# Best Response Dynamics and Neural Networks

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December 14, 1998

**Abstract**

We consider a population of players in a setting that allows to analyze local as well as global interaction. Using the formalism of automata networks we show that best response is a special case of a biased majority (minority) imitation. For a population of best response players we first discuss the known properties of the deterministic dynamics as a preparation and reference for stochastic dynamics. The stochastic dynamics of the system will always have a stationary distribution. It turns out that in a special case of asynchronous updating and logistic noise this distribution is of Boltzmann type. We further show that with a Boltzmann distribution, the long-run equilibria are associated with a minimum of a cost function defined in the paper. Comparison of our results with the existing literature suggests robustness of the previous long-run equilibrium results. In the case of differentiation games, we demonstrate the sensivity of long-run equilibrium to the choice of interaction structure.

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# 1 Introduction

The appeal of evolutionary game theory to social scientists in general and economists in particular rests on the fact that it allows to investigate the dynamics and long-run properties of a population of interacting boundedly rational players. Methods borrowed from biology have helped to address questions of equilibrium selection and stability as documented in several monographs: van Damme (1991), Weibull (1995), Samuelson (1997).

Closer to the rationality paradigm of mainstream economics are best response dynamics where at each time, every player plays a (static) best response against the distribution of the last strategies played by his neighbors. This constitutes rational behavior impaired by myopia. Myopia in the temporal sense means that the player is not forward looking, does not take into account that other players might be changing their strategies. This trait is shared, for example, by naive Bayesian learners [Eichberger *et al.* (1993)]. Myopia in the spatial sense, if applicable, means that the player is only influenced by his local environment.

In the young tradition of Kandori, Mailath, and Rob (1993), abbreviated KMR in the sequel, Ellison (1993) and Rhode and Stegeman (1996), we study best response dynamics of $2 \times 2$ population games. The work of Berninghaus and Schwalbe recognizes, explicitly or implicitly, that the dynamics of population games — where each player interacts (locally or globally) with finitely many others — can be conveniently modelled as automata networks. These two authors deal with deterministic dynamics. In Berninghaus and Schwalbe (1996a), they focus on convergence issues, using methods from the theory of iterated discrete functions. In Berninghaus and Schwalbe (1996b), they demonstrate that the theory of neural networks can be successfully applied to analyze deterministic best response dynamics with global or local interaction.

Here we go beyond Berninghaus and Schwalbe (1996b) and introduce noise into the system. This has been done before, notably by KMR and Ellison (1993) for best response and other dynamics.[1] We use a so-called "Boltzmann machine", a standard type of stochastic neural network, to model noise in a new way. In our model, a state of the system describes a spatial pattern of play, not merely a summary statistics as in KMR and Ellison. The probability of a "flip", i.e. non-best response play depends, in a specific way, on the state and the player, unlike in KMR and Ellison where the probability is independent of state and player.

Running the "Boltzmann machine", we obtain an explicit formula for the stochastic steady state (invariant distribution) in terms of the parameters of the model,

[1] See also the comment on KMR by Rhode and Stegeman (1996). See further Foster and Young (1990), Young and Foster (1991), and Young (1993).

regardless of the (regular) interaction structure. We are able to relate the long-run equilibria à la KMR to the deterministic steady states. We can, in principle, compute all long-run equilibria for any symmetric $2 \times 2$ game and any regular interaction structure, by solving a discrete optimization problem. By and large, the properties of long-run equilibria reported by KMR and Ellison are confirmed which shows a certain robustness of the model. However, our findings are more concise. We not only know how often a strategy occurs in a steady state, or long-run equilibrium, but also the spatial pattern of play. We further find as a novel result for differentiation games that the long-run equilibria differ significantly across interaction structures.

Instead of being rational, a player can follow other principles. Imitation is but one possibility. Imitation among humans is the premise underlying social learning theory [Bandura (1977)]. In subsection 4.1 we point out that best response dynamics and biased majority imitation are very similar and can both be modelled by means of neural networks. In contrast, best performance imitation cannot be modelled by means of neural networks. Best performance imitation, akin to fitness criteria in biology, has been studied by, among others, Nowak and May (1993) who find that local interactions are dramatically different from global ones. In subsection 3.1, we report on Outkin (1997) who finds that best response play and best performance imitation lead to drastically different dynamics.

In sum, we forward a modelling approach to best response dynamics that

(a) allows for rather general interaction structures;

(b) exhibits spatial patterns of play;

(c) exhibits differentiated noise;

(d) links stochastic and deterministic dynamics;

(e) encompasses majority imitation.

The article is organized as follows. The next two sections develop the deterministic model. This material overlaps considerably with Berninghaus and Schwalbe (1996b). But it seems indispensable, providing a foundation and a reference point for the more elaborate stochastic model. Section 4 deals with imitation. Section 5 introduces and analyzes noise in the form of a Boltzmann machine. Section 6 introduces logistic noise into the evolutionary game model which gives rise to a Boltzmann machine. Subsection 6.1 is devoted to long-run equilibria. As a noteworthy curiosity, subsection 6.2 relates the invariant distribution to the outcome of random social choice. Section 7 concludes.

## 2 Networked Models

Every stationary population game dynamics with finite strategy sets and finite neighborhoods corresponds to an automata network. This by itself does not allow strong conclusions. To arrive at interesting results, more structure has to be imposed. Here we consider dynamics that can be represented by means of neural networks. For extensive reading on automata networks, see Coughlin and Baran (1995), Goles and Martinez (1990, 1992), Haykin (1994). We commence with a few formal definitions, of an Automata Network (AN) and a Neural Network (NN). Throughout, $I$ is a finite or infinite set.

**Definition 1** *An* **Automata Network** *on* $I$ *is a triple*

$$A = (G, \Sigma, (f_i, i \in I)) \text{ where:}$$

$G = (I, V)$ *is a (undirected) graph on* $I$ *with connections given by the set* $V \subset I \times I$. *We assume* $G$ *to be locally finite, which means that every neighborhood* $V_i = \{j \in I : (j, i) \in V\}$ *is finite*[2].

$\Sigma$ *is the set of states of a node of the graph* $G$. *It is usually assumed to be finite. A state of a node* $i$ *is denoted by* $s_i$.

$f_i : \Sigma^{V_i} \to \Sigma$ *is the transition function associated with the vertex* $i$. *The evolution in discrete time* $t$ *of the global state* $s(t) \in \Sigma^I$ *is governed by the global transition function* $F : \Sigma^I \to \Sigma^I$ *obtained as composition of all the local ones. Each* $f_i$ *is also called a (memory-less) automaton.*

The use of automata networks is quite common in computer science. For instance, the interaction within a computer network falls into this category. Parallel processing is another example. The approach also suggests itself for decentralized models of robots interacting in a production process. Maes (1989) aims at "the building of an intelligent system as a society of interacting mindless agents, each having their own specific competence." Similarly, in the area of industrial organization, automata networks appear well suited for the description of information flows between units or divisions of an organization (firm, industry).

