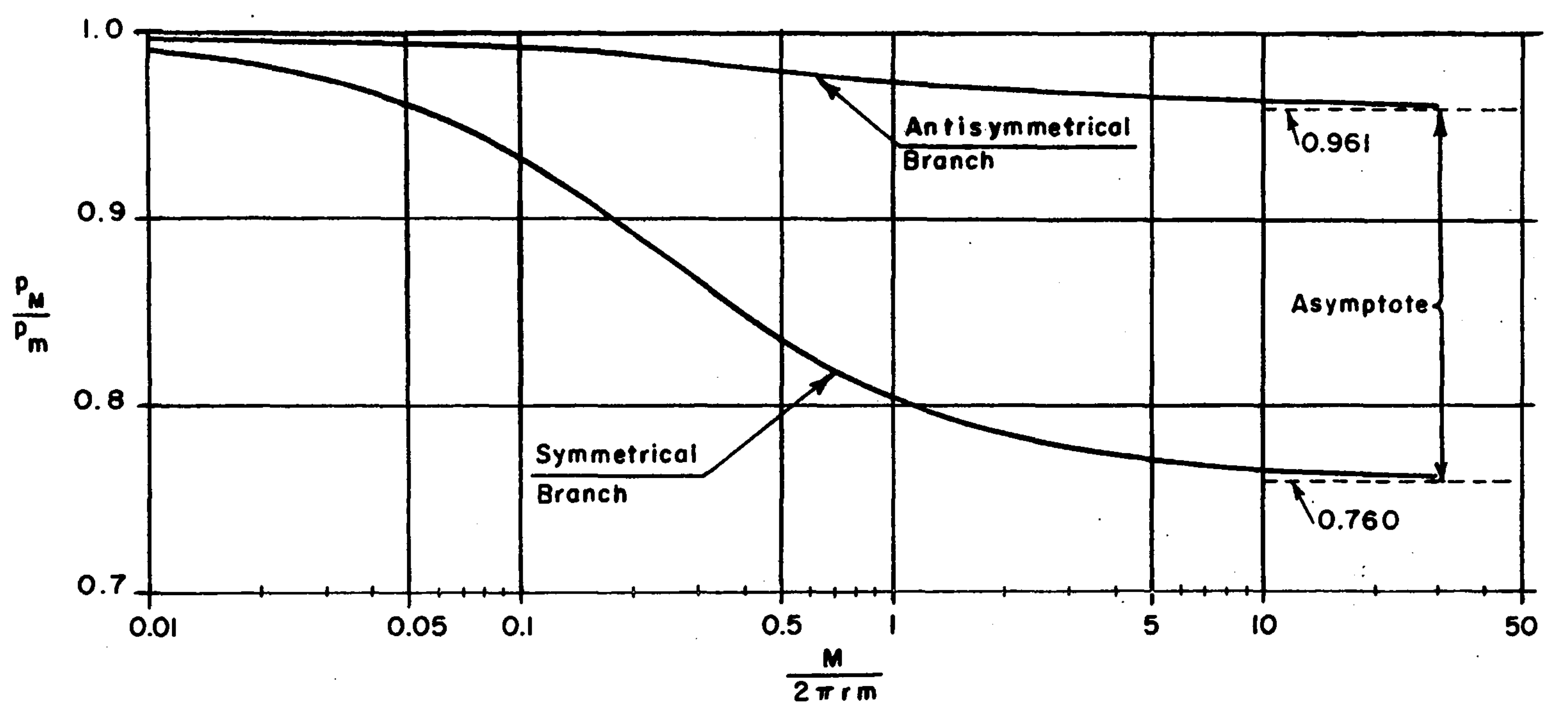

Fig. 7 Variation of Frequency with Mass, Second Mode

TABLE 3 Mode Constants A_K and B_K

Antisymmetrical Branch	Symmetrical Branch

First Mode

$$\frac{A_2}{A_1} = \phi - \lambda i \qquad \frac{B_2}{B_1} = \beta - \gamma i$$

$$\frac{A_3}{A_1} = \phi + \lambda i \qquad \frac{B_3}{B_1} = \beta + \gamma i$$

n_1	ϕ	λ	β	γ
2.00	0	0	0	0
1.99	-0.00992	0.00299	-0.000468	0.00598
1.975	-0.02483	0.00840	-0.000813	0.01517
1.95	-0.04971	0.01984	-0.000456	0.03104
1.94	-0.06050	0.02503		
1.90			0.00368	0.06482
1.80			0.02537	0.14021
1.70			0.06257	0.22638
1.68			0.07150	0.24500

Second Mode

n_1	$\frac{A_2}{A_1}$	$\frac{A_3}{A_1}$	$\frac{B_2}{iB_1}$	$\frac{B_3}{iB_1}$
3.00	0	0	0	0
2.99	0.02132	-0.0000813	0.00742	-0.0000658
2.975	0.05313	-0.000219	0.01860	-0.000176
2.95	0.10543	-0.000494	0.03730	-0.000396
2.80			0.14751	-0.00318
2.70			0.21245	-0.00769
2.675			0.22700	-0.00946

2. Eqs. (57) through (60) were used to determine the mode shapes for four values of the mass ratio: $\frac{M}{2\pi r m} = 0, 0.26, 1.0$ and ∞. In determining the mode constants from one mode to another as well as between branches of a given mode for a selected value of the mass ratio, data of Table 3 were plotted versus the mass ratio. One such graph is shown in Fig. 8; the others are similar. The relationship between the mass ratio and Q was obtained from Figs. 6 and 7 and, although not presented herein, adequate values of the mode parameters n_K were then found by plotting the data of Table 1.

The displacement functions U and V of both branches of the first two modes are shown in Figs. 9 through 16 for θ between 0° and 180°. Perhaps a better illustration of the influence of the mass on mode shape is given in Figs. 17 through 20 where complete shapes are plotted for mass ratios of 0 and 1.0. Graphs of the displacement functions in Figs. 9 through 16 and mode shapes in Figs. 17 through 20 were not normalized.