For an automata network, two polar updating rules are possible: synchronous and asynchronous iteration. In the synchronous mode we assume that all agents update their strategies at the same time, and in asynchronous mode we assume that at a given time only one agent can update his strategy.

**Definition 2** *A* **Neural Network** *or* **Threshold Automata Network** *is a particular type of automata network. Its state space is binary, e.g.* $\Sigma = \{-1, 1\}$. *Here we assume for convenience that* $\Sigma = \{0, 1\}$. *The network's transition*

[2] As can be seen from the definition, $i \notin V_i$.

*function and, implicitly, its graph are based on a weight structure, given by a matrix $A = (a_{ij})$. Namely: every arc $(i, j) \in V$ is associated with a real number $a_{ij} \in \mathbb{R}$ which represents its weight. If $i$ and $j$ are not neighbors, we put $a_{ij} = 0$. The transition function takes on the following form:*

$$s_i(t+1) = L\left(\sum_{j=1}^{N} a_{ij} s_j(t) - b_i\right) \tag{1}$$

*where $b_i$ is a threshold and the function $L(\cdot)$ is given by:*

$$L(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases} \tag{2}$$

*An automaton described by (1) is called a* **neuron** *or* **threshold automaton***.*

As the term suggests, neural networks can be used to model the interaction between brain or nerve cells: If and only if a receptor-transmitter receives a strong enough stimulus, it will emit a signal of its own. In a similar vein, in a social environment, an individual's decision to be violent or not may depend on the amount of violence in the neighborhood. Our current interest in neural networks rests on the fact that simple best response dynamics, among others, can be described in terms of a neural network.[3]

# 3 A Synchronous Best Response Model

Like Berninghaus and Schwalbe (1996b), let us put the best response dynamics into the framework of an NN. We will identify the set $I$ above with the set of players. For most of the following we will assume that $I$ is finite: $I = \{1, 2, ..., N\}$. We will also assume that all agents have the same number of neighbors $n$, $n = |V_i|$, $i \in I$. In other words, the graph is $n$-regular.

Consider a symmetric two person game:

$$\begin{array}{ccc} & \sigma_1 & \sigma_2 \\ \sigma_1 & a, a & b, c \\ \sigma_2 & c, b & d, d \end{array} \tag{3}$$

We will assume that only pure strategies are played. The set of pure strategies available for each player is the same as in Definition 2, $\Sigma = \{\sigma_1, \sigma_2\} = \{0, 1\}$, and the state space of the whole system is represented by $\Sigma^I = \{0, 1\}^I$.

[3]Majority imitators or opportunists in the sense of Berninghaus and Schwalbe (1996a), i.e. players who adopt the strategy most frequently played in the neighborhood, can also be represented by means of threshold automata. See Section 4.

The neighborhood structure is given by Definition 1. In other words, the player interacts only with his neighbors and has information only about the strategies played in his neighborhood. We will assume that a player employs a myopic best response, i.e. the strategy he chooses for the period $t+1$ is a best response against the distribution of strategies played in his neighborhood at time $t$. One can find a discussion of validity of the myopia assumption in KMR and Ellison (1993).

We denote the payoff to player $i$ at time $t$ by $\pi_i(s_i, s_{-i}(t))$, where $s_{-i}(t)$ is the distribution of strategies in $V_i$ at time $t$.

The decision rule for a population of best response players can be written in a threshold form:

$$s_i(t+1) = L[\pi_i(\sigma_2, s_{-i}(t)) - \pi_i(\sigma_1, s_{-i}(t))] \tag{4}$$

With the additional notation $z_i = \sum_{j \in V_i} s_j$, $z_i$ is the number of players using strategy $\sigma_2$ in $V_i$ and $n - z_i$ is the number of players using $\sigma_1$ in $V_i$. We now can express $\pi_i(\sigma_1, s_{-i}(t))$ and $\pi_i(\sigma_2, s_{-i}(t))$ as follows:

$$\pi_i(\sigma_1, s_{-i}(t)) = (n - z_i)a + z_i b$$

$$\pi_i(\sigma_2, s_{-i}(t)) = (n - z_i)c + z_i d$$

Therefore, (4) amounts to:

$$\begin{aligned} s_i(t+1) &= L\left[z_i(a-c+d-b) + n(c-a)\right] = \\ &= L\left[\sum_{j \in V_i}(a-c+d-b)s_j - n(a-c)\right] \end{aligned}$$

Hence:

$$s_i(t+1) = L\left[\sum_{j \in I} w_{ij} s_j - n\beta\right] \tag{5}$$

where $\beta = (a-c)$ and

$$w_{ij} = \begin{cases} a-c+d-b & \text{, if } j \in V_i \\ 0 & \text{, if } j \notin V_i \end{cases}$$

Since $w_{ij} = w_{ji}$, we can apply Corollary 3.1 in Goles and Martinez (1990) and obtain

**Proposition 1** *For a population of best response players, the orbits of synchronous iteration are only fixed points and/or two-cycles.*

Next set $w = a - c + d - b$. Then (5) can be equivalently written as

$$s_i(t+1) = L\left[w \sum_{j \in V_i} s_j - n\beta\right] \tag{6}$$

A trivial conclusion one can draw from (6) is that for some games the threshold will never be crossed (from above or from below). Then a deterministic dynamics reaches a steady state right away. Take, for instance, $|w| < |\beta|$ or $w\beta < 0$.

If we ask, however, whether the threshold can be crossed depending on the size of the neighborhood, we observe that in this narrow sense the neighborhood size does not matter much: if the threshold can be crossed at a particular size of the neighborhood, then it can also be crossed at every other size. This statement becomes especially clear if we write the equation (6) in a slightly different form:

$$s_i(t+1) = L\left[n(w\ \bar{s}_{-i} - \beta)\right] \tag{7}$$

where $\bar{s}_{-i}$ is the relative frequency of the strategy $\sigma_2$ in the neighborhood $V_i$. In case the threshold can be crossed in both directions, $w\beta > 0$ has to hold, say $w > 0$ and $\beta > 0$. If $w\ \bar{s}_{-i} - \beta > 0$, then $w\bar{v}^i_{-i} - \beta = w - \beta > 0$ for each vector $v^i$ with coordinates $v^i_j = 1$ for all $j \neq i$ and any number of neigbors. If $w\ \bar{s}_{-i} - \beta \leq 0$, then $w\bar{e}^i_{-i} - \beta = -\beta \leq 0$ for $e^i$, the $i$-th unit vector in $\mathbb{R}^I$, and any number of neighbors.

Nevertheless, it is intuitively clear and can also be shown by examples that the size of the system and the neighborhood structure do matter in a broader sense: the underlying graph, the size and shape of neigborhoods can affect the nature of limit cycles and steady states, and also their stability and convergence properties.

**Example 1 (Stability):** Let us consider the following coordination game:

| | $\sigma_1$ | $\sigma_2$ |
|---|---|---|
| $\sigma_1$ | $3, 3$ | $0, 0$ |
| $\sigma_2$ | $0, 0$ | $4, 4$ |

Assume a population of $N$ players on a circle is coordinating on a "wrong" equilibrium - everyone is playing $\sigma_1$. If every agent has two neighbors - one on the left and one on the right, then deviation of a single player will lead the population away from the "wrong" convention. One can check, however, that the same convention will be stable against a deviation of a single player in case everyone has four neighbors - two on the left and two on the right. This shows that the size of neighborhoods matters.

**Example 2 (Steady States):** Let us reconsider the previous coordination game. Assume a population of $N = 36$ players. Berninghaus and Schwalbe

(1996b) compare two neighborhood structures, each consistent with $n = 4$. In the first structure, the players are located on a circle and as before, each player has two neighbors to the left and two to the right. In the second structure, the players are located on a torus and each player has one neighbor to the left, the right, "above", and "below", respectively. It turns out that the different neighborhood shapes lead to different steady states. The second structure allows for the co-existence of two conventions whereas the first does not. This shows that the shape of neighborhoods matters as well.

**Example 3 (No Mixed Steady States):**
We consider an arbitray coordination game, i.e. $a - c > 0$ and $d - b > 0$ or, equivalently, $w > \beta > 0$. We assume first that the players are located on a circle and each player has $n/2$ neighbors on the right and on the left. Let us assume that we are in a mixed steady state - when a positive number of players play strategy 1 and a positive number of players play strategy 0. Then we can find two players $i_0$ and $i_1$ located next to each other and such that $i_0$ plays 0 and $i_1$ plays 0. The neighborhoods of the players $i_0$ and $i_1$ are the same, except for the fringe players $j_0$ and $j_1$ in the following flattened piece of the circle:

$$j_0 \text{-------} i_0 i_1 \text{-------} j_1$$

Let $j_0$ play $x_0$ and $j_1$ play $x_1$. Then the pattern of play is:

$$x_0 \text{-------} 01 \text{-------} x_1$$

From the equilibrium condition we know that neither of players $i_0$ or $i_1$ will want to change their strategy. This implies the following two conditions:
For the player $i_0$: $\sum\limits_{j \in I} w_{i_0 j} s_j - n\beta = wn_0 - n\beta \leq 0$,
For the player $i_1$: $\sum\limits_{j \in I} w_{i_1 j} s_j - n\beta = wn_1 - n\beta > 0$
where $n_0$ and $n_1$ are the total number of individuals playing the strategy 1 in the neighborhoods of players $i_0$ and $i_1$, respectively. We can conclude that to satisfy the steady state conditions inequality $n_1 > n_0$ has to hold. ¿From the picture above we derive $n_1 = n_0 - 1 - x_0 + x_1 \leq n_0$, a contradiction. Therefore, there are no mixed steady states in the symmetric one-dimensional case. A similar argument can be used to prove the same statement for an arbitrary fully connected set of players: Suppose a steady state $s$ has $k > 0$ ones and $N - k > 0$. Then $kw - (N-1)\beta \leq 0$ and $(k-1)w - (N-1)\beta > 0$, a contradiction.

# 4 Majority Imitation and Its Relation to Best Response

Imitation broadly defined means that a player selects a strategy previously played in her neighborhood or by herself. As a rule, she is allowed to self-imitate. Thus the dynamics is subject to the constraint

$$s_i(t+1) \in \{s_j(t) : j \in V_i \cup \{i\}\}. \tag{8}$$

As before, we assume that each player makes identical choices in situations which look identical to her so that the dynamic process is stationary. With sizeable, yet finite neighborhoods, a player's set of potential selection criteria is large but finite, and not all are plausible. The simplest form of imitation assumes that each player $i$ has a role model $j(i) \in V_i \cup \{i\}$. The selection criterion stipulates to imitate the role model's previous action:

$$s_i(t+1) = s_{j(i)}(t)$$

The two prominent imitation rules, majority imitation and best performance imitation, are based on conformity and fitness, respectively.

## 4.1 Majority Imitation

Berninghaus and Schwalbe (1996a) use the term opportunism instead. It turns out that majority imitation fits into the formal framework of neural networks. Moreover, for many payoff parameter values, best response reduces to a biased majority imitation criterion. Hence the neural network approach can accommodate a population composed of best response players and majority imitators. In particular, Proposition 1 still applies.

This rule stipulates that player $i$ choose the strategy played by a majority of individuals in $V_i \cup \{i\}$. The player following this criterion might presume that the neighbors face a similar environment and the majority makes the right or an adequate choice. Majority imitation can be formulated by means of threshold automata:

$$s_i(t+1) = L\left[\sum_{j \in I} w_{ij} s_j(t) - n/2\right] \tag{9}$$

where
$w_{ij} = 1$, if $i \neq j$ are neighbors;
$w_{ij} = 0$, if $i \neq j$ are not neighbors;
$w_{ii} = 1$, if $i$ can self-imitate;
$w_{ii} = 0$, if $i$ cannot self-imitate.
Replacing $n/2$ in (9) by a threshold $b_i$ with
$0 \leq b_i < n$ (for $w_{ii} = 0$) or
$0 \leq b_i < n+1$ (for $w_{ii} = 1$),
yields a **biased majority imitation criterion**. Incidentally, we observe the following

**Fact.** *In certain cases best response play coincides with a biased minority imitation criterion.*

Namely, consider the case $w > \beta \geq 0$. Then (6) can be rewritten as

$$s_i(t+1) = L\left[w\left\{\sum_{j\in V_i} s_j - \frac{n\beta}{w}\right\}\right] = L\left[\sum_{j\in V_i} s_j - \frac{n\beta}{w}\right] \tag{10}$$

In contrast, if $w < \beta < 0$, then the best response player's selection criterion is rather nonconformist: choose the strategy played by a qualified minority.

## 4.2 Best Performance Imitation

As a rule, best performance imitation does not fit into the formal framework of neural networks. It is the prevalent imitation criterion in the literature. A player adopts the "most successful" strategy played in the neighborhood. The explanation is similar to that of majority imitation: If a player is not quite sure what strategy will work, he might observe the performance of his neighbors under seemingly similar circumstances and trust that what proves best (or adequate) for them will be best (or adequate) for him. Formally, let again $\pi_i(s(t)) = \pi_i(s_i(t), s_{-i}(t))$ denote the payoff to player $i$ at time $t$ where $s_{-i}(t)$ is the strategy profile of players $j \in V_i$ at time $t$. Then the best performance (fitness) criterion prescribes

$$s_i(t+1) \in \{s_j(t) | j \in \arg\max\{\pi_k(s(t)) : k \in V_i \cup \{i\}\}, \tag{11}$$

possibly with a tie-breaking convention. Imitative behavior given by (11) reflects quite sophisticated feedback. If $j \in V_i$, then the behavior of the neighbor's neighbors $l \in V_j$ enters the calculation of $\pi_j(s(t))$. Thus the strategy of a player in the next period depends not solely on strategies and payoffs in his neighborhood, but involves a larger set of players (the neighbors' neigbors). This makes a significant difference. Whereas best response dynamics for a Prisoners' Dilemma game converge to full non-cooperation in one iteration, best performance based dynamics do support cooperation. For the best performance based dynamics with players situated on a circle and each interacting with the two adjacent players, Outkin (1997) finds among other things:

- From a mixed state (where both strategies are played), the dynamics need not converge to full non-cooperation. Full cooperation can be a locally attractive state.
- A mixed steady state has groups of non-cooperators surrounded by larger groups of cooperators where the "most successful" cooperator obtains a higher payoff than the "most successful" non-cooperator.