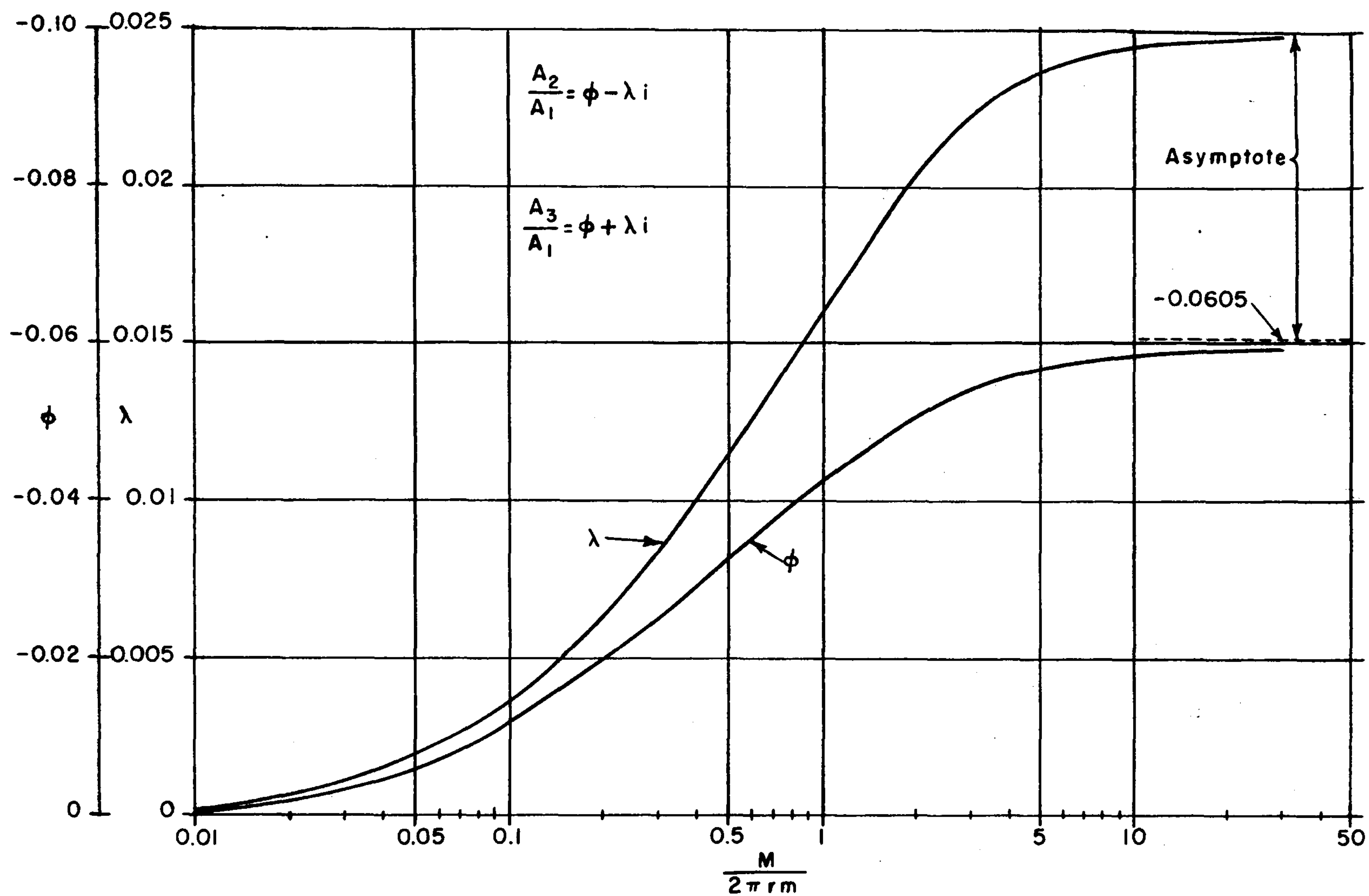

Fig. 8 Variation of Mode Constants with Mass, Antisymmetrical Branch of First Mode

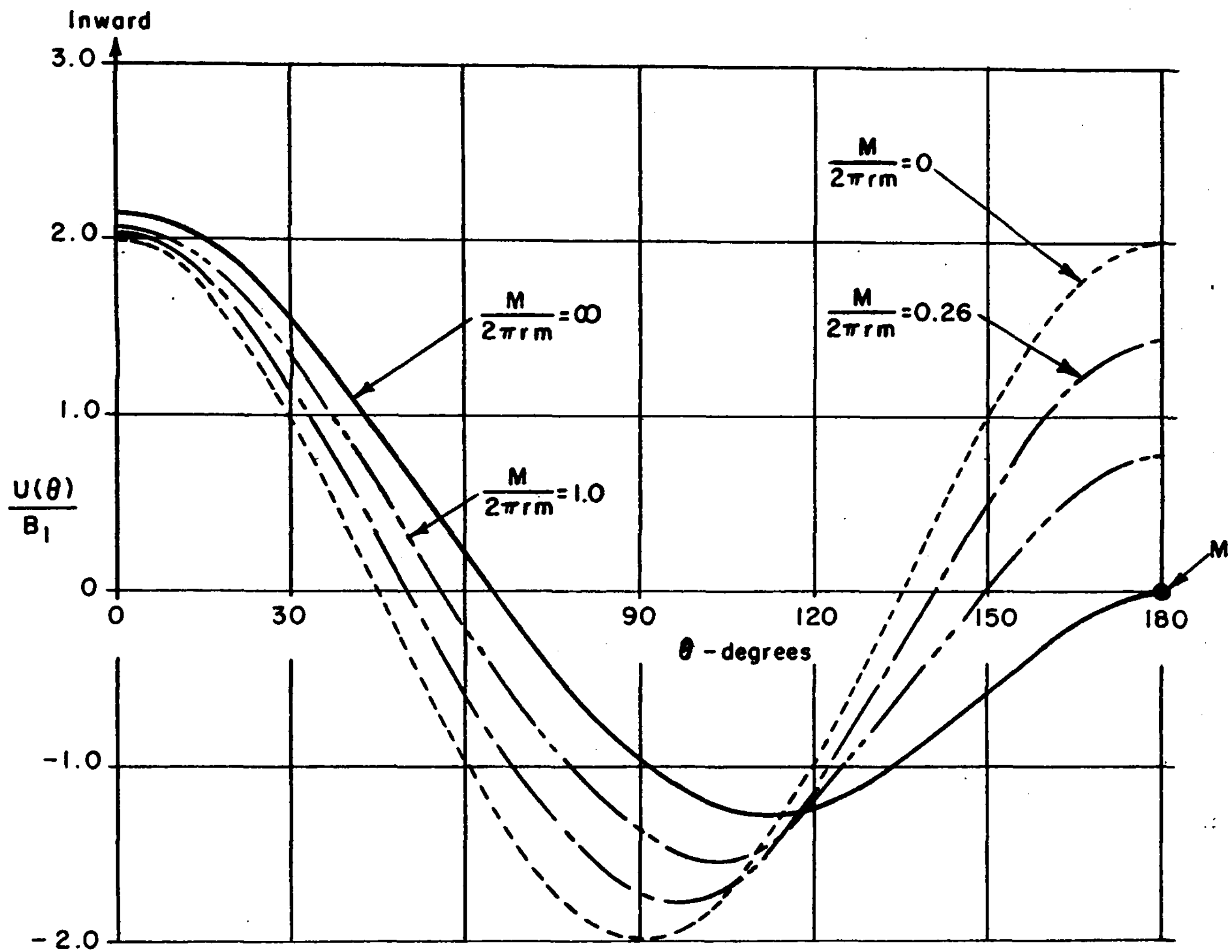

Fig. 9 Radial Displacements, Symmetrical Branch of First Mode

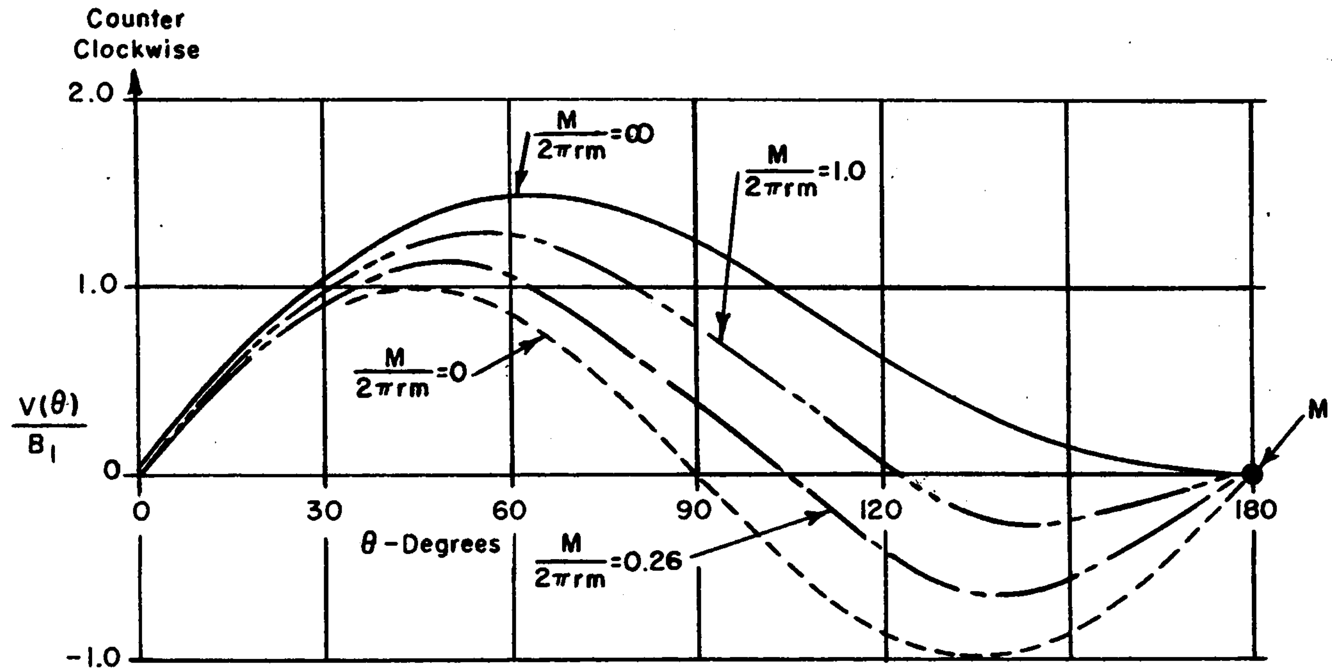

Fig. 10 Tangential Displacements, Symmetrical Branch of First Mode

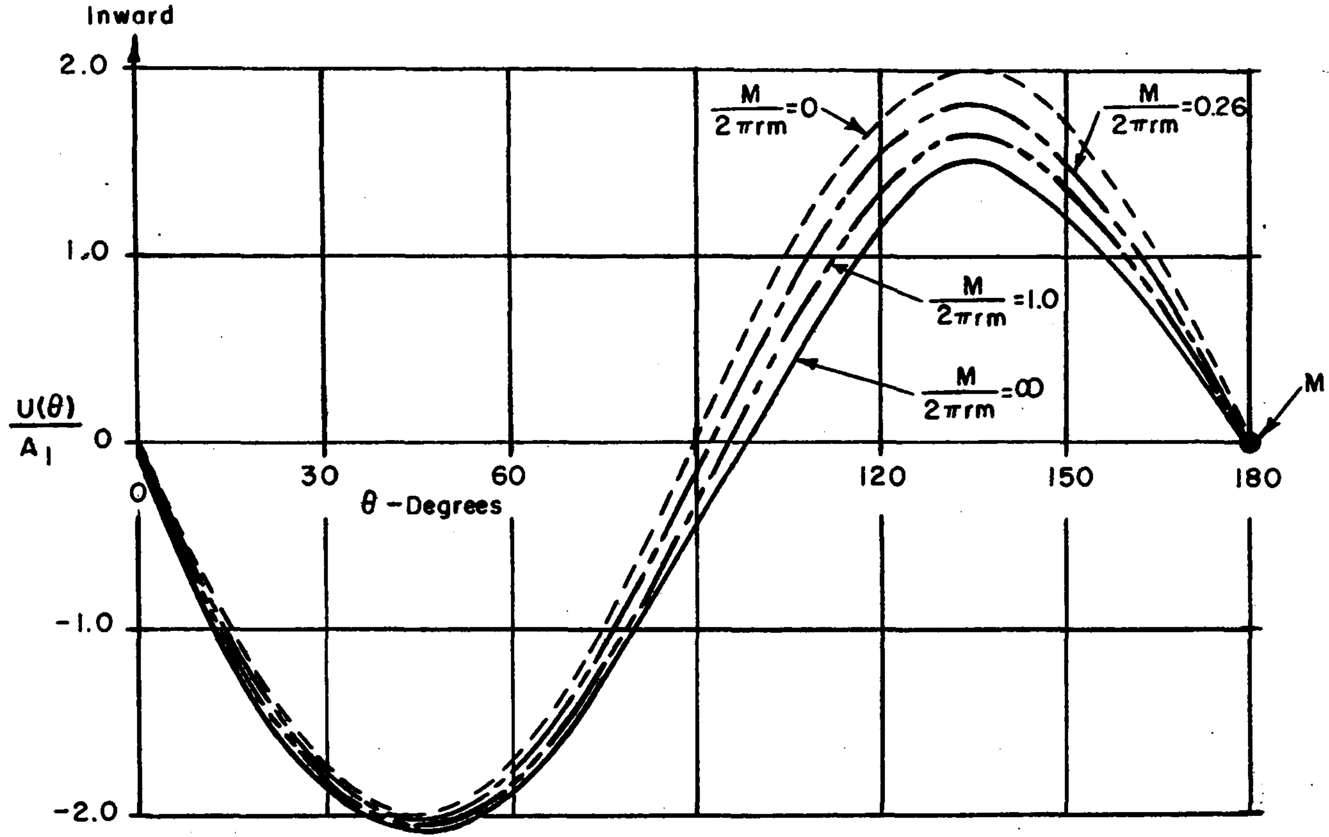

Fig. 11 Radial Displacements, Antisymmetrical Branch of First Mode

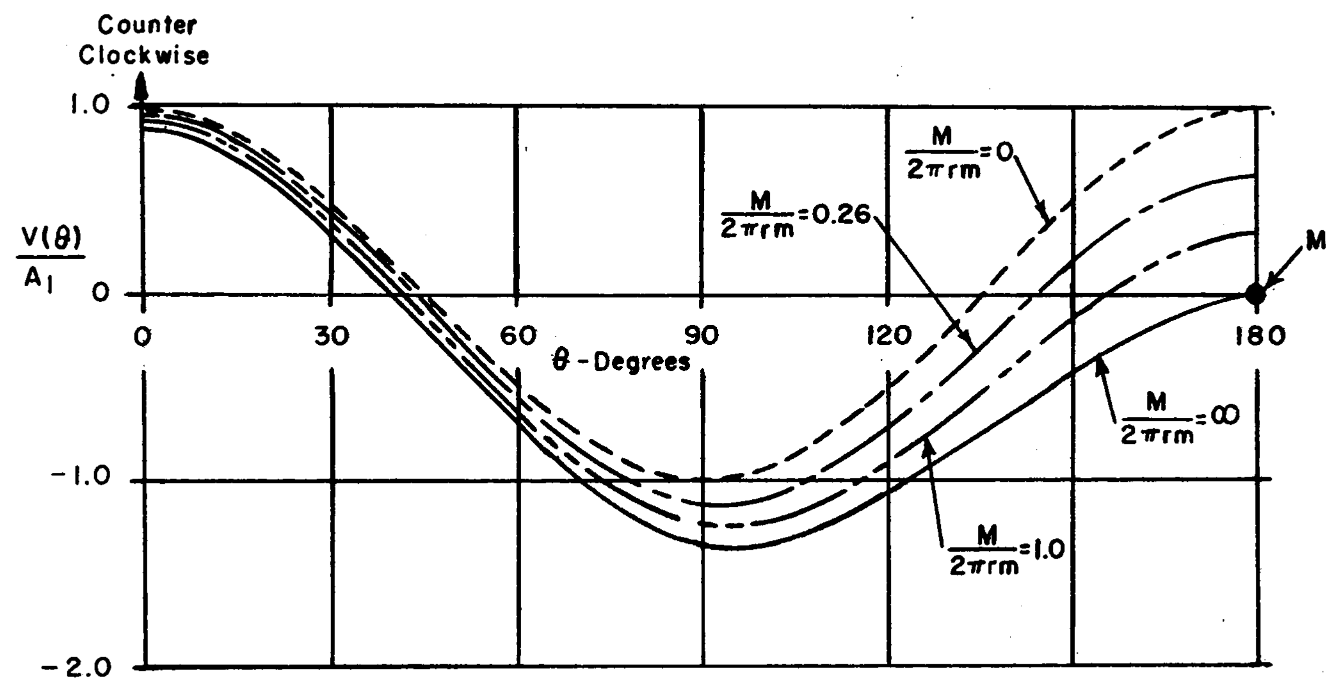

Fig. 12 Tangential Displacements, Antisymmetrical Branch of First Mode

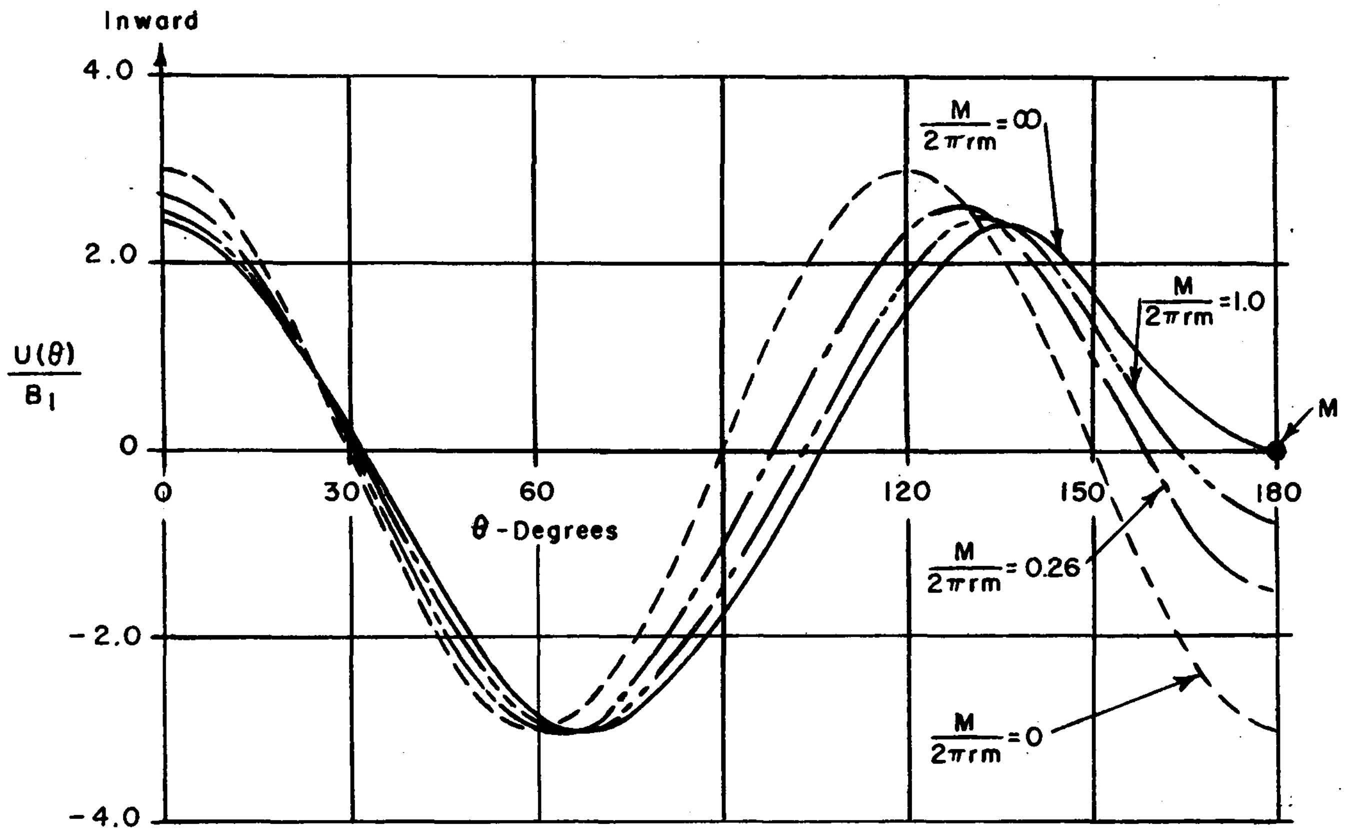

Fig. 13 Radial Displacements, Symmetrical Branch of Second Mode

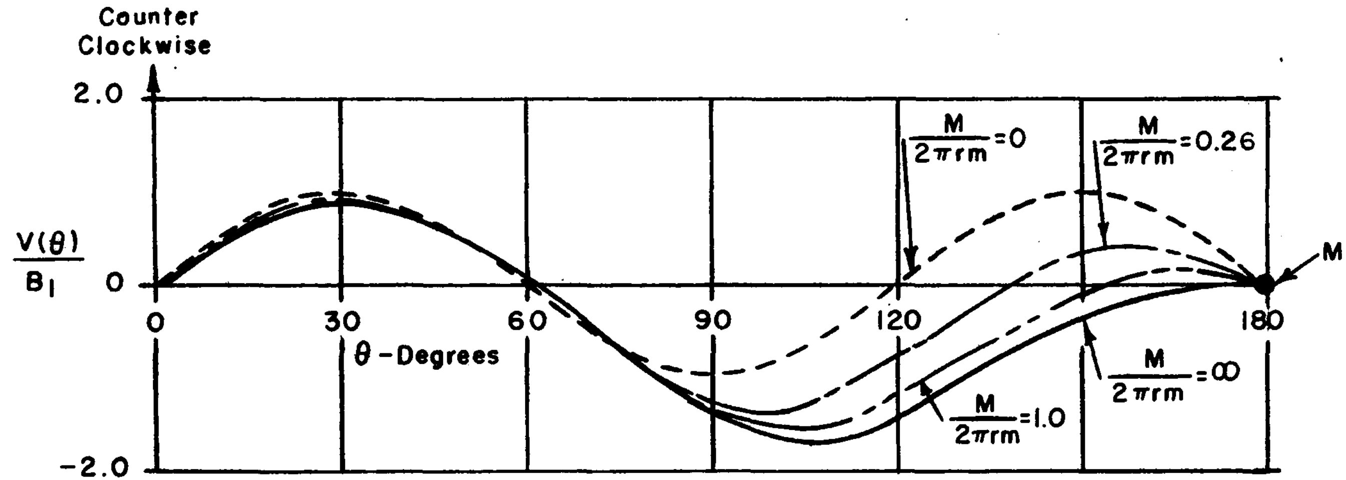

Fig. 14 Tangential Displacements, Symmetrical Branch of Second Mode

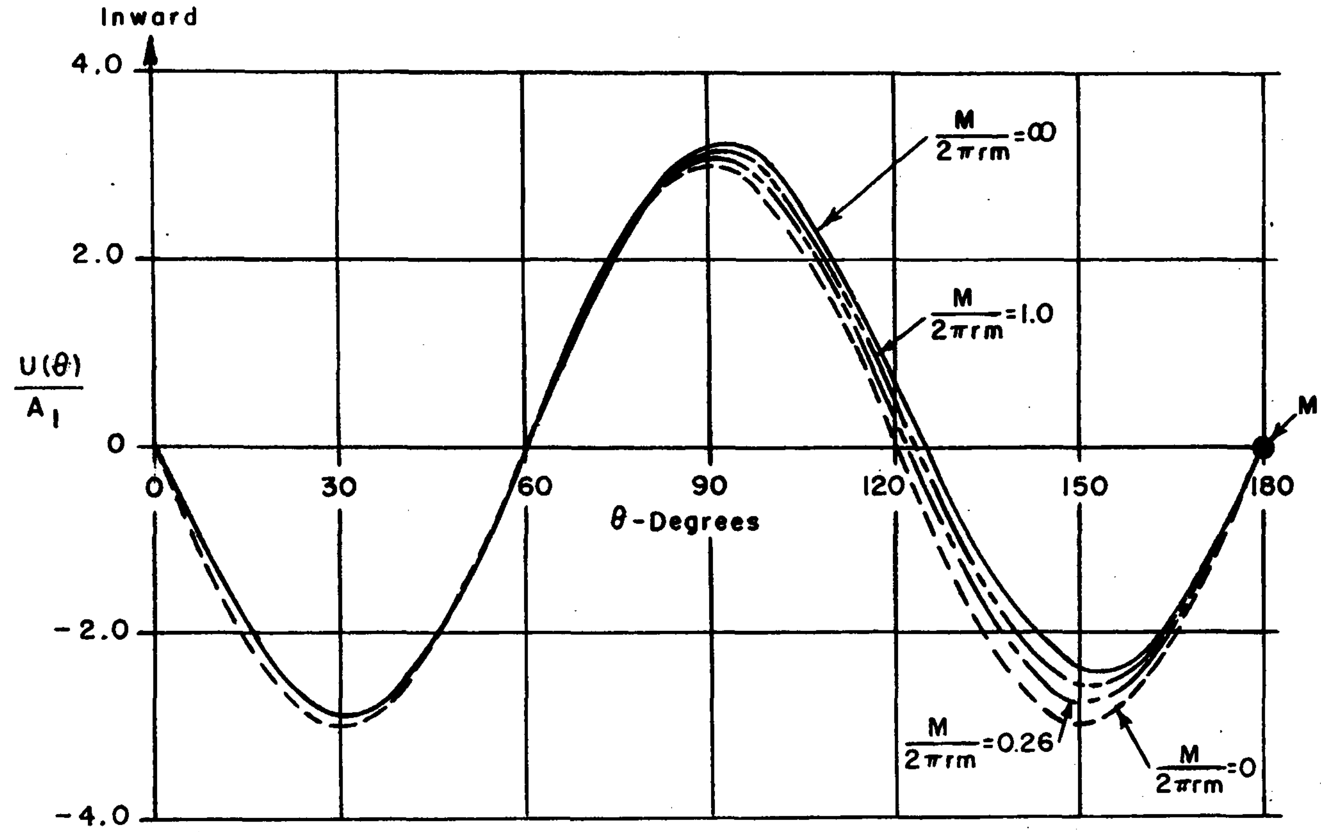

Fig. 15 Radial Displacements, Antisymmetrical Branch of Second Mode

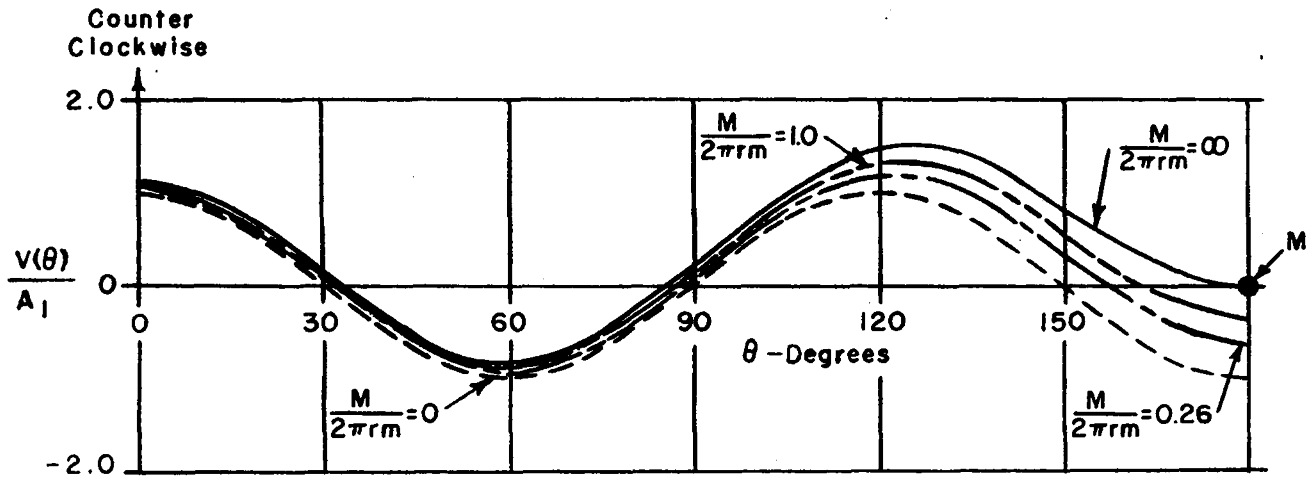

Fig. 16 Tangential Displacements, Antisymmetrical Branch of Second Mode

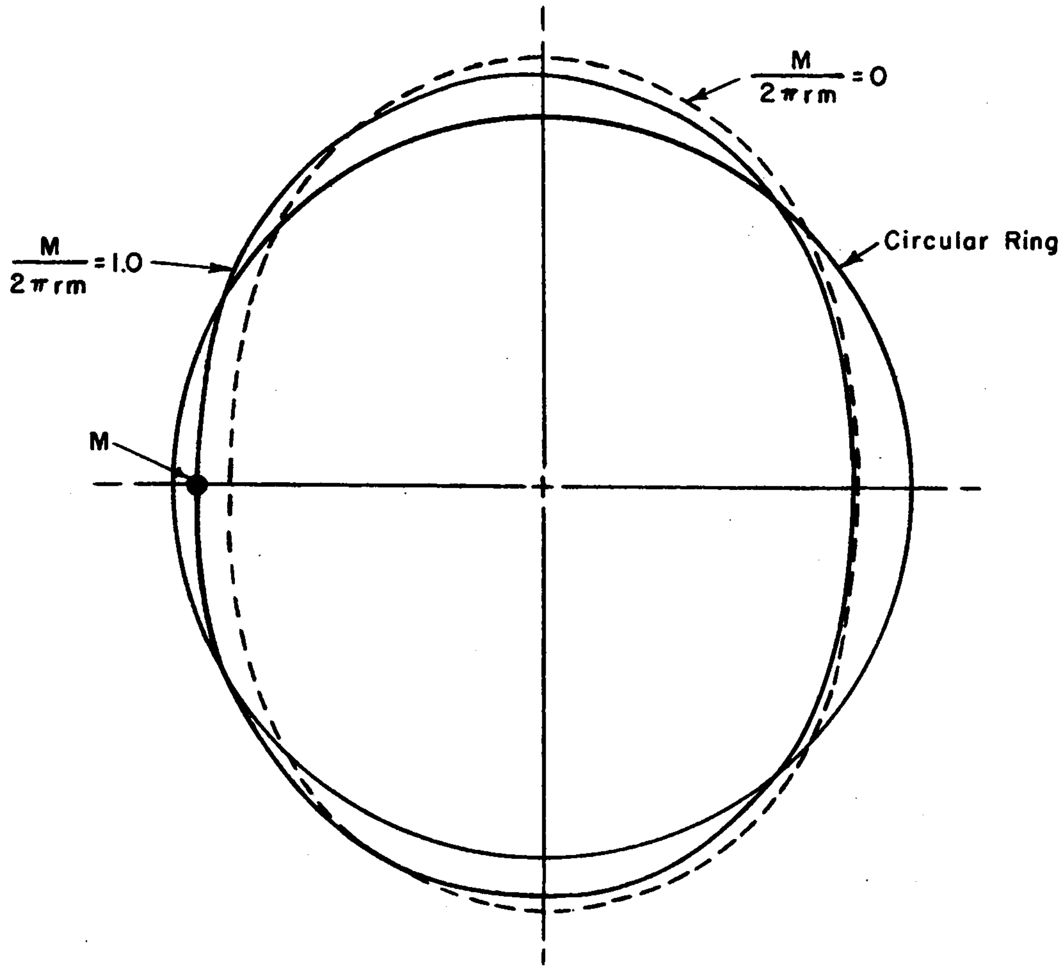

Fig. 17 Mode Shape, Symmetrical Branch of First Mode

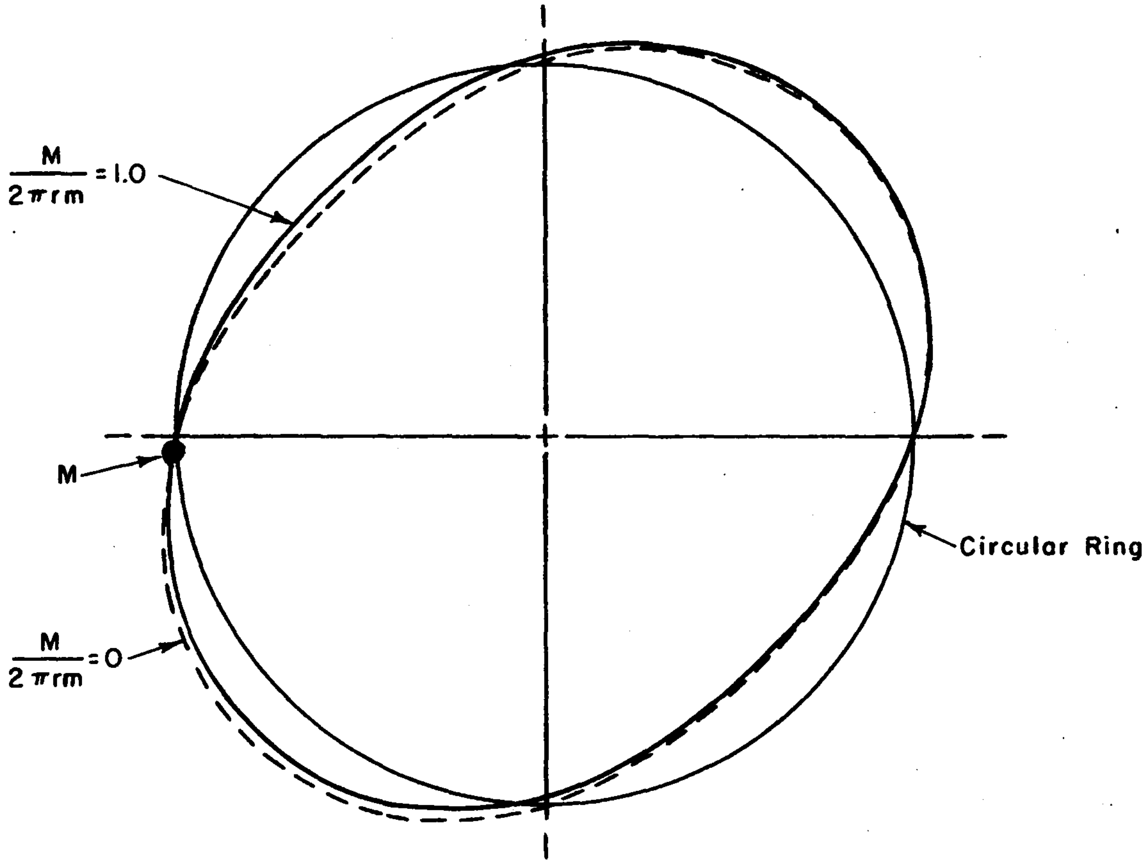

Fig. 18 Mode Shape, Antisymmetrical Branch of First Mode

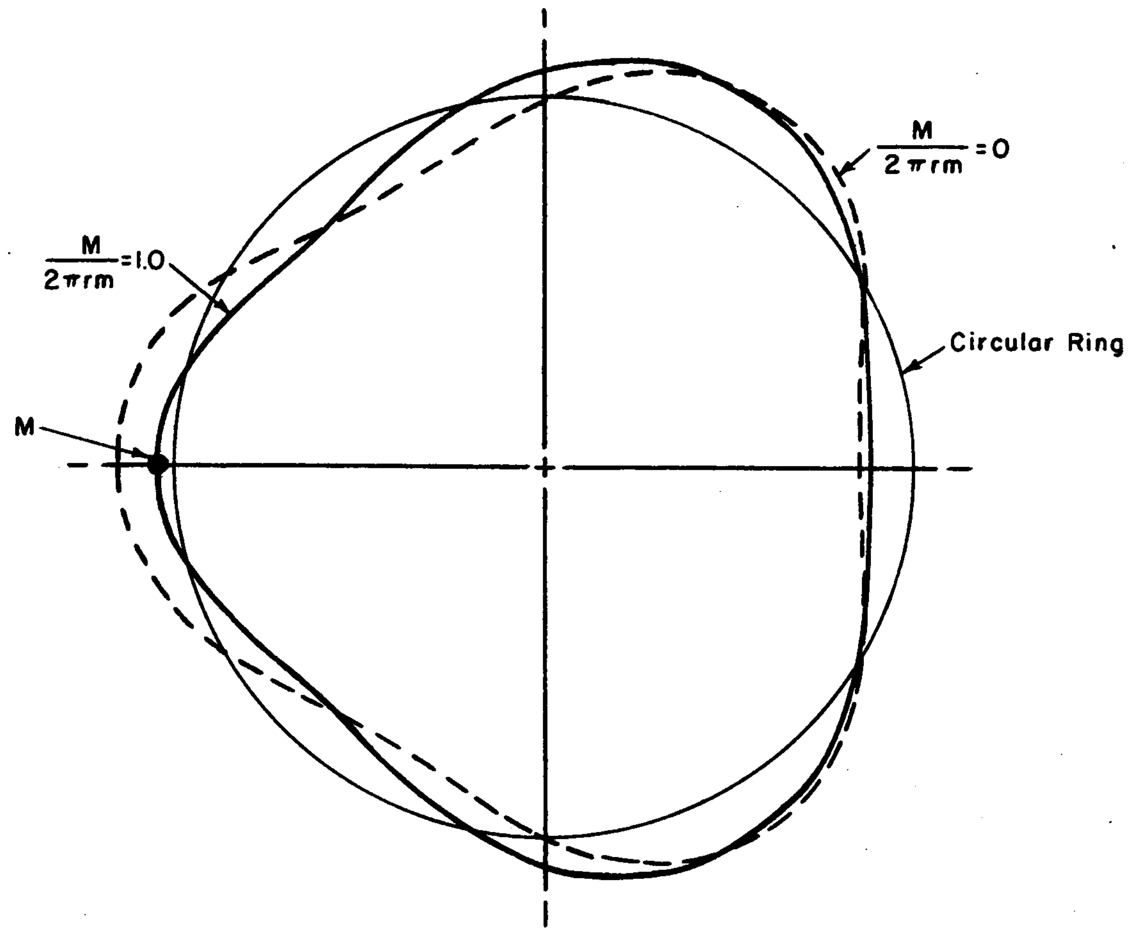