Hence best performance imitation implies very compelling dynamic properties. But it requires high observational and computational skills on the part of the players and lacks the "rational" appeal of best response behavior. Analytically, it cannot be treated within the context of neural networks.

## 5 Noisy Best Response Dynamics

In many cases it is beneficial to introduce randomness into the model. For example, deterministic models may have cycles or multiple equilibrium points. Frequently, under reasonable assumptions, a random Markov process on the same system has a unique stationary distribution. Also, we know from computer science that the learning capabilities of a stochastic network can be substantially better than those of a deterministic one[4]. Whenever applicable, we keep the previous notation.

**Synchronous Updating.** We start with a model of synchronous updating, described by (5) with noise added, represented by a family of random variables $\varepsilon_i(t), i = 1, \ldots, N, t = 0, 1, 2, \ldots$:

$$s_i(t+1) = L\left[w \sum_{j \in V_i} s_j - \beta n + \varepsilon_i(t)\right] = L\left[w\left\{\sum_{j \in V_i} s_j - n\frac{\beta}{w}\right\} + \varepsilon_i(t)\right] \quad (12)$$

The noise can come from several sources: noise in the level of the threshold, in the strategy played or perhaps in the payoff parameters. We will concentrate on the processes with stationary Markovian dynamics, i.e. dynamics fully determined by a transition matrix $W$, whose elements $W(s'|s)$ represent the probability for the system to be in state $s'$ at time $t+1$ given it is in state $s$ at time $t$. Then the evolution of the probability distribution will be governed by the following master equation:

$$\rho(s', t+1) = \sum\nolimits_s W(s'|s)\rho(s,t) \quad (13)$$

and in matrix notation: $\rho(t+1) = W\rho(t)$, where $\rho(t) = \rho(s,t)$ and two equalities have to be satisfied:

$$\sum_{s'} W(s'|s) = 1 \quad \textit{and} \quad \sum_{s} \rho(s,t) = 1. \quad (14)$$

A **steady state distribution (invariant measure)** $\bar{\rho}$ is of particular interest. It is a probability and right eigenvector of the matrix $W$ for the eigenvalue $\lambda = 1$:

$$W \cdot \bar{\rho} = \bar{\rho}. \quad (15)$$

As a rule, $\bar{\rho} = \rho(t \to \infty)$.

The main question is whether and when stochastic automata networks described by the above formalism serve our purposes well. Let us focus on the

[4] Regarding human decisions, people might simply make mistakes.

neural network (threshold automata network) whose transition functions are given by (12) and modifications thereof. Clearly, (12) gives rise to a special system of the form (13) and the art consists in finding the invariant distribution(s) or limit cycles. To start with, (12) constitutes a case of **simultaneous or synchronous updating**.

We shall proceed under the following

**ASSUMPTIONS:**

(i) The random variables $\varepsilon_i(t)$ are independent (across $i$ and $t$) and given any $i$, identically distributed.

(ii) The event $\{\varepsilon_i(t) > n \cdot (|w| + |\beta|)\}$ has positive probability for each pair $(i, t)$.

(iii) The event $\{\varepsilon_i(t) < -n \cdot (|w| + |\beta|)\}$ has positive probability for each pair $(i, t)$.

Under (i), the dynamics determined by (12) is, indeed, a stationary Markov process on $\Sigma^N$ whose transition matrix we denote $W$. Moreover:

**Proposition 2** *If (i)-(iii) hold, then the Markov chain generated by $W$ is irreducible and aperiodic.*

**Proof.** Consider two states $s$ and $s'$. Suppose $s(t) = s$. Then with positive probability, $s(t+1) = s'$. Because of (i), it suffices to show that for each $i \in I$, $s_i(t+1) = s'_i$ holds with positive probability. So let $i \in I$.

If $s'_i = 1$, then by (ii), with positive probability $\varepsilon_i(t) > n \cdot (|w| + |\beta|)$, hence

$$w \cdot \sum_{j \in V_i} s_j - n\beta + \varepsilon_i(t) > w \cdot \sum_{j \in V_i} s_j - n\beta + n \cdot (|w| + |\beta|) \geq 0$$

and $s_i(t+1) = s'_i$.

If $s'_i = 0$, then by (iii), with positive probability $\varepsilon_i(t) < -n \cdot (|w| + |\beta|)$, hence

$$w \cdot \sum_{j \in V_i} s_j - n\beta + \varepsilon_i(t) < w \cdot \sum_{j \in V_i} s_j - n\beta - n \cdot (|w| + |\beta|) \leq 0$$

and $s_i(t+1) = s'_i$. Since $s$ and $s'$ were arbitrary, this shows both irreducibility and aperiodicity. □□

Now well known results for discrete time Markov processes with finite state space yield

**Proposition 3** *If (i)-(iii) hold, then the Markov chain has a unique invariant distribution $\bar{\rho}$ which has full support. Moreover, the distributions $\rho(\cdot, t)$ converge in distribution to $\bar{\rho}$ regardless of the initial distribution of states.*

Among the classical references for this sort of result are Doob (1953), Feller (1968), and Loève (1960). Grimmet and Stirzaker (1982) and Seneta (1981) provide easy access to the material.

**Asynchronous Updating.** For further discussion we will concentrate on asynchronous updating rules, i.e. the state of only one neuron will be updated at a time. We can express this by means of a formula similar to (12):

$$s_i(t+1) = \left\{ \begin{array}{ll} L\left[w\left\{\sum_{j\in V_i} s_j - n\frac{\beta}{w}\right\} + \varepsilon_i(t)\right] & \text{, if } i = K(t) \\ \\ s_i(t) & \text{, otherwise} \end{array} \right\} \tag{16}$$

where $\{K(t),\ t = 0, 1, 2, ...\}$ is a sequence of independent and identically distributed $I$-valued random variables with full support $I$. $K(t)$ determines which of the agents will have a chance to change his strategy at time $t$.

The noise $\{\varepsilon_i(t), t = 0, 1, 2, ...\}$ is again a sequence of independent, identically distributed (i.i.d.) random variables satisfying (i)-(iii). In addition, the noise process and the agent-picking process are independent.

In a slightly more complicated way, one can establish analogues of Propositions 2 and 3 with asynchronous updating. However, this does not mean that the invariant distribution is the same under asynchronous updating as with simultaneous updating. Furthermore, the speed of convergence may differ between the two regimes.

**Invariant Boltzmann Distributions.** The steady state distribution can often be found without solving (15) explicitly. Such is the case for distributions of the Gibbs-Boltzmann type. Most of the literature is instructive, but rather sketchy and/or preoccupied with deriving certain properties from first principles of statistical mechanics. Therefore, we provide a brief, yet self-contained treatment of our own, following Haykin (1994, section 8.12) to some degree.

We assume that the transition process can be factored into two steps:

$$W(s'|s) = r(s'|s) \cdot q(r'|s) \text{ for } s \neq s' \tag{17}$$

where $r(s'|s)$ is the probability of an opportunity for a transition from state $s$ to state $s'$ and $q(s'|s)$ is the probability of a transition conditional on the event that an opportunity arises. Further restrictions are:

- **Symmetry**

$$r(s'|s) = r(s|s') \text{ for all } s \neq s'$$

- **Normalization**

$$\sum_{s' \neq s} r(s'|s) = 1$$

- **Complementarity**

$$q(s'|s) + q(s|s') = 1 \text{ for } s \neq s'$$

The key result is

**Proposition 4** *Suppose that:*

(a) *There is a unique invariant measure $\bar{\rho}$ that has full support.*

(b) *There is a "cost function" $G : \Sigma^I \to \mathbb{R}$ such that*

$$q(s'|s) = \frac{1}{1 + \exp(G(s') - G(s))} = \frac{\exp(G(s))}{\exp(G(s)) + \exp(G(s'))}. \qquad (18)$$

*Then $\bar{\rho}$ assumes the Gibbs-Boltzmann form:*

$$\bar{\rho}(s) = \frac{\exp(-G(s))}{\sum_{s'} \exp(-G(s'))}. \qquad (19)$$

**Proof.** Suppose $\bar{\rho}$ is given by (19). Then a strong "detailed balance principle" holds:

$$q(s'|s)\bar{\rho}(s) = q(s|s')\bar{\rho}(s') \text{ for all } s \neq s'. \qquad (20)$$

As an immediate consequence of (20), (17), and symmetry, we obtain the usual "detailed balance principle":

$$W(s'|s)\bar{\rho}(s) = W(s|s')\bar{\rho}(s') \text{ for all } s \neq s'. \qquad (21)$$

But (21) combined with (14) and the normalization condition on the probabilities of opportunity implies (15): For any $s'$,

$$\begin{aligned} \textstyle\sum_s W(s'|s)\bar{\rho}(s) &= \textstyle\sum_s W(s|s')\bar{\rho}(s') \\ &= \bar{\rho}(s') \textstyle\sum_s W(s|s') \\ &= \bar{\rho}(s'), \end{aligned}$$

that is (15). □□

Notice that because of (18), complementarity has been used implicitly in the proof. Also notice that (21) implies (20), in case $r(s'|s) > 0$ for all $s \neq s'$. This will not be the case in our application. But suppose that it is the case and that (21) can be assumed on *a priori* grounds. Then like in the literature, the order of crucial arguments can be reversed. In particular, (18) need not be assumed any longer, but is rather a consequence. Namely, first (21) implies (20). But then, by complementarity, (18) holds with $G(s) \equiv -ln\bar{\rho}(s)$:

$$q(s'|s) = \frac{1}{1 + \bar{\rho}(s)/\bar{\rho}(s')} = \frac{1}{1 + \exp(G(s') - G(s))}$$

# 6 Logistic Noise

We are now prepared to apply the Boltzmann proposition to a special game-theoretical case. We assume here that each noise variable $\varepsilon_i(t)$ is a logistic random variable with mean $\mu_i$ and common scaling parameter $T$, i.e. the cumulative distribution function (c.d.f.) is given by:

$$\Pr[\varepsilon_i(t) \leq \varepsilon] = \frac{1}{1 + \exp[-(\varepsilon - \mu_i)/T]}, \forall\, i, t \tag{22}$$

One can prove:

**Proposition 5** *The invariant distribution of strategies in a population of best response players, whose dynamics is defined by (16) with the noise being distributed according to (22) is given by the Gibbs/Boltzmann law (19) where* $G(s) = H(s)/T$ *and* $H$ *is defined as*

$$H(s) = \sum_{k \in I} s_k \left[ -\frac{1}{2} w \sum_{j \in V_k} s_j - (\mu_k - n\beta) \right]. \tag{23}$$

**Proof.** Let us verify that the system has the two-step factorization property. Let $s \neq s'$. A direct transition from $s$ to $s'$ requires that $s$ and $s'$ differ in exactly one coordinate, say the $i$-th. Then an "opportunity" for a transition from $s$ to $s'$ arises if and only if it is $i$'s turn to move. Also, an "opportunity" for a transition from $s'$ to $s$ arises if and only if it is $i$'s turn to move. Thus $r(s'|s) = r(s|s') = \text{Prob}(K(0) = i)$. In case $s$ and $s'$ differ in more than one coordinate, let us set $r(s'|s) = r(s|s') = 0$. Suppose that at time $t+1$, the current state is $s$ and player (neuron) $i$ has the opportunity to change $s_i$ from 1 to 0, resulting in the new state $s'$. This transition only happens if

$$w \sum_{j \in V_i} s_j - n\beta + \varepsilon_i(t) \leq 0 \text{ or}$$

$$\varepsilon_i(t) \leq n\beta - w \sum_{j \in V_i} s_j.$$

The probability of the latter event is

$$q(s'|s) = \frac{1}{1 + \exp[(\mu_i + w\sum_{j \in V_i} s_j - n\beta)/T]} = \frac{1}{1 + \exp[(H(s') - H(s))/T]}.$$

Clearly, complementarity applies: If the current state is $s'$ and player (neuron) $i$ has the opportunity to change $s'_i$ from 0 to 1, then the probability of this happening is $q(s|s') = 1 - q(s'|s)$ and, therefore,

$$q(s|s') = \frac{1}{1 + \exp[(H(s) - H(s'))/T]}.$$

Since $s$ and $s'$ were arbitrary, this covers all relevant contingencies. For the sake of completeness, we may extend the formula for $q(s'|s)$ to the case where $s$ and $s'$ differ in more than one coordinate. By Proposition 4, the assertion follows. □□

## 6.1 Long-Run Equilibria

In case $T \to \infty$, the c.d.f. (22) becomes flat and half of the mass is moved towards either tail. The noise becomes the sole driving force of the dynamics. Accordingly, $H(s)/T \to 0$ and in the limit, the invariant measure assigns equal probability to all states.

The case $T \to 0$ shifts all the mass towards the mean of the noise distribution and, thus, constitutes a gradual removal of noise. The support of the resulting **limit distribution** $\rho^*$ consists of the states $\bar{s}$ at which $H$ is minimized, i.e. $H(\bar{s}) \leq H(s)$ for all $s$. Namely, let $\bar{H}$ denote the minimum and $M$ denote the number of minimizers of $H$. Let $s$ denote generic states, $\bar{s}$ and $s'$ denote states at which $H$ is minimized, and $s''$ denote states at which $H$ is not minimized.