Fig. 19 Mode Shape, Symmetrical Branch of Second Mode

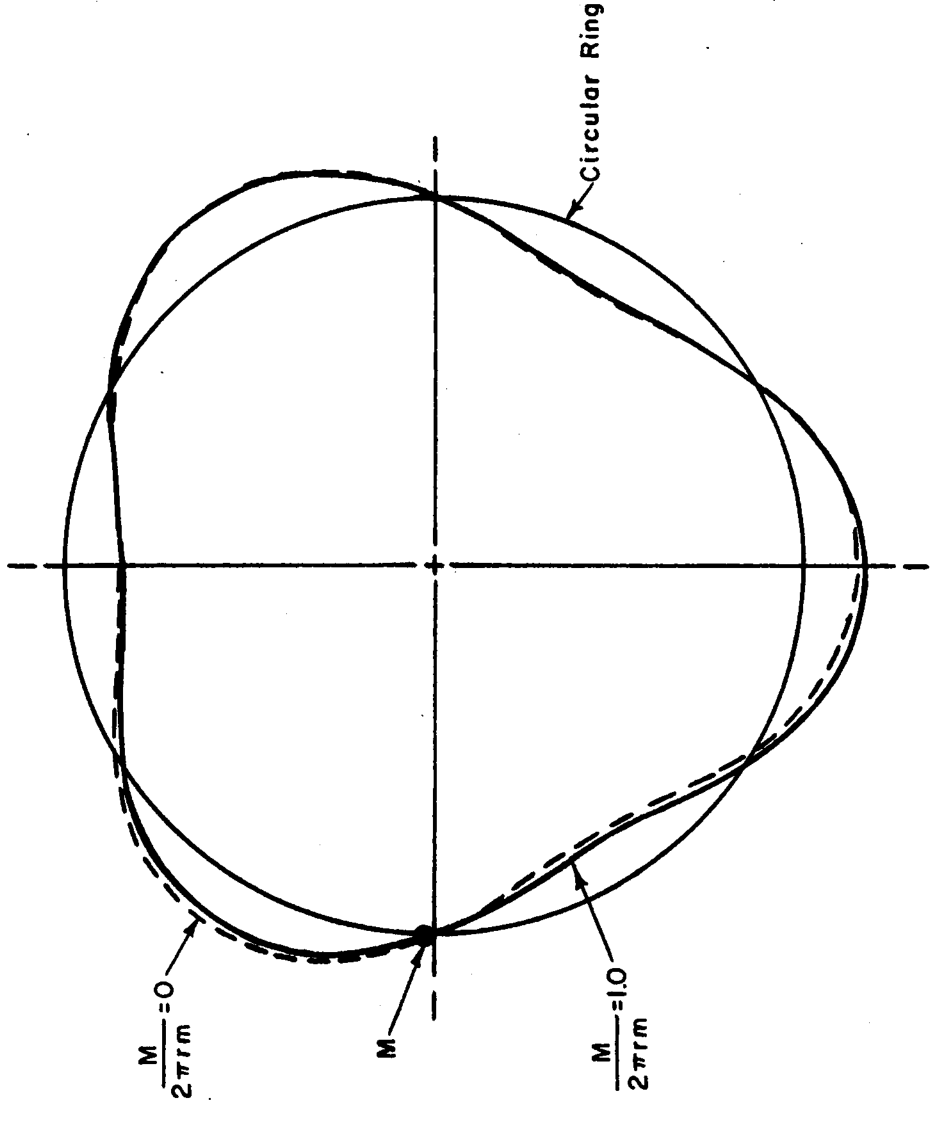

Fig. 20 Mode Shape, Antisymmetrical Branch of Second Mode

V. DISCUSSION

A. INFLUENCE OF MASS ON FREQUENCY

The most noticeable feature here is the difference in frequency between the symmetrical and antisymmetrical branches of a normal mode, as is shown in Figs. 6 and 7. For a ring with no mass both branches are of the same frequency, but as the point mass increases the difference in frequency becomes greater and is greatest for the extreme case of a ring with an infinite point mass.

In a given mode, as the mass ratio increases, the reduction in frequency is greater in the symmetrical branch than in the antisymmetrical branch, or for a given value of the mass ratio the higher frequency occurs in the antisymmetrical branch while in the symmetrical branch it is lower.

Comparing Figs. 6 and 7 it may be noted that for the same mass ratio the reduction in frequency is less for both the symmetrical and antisymmetrical branches in the second mode than in the first. In other words, the mass has a lesser effect in the second mode than in the first.