Then

$$\begin{aligned}
\bar{\rho}(\bar{s}) &= \frac{\exp[-H(\bar{s})/T]}{\sum_s \exp[-H(s)/T]} \\
&= \frac{\exp[-H(\bar{s})/T]}{\sum_{s'} \exp[-H(s')/T] + \sum_{s''} \exp[-H(s'')/T]} \\
&= \frac{\exp[-\bar{H}/T]}{\sum_{s'} \exp[-\bar{H}/T] + \sum_{s''} \exp[-\bar{H}/T] \cdot \exp[(\bar{H} - H(s''))/T]} \\
&= \frac{1}{M + \sum_{s''} \exp[(\bar{H} - H(s''))/T]} \\
&\to \frac{1}{M} \text{ as } T \to 0
\end{aligned}$$

which shows the claim. The points in the support (carrier) of $\rho^*$ have been called **long-run equilibria** by KMR. It turns out that in general, the long-run equilibria form a subset of the steady states of the associated deterministic dynamics. For a concise formulation of the result, let us say that there are **ties**, if $w \sum_{j \in V_i} s_j - n\beta = 0$ for some $i \in I$ and $s \in \Sigma^N$. We say further that $H$ attains a **local minimum** at state $s$, if $H(s) \leq H(s')$ for all $s'$ that differ from $s$ in only one component.

**Proposition 6** *Suppose that there are no ties and $\mu_1 = \ldots = \mu_N = 0$. Then for any state $\bar{s} \in \Sigma^I$, properties (i) and (ii) are equivalent and properties (iii) – (v) are equivalent.*

(i) *$\bar{s}$ is a long-run equilibrium.*

(ii) *$H(\bar{s})$ is a minimum of $H$.*

(iii) *$H(\bar{s})$ is a local minimum of $H$.*

(iv) *$\bar{s}$ is a steady state of the synchronous deterministic dynamics.*

(v) *$\bar{s}$ is a steady state of the asynchronous dynamics without noise.*

**Proof.** The equivalence of (i) and (ii) has already been established. $\bar{s}$ is a steady state of the (synchronous or asynchronous) deterministic dynamics, if and only if none of the players wishes to deviate unilaterally from $\bar{s}$. The only difference is that in the synchronous case the opportunity to deviate from $\bar{s}$ occurs, with certainty, for all players simultaneously whereas in the asynchronous case an opportunity to deviate from $\bar{s}$ occurs with positive probability for each player. This demonstrates the equivalence of (iv) and (v).[5] Now consider states $s$ and $s'$ which differ only in one component, $i$. A unilateral change from $s_i$ to $1-s_i$ constitutes a transition from $s$ to $s'$, with the follwing change in the value of $H$:

$$\triangle H(s) = H(s') - H(s) = -(1-2s_i)\left\{\sum_{j\in I} w_{ij}s_j - \beta n\right\}$$

For $H(s)$ to be a local minimum, it has to be the case that $\triangle H(s) = H(s') - H(s) \geq 0$ for all $s'$ which differ from $s$ in exactly one coordinate. Since there are no ties, these inequalities are strict. For $\bar{s}$ to be a steady state in the absence of ties, it has to be the case that

$\sum_{j\in I} w_{ij}s_j - n\beta < 0$, if $s_i = 0$;

$\sum_{j\in I} w_{ij}s_j - n\beta > 0$, if $s_i = 1$.

It follows that the conditions for a local minimum and a steady state coincide. This shows the equivalence of (iii) and (iv). □□

As an immediate consequence, we obtain

**Corollary 1** *Suppose that there are no ties and $\mu_1 = \ldots = \mu_N = 0$. Then each long-run equilibrium is a steady state of the associated deterministic dynamics.*

**Proof.** Since a minimum is a local minimum, the assertion follows from Proposition 6. □□

If there are ties, then not every local minimum is a steady state. Namely, the condition $L(0) = 0$ breaks ties in favor of 0. It therefore can happen that a transition from $s_i = 1$ to $s_i = 0$ does not affect the value of $H$. We can conclude, however, that every long-run equilibrium is a deterministic steady state with respect to some, possibly personalized, tie-breaking rule. Notice that the choice of tie-breaking rule has no impact on the stochastic dynamics, since there ties are zero probability events.

Clearly, $\mu_i \neq 0$ would cause a distortion between deterministic and stochastic dynamics. We continue under the above assumption that

$$\mu_1, = \ldots = \mu_N = 0. \tag{24}$$

It remains to determine the minimizers of $H$. *Prima facie*, this promises to be a formidable discrete optimization task. However, we can find the minimizers of $H$ in special cases that include the cases previously studied in the literature. In particular, we consider:

(A) Global interaction.

[5] The equivalence of (iv) and (v) does not mean that the dynamics are identical.

(B) Local interaction on a circle where each player has a neighbor to the left and a neighbor to the right.

The details for all conceivable cases with interaction structures A and B are worked out in the appendix. In part C of the appendix, we further analyze Ellison's $2k$ neigbor case, i.e. the case of a coordination game where the players are located on a circle and each player has $k$ neighbors to the left and $k$ neighbors to the right. These findings allow us to identify the "long-run equilibria" for certain prototype games. Denote $s(0) = (0, \dots, 0)$, the state of all zeros, and $s(N) = (1, \dots, 1)$, the state of all ones. The corresponding states in KMR are $z = N$ and $z = 0$. Further, for a real number $x$, let $[x]_-$ denote the largest integer not exceeeding $x$ and $[x]_+$ denote the smallest integer not less than $x$. If $x$ happens to be an integer, then $x = [x]_- = [x]_+$. If $x$ is not an integer, then $[x]_+ = [x]_- + 1$ and $[x]_- < x < [x]_+$. Finally, if $N$ is even, let $s^*$ denote those states where zeros and ones alternate. If $N$ is odd, let $s^-$ denote the states with a pattern $\dots 101010010101 \dots$, i.e. with a neighboring pair of zeros surrounded by alternating components, and let $s^+$ denote the states exhibiting a pattern $\dots 010101101010 \dots$, i.e. a neighboring pair of ones surrounded by alternating components.

**Dominant Strategy Games.** These games are characterized, up to relabelling of strategies, by $a - c > 0$ and $d - b \leq 0$ which is equivalent to $\beta > 0$ and $w \leq \beta$. In this case, under both interaction structures, $s(0)$ is the unique long-run equilibrium, i.e. the dominant strategy is played by all players. For global interaction and $w < \beta$, this finding coincides with Theorem 2 of KMR.

**Coordination Games.** These games are characterized by $a - c > 0$ and $d - b > 0$, hence $w > \beta > 0$. The inequality $a - c > d - b$ corresponds to $\beta > w/2$ in which case $s(0)$ is the long-run equilibrium under both interaction structures. Notice that in this case, $(0, 0)$ is the risk dominant, not necessarily the payoff dominant equilibrium point of the constituent bi-matrix game. Similarly, $a - c < d - b$ corresponds to $\beta < w/2$ in which case $s(N)$ is the long-run equilibrium under both interaction structures. In this case, $(1, 1)$ is the risk dominant, but not necessarily the payoff dominant equilibrium point of the constituent bi-matrix game. Our borderline case is given by $a - c = d - b$ or $\beta = w/2$ in which case $s(0)$ and $s(N)$ are the long-run equilibria under both interaction structures.