Another interesting feature of Figs. 6 and 7 is the range of values of the mass ratio: for $\frac{M}{2\pi rm} < 0.05$ the frequencies are about the same as for a ring without a point mass, and for $\frac{M}{2\pi rm} > 5$ the frequencies are essentially those for a ring with an infinite point mass.

B. INFLUENCE OF MASS ON MODE SHAPES

Although the curves are not normalized the influence of the mass on the mode shapes is more or less self-evident from Figs. 9 through 20. The most prominent feature is the reduction in movement of the mass as the mass ratio increases. In the case of an infinite mass there is no motion of the mass and it becomes a stationary point. It should be noted also that the mass has appreciably less influence on the radial displacements of the antisymmetrical shapes than on the symmetrical as illustrated in Figs. 18 and 20. The major influence of the mass for antisymmetrical vibrations occurs in the tangential displacements, since for these shapes the mass is located at a node.

VI. SUMMARY AND CONCLUSIONS

Perhaps the most convenient way to summarize the results of this investigation is to start with a ring without a point mass and proceed from there:

A. RING WITHOUT A POINT MASS

1. The solution was developed from the general equations for a ring with a point mass by taking the mass to be zero.

2. The solution was found to be a degenerate case of the general solution of a ring with a point mass since it could be expressed as a superposition of symmetrical and antisymmetrical vibrations, each having the same frequency.

3. The node positions were found to be completely arbitrary and to depend only on the manner in which the vibrations were started: such would be the case for any body of revolution with perfect physical and geometrical circumferential symmetry.

B. RING WITH A POINT MASS

1. With the addition of a small point mass to the ring, the frequency of each branch was reduced to a different degree. Symmetrical branch frequencies were affected to a greater extent than those of the antisymmetrical branch.