KMR study only the global or uniform interaction structure. In their context, they obtain results similar to, but not identical with ours. Their borderline case [in their Theorems 3 and 4 and Corollary 1] is given by $a - c = d - b + (a - d)/N$. For sufficiently large $N$, depending on the payoff parameters, the discrepancy disappears and their long-run equilibria coincide with ours — except when $a - c = d - b$.

Ellison (1993), in a model similar to KMR's, considers global (uniform) interaction as well as local interaction on the circle where each player has $k$ neighbors to the left and $k$ neighbors to the right. For $a-c \neq d-b$ and $N$ large enough, he shows that the long-run equilibrium reflects risk dominant play. In part C of the appendix, we arrive at the same conclusion without restrictions on $N$.

**Differentiation Games.** These games are characterized by $a-c<0$ and $d-b<0$, hence $w<\beta<0$ and $-w>-\beta>0$. The unique symmetric Nash equilibrium (in mixed strategies) has 1 played with probability $\lambda=(c-a)/(c-a+b-d)=\beta/w$ and 0 played with probability $1-\lambda=(b-d)/(c-a+b-d)$.

Under the global interaction structure, the long-run equilibria are the states with $[k^*]_-$ ones where $(k^*-1)w=(N-1)\beta$ or $k^*=1+(\beta/w)\cdot(N-1)=\lambda N+1-\lambda$, hence $1<k^*<N$ and $[k^*]_-=[\lambda N]_-$ or $[k^*]_-=[\lambda N]_+$.

In case $a-c\neq d-b$, Theorem 5 of KMR states essentially the same: *With* $N\geq 4$, $1\leq(1-\lambda)N\leq N-1$, *and a contraction condition, the long-run equilibria are the states with* $[(1-\lambda)N]_-$ *or* $[(1-\lambda)N]_+$ *zeros which amounts to* $[\lambda N]_+$ *or* $[\lambda N]_-$ *ones.*

In the case of $a-c=d-b$ and best response dynamics, KMR draw a conclusion markedly different from ours. They assert that $s(0)$ and $s(N)$ are the long-run equilibria, each occurring with probability 1/2.

There is a striking difference between global and local interaction. Under global interaction, the relative frequency of ones in a long-run equilibrium is closely related to the symmetric equilibrium in mixed strategies. With local interaction, the relative frequency of ones in a long-run equilibrium is roughly 1/2 and appears unrelated to the symmetric equilibrium in mixed strategies. Specifically, if $N$ is even, then the long-run equilibria assume the form $s^*$. If $N$ is odd and $d-b>a-c$, the long-run equilibria are of the form $s^+$. If $N$ is odd and $d-b=a-c$, the long-run equilibria are of the form $s^+$ or $s^-$. And if $N$ is odd and $d-b<a-c$, the long-run equilibria are of the form $s^-$.

## 6.2 Random Social Choice

Modern social choice theory typically proceeds in two steps. First it postulates certain economically or morally desirable properties of social outcomes. The axiom system imposed may prove inconsistent or irrelevant or, indeed, select a proper subset of social outcomes. The second step investigates the plausibility and viability of select outcomes. This investigation can rely on static (implementation) or dynamic (stability) concepts. The implementation or mechanism design literature is firmly rooted in strategic game theory and concerned with game forms or mechanisms which, for every conceivable profile of individual characteristics, yield the select outcome (a set of select outcomes) as equilibrium point(s). The stability literature has a long tradition in economics and studies the long-run behavior of adjustment processes, like adjustments of disequilibrium prices in the Walras-Arrow-Debreu model or quantity-adjustments in a Cournot model. The social outcome is considered stable with respect to a particular adjustment process, if the process converges — in a sense to be specified — to the outcome.

Here the social outcomes are the states $s \in \Sigma^I$. Taking the second step first, we find ourselves in a situation where we know the limit distribution $\bar{\rho}(\cdot)$ of a stochastic adjustment process. By Proposition 3, the distributions $\rho(\cdot, t)$ converge in distribution to $\bar{\rho}$ regardless of the initial distribution of states. Furthermore, the process is ergodic and, therefore, almost certainly the average frequency of a state $s$ along a path converges to $\bar{\rho}(s)$. Hence $\bar{\rho}(\cdot)$ is a stable outcome in a stochastic sense. But is it distinguished otherwise? It turns out that $\bar{\rho}$ can be interpreted as the probability of $s$ being chosen by a social planner with a random social welfare function of the form

$$\tilde{U}(s) = -H(s) + \epsilon(s). \tag{25}$$

The precise answer lies in the resemblance of formula (19) and the multinomial logit (MNL) model of discrete choice theory. The derivation of the MNL assumes a social planner whose preferences over social outcomes $s \in \Sigma^I$ are stochastic and represented by random utilities

$$\tilde{U}(s) = u(s) + \epsilon(s) \tag{26}$$

where the $u(s), s \in \Sigma^I$, are constants whereas the $\epsilon(s)$ are random variables with zero mean. The following result is taken from Anderson, la Palma, and Thisse (1992, Theorem 2.2) and attributed to Holman and Marley by Luce and Suppes (1965, pp. 338).

**Proposition 7** *Assume that the $\epsilon(s)$ are i.i.d. according to the double exponential distribution,*

$$Pr(\epsilon(s) \leq \eta) = \exp\{-\exp[-(\eta/\tau + \gamma)]\}$$

*where $\gamma$ is Euler's constant*[6] *and $\tau$ is a positive constant. Then the resulting choice probabilities are given by*

$$P(s) = \frac{\exp(u(s)/\tau)}{\sum_{s'} \exp(u(s')/\tau)}.$$

Setting $u(s) = -H(s)$ and $\tau = T$ yields the Gibbs-Boltzmann formula: $P(s) = \bar{\rho}(s)$. Hence if $u(\cdot) = -H(\cdot)$ and $\tau = T$, the random social choice of a decision maker with the characteristics hypothesized in Proposition (7) is summarized by $\bar{\rho}$.

[6] $\gamma = \lim_{K\to\infty}[\sum_{k=1}^{K} 1/k - \ln K]$ is approximately equal to 0.577215.

## 7 Conclusion

The paper demonstrates both the power and the limitation of neural network theory when applied to best response dynamics. On the positive side, it offers a straightforward way to reproduce, generalize, and refine the results in the literature. It seems to us that the homogeneity of the population maintained in the paper is convenient, but not absolutely necessary. On the negative side, the theory presumes that every player has a binary choice — a potential shortcoming shared with some of the most prominent alternative approaches. Also, it may be ill suited to address the issue of speed of convergence.

In section 4, we sort out the common features as well as the differences of best response dynamics and imitation dynamics. This clearly is not the last word on the comparison of the two dynamics.

There are several, more or less related literatures that we barely touched. First, local interaction models have been successfully employed to explain contagion effects. Second, the potential fruitfulness of statistical mechanics approaches to socioeconomic interaction has been demonstrated by Föllmer (1974), Blume (1993) and Durlauf (1993, 1997), among others. See also Verbrugge (1998). Finally, automata have been used before for other game-theoretical modelling purposes: A small strand of literature, pioneered by Neyman (1985) and Rubinstein (1986) has used finite automata to model the complexity of strategies and bounded rationality in repeated games.