2. Since the frequency of each branch was different, the two branches could not be superimposed to determine the displacements for the mode shapes. The normal mode shapes for each branch were determined separately and were found to be different. Symmetrical branch shapes were influenced to a greater degree than antisymmetrical.

3. Node positions were not arbitrary but were found to have definite orientations with respect to the point mass. In symmetrical vibrations

the mass was located at an antinode whereas in antisymmetrical vibrations it was at a node which somewhat briefly explains the greater influence of the point mass in the symmetrical branch.

4. For a very large point mass, $\frac{M}{2\pi r m} > 5$, both the radial (symmetrical branch) and the tangential (antisymmetrical branch) displacements of the mass approached zero and the mass became nearly a stationary point. In this case the frequency and mode shape for both branches of a particular mode were very close to the limiting values associated with an infinite mass.

5. It was found that a given mass exerted less influence on the frequency in the second mode than in the first.

Briefly the results of this investigation may be summarized as follows:

The symmetrical and antisymmetrical branches of vibration for a perfect ring without a point mass have the same frequency and mode shape and arbitrary node locations whereas for a ring with a point mass they have different frequencies and mode shapes and a definite orientation of the nodes with respect to the mass.

Similar phenomena have been observed experimentally [1, 2, 3, 4] in vibrations of imperfect bodies of revolution. It has been noted that the presence of small asymmetry in such bodies tended to resolve a single natural frequency into two nearby values; the larger the asymmetry or irregularity, the farther apart and more distinct the values became. Forced vibration studies have revealed preferential planes; that is, if the body was excited in one of these planes only one frequency appeared, but if excited at any other position around the body, twopeaks occurred. In general, the results of this investigation offer a preliminary theoretical explanation for such phenomena.

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