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## APPENDIX

Here we determine the minimizers of the function $H(s)$ of Section 6. With (24), (23) can be rewritten as

$$H(s) = -\frac{1}{2} s\Gamma s + \gamma s \tag{27}$$

where $\Gamma \equiv (w_{ij})$ and $\gamma = (n\beta, \ldots, n\beta)$. For $k = 0, 1, \ldots, N$, let us denote by $s(k) \in \Sigma^I$ the vector given by $s_i(k) = 1$ for $i \leq k$ and $s_i(k) = 0$ for $i > k$. Then

$$H(s(k)) = -Q(k) \cdot w + k \cdot n\beta \tag{28}$$

where $Q(k)$ is half the number of $w$ residing in the $k$-th successive principal minor sub-matrix of $\Gamma$. Especially, we get

$$H(s(0)) = 0;\ H(s(N)) = -\frac{1}{2} Nnw + Nn\beta = (-\frac{1}{2} w + \beta) \cdot Nn. \tag{29}$$

Given the interaction structure, the dynamics is completely determined by the parameters $\beta$ and $w$. We distinguish between the two interaction structures A and B.

The 4-th principal minor for interaction structure A (B) is shown in Figure A (B).

| | | | | |
|---|---|---|---|---|
| | $w$ | $w$ | $w$ | $\ldots$ |
| $w$ | | $w$ | $w$ | $\ldots$ |
| $w$ | $w$ | | $w$ | $\ldots$ |
| $w$ | $w$ | $w$ | | $\ldots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

**Figure A**

| | | | | |
|---|---|---|---|---|
| | $w$ | | | $\ldots$ |
| $w$ | | $w$ | | $\ldots$ |
| | $w$ | | $w$ | $\ldots$ |
| | | $w$ | | $\ldots$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |

**Figure B**

### A. Global or Uniform Interaction

Let us begin with the interaction structure A, i.e. $n = N - 1$. By Figure A and (28), we have

$$H(s(k)) - H(s(k-1)) = (N-1)\beta - (k-1)w.$$

Furthermore, $H(s)$ depends only on the number of ones played in $s$.

First, suppose that $\operatorname{sign}\beta \neq \operatorname{sign} w$. If $\beta < 0$ and $w \geq 0$, then $H$ attains its minimum at $s(N)$ and its maximum at $s(0)$. If $\beta > 0$ and $w \leq 0$, then $H$ attains its minimum at $s(0)$ and its maximum at $s(N)$. If $\beta = 0$ and $w > 0$, then $H$ is minimized at $s(N)$ and is maximized at each state where none or exactly one player plays 1. If $\beta = 0$ and $w < 0$, then $H$ is minimized at each state where none or exactly one player plays 1 and $H$ is maximized at $s(N)$.

Second, suppose $\operatorname{sign}\beta = \operatorname{sign} w$. If $\beta = w = 0$, then every $s$ is a minimizer and a maximizer of $H$. For the rest, it suffices to analyze the case $\beta > 0$ and $w > 0$. We obtain several subcases.

$\beta > w/2 > 0$. Then there are two possibilities. Either $H(s(k))$ is strictly increasing in $k$ or unimodal. By (29), $H(s(0)) < H(s(N))$. Therefore, $s(0)$ is the unique minimizer. In case $w < \beta$, $s(N)$ is the maximizer of $H$. In case $w = \beta$, $s(N-1)$ and $s(N)$ are maximizers of $N$. In case $w > \beta$, solve the equation $(N-1)\beta = (k-1)w$ for $k$. If $k^*$ is the real-valued solution, then $H$ is maximized at $s([k^*]_-)$.[7]

$w/2 > \beta > 0$. The sequence of $H(s(k)), k = 0, \ldots, N$, is unimodal and, by (29), $H(s(N)) < H(s(0))$. Therefore $s(N)$ is the unique minimizer. Since $w > \beta$, the peak is reached at $s([k^*]_-)$ — with the qualification of footnote 5.

$w/2 = \beta > 0$. Then by (29), $H(s(0)) = H(s(N)) = 0$. Further the sequence $H(s(k)), k = 0, \ldots, N$, is unimodal. Hence $H$ is minimized at $s(0)$ and $s(N)$. Since $w > \beta$, the peak is reached at $s([k^*]_-)$ — with the qualification of footnote 5.

[7] We have $1 \leq k^* < N$. In case $k^*$ is an integer, $[k^*]_- = k^*$ and both $s(k^*)$ and $s(k^*+1)$ are maximizers.

**B. Local Interaction: 2 Neighbors**

Let us consider next the interaction structure B, i.e. $n = 2$. By Figure B and (28), we have

$$H(s(k)) - H(s(k-1)) = \begin{cases} 2\beta & \text{if } k = 1; \\ 2\beta - w & \text{if } k = 2, \ldots, N-1; \\ 2\beta - 2w & \text{if } k = N. \end{cases}$$

First, suppose that $\operatorname{sign}\beta \neq \operatorname{sign} w$. If $\beta < 0$ and $w \geq 0$, then $H$ attains its minimum at $s(N)$ and its maximum at $s(0)$. If $\beta > 0$ and $w \leq 0$, then $H$ attains its minimum at $s(0)$ and its maximum at $s(N)$. If $\beta = 0$ and $w > 0$, then $H$ is minimized at $s(N)$ and is maximized at each state where any two players playing 1 are not neighbors, e.g. when none or exactly one player plays 1. If $\beta = 0$ and $w < 0$, then $H$ is minimized at each state where any two players playing 1 are not neighbors and $H$ is maximized at $s(N)$.

Second, suppose $\operatorname{sign}\beta = \operatorname{sign} w$. If $\beta = w = 0$, then every $s$ is a minimizer and a maximizer of $H$. For the rest, it suffices again to analyze the case $\beta > 0$ and $w > 0$.

*B.1. Minimizers.* We distinguish several subcases.

$\beta > w/2 > 0$. Then by (29), $H(s(0)) < H(s(N))$. Moreover, $H(s(0)) < H(s(1)) < \ldots < H(s(N-1))$, hence $H(s(0)) < H(s(k))$ for $k = 1, 2, \ldots, N$. But then $H$ is minimized at $s(0)$. For if $s \neq s(0)$ has $k$ ones and $N - k$ zeroes, then $\frac{1}{2}sBs \leq Q(k)$ and, therefore, $H(s) \geq H(s(k)) > H(s(0))$. Thus $s(0)$ is the unique minimizer of $H$.

$w/2 > \beta > 0$. Then by (29), $H(s(N)) < H(s(0))$. Moreover, $H(s(0)) < H(s(1))$ and $H(s(N)) < H(s(N-1)) < \ldots < H(s(1))$. Again, if $s \neq s(0)$ has $k$ ones and $N - k$ zeros, then $H(s) \geq H(s(k))$. We conclude that $H$ is minimized at $s(N)$.

$w/2 = \beta > 0$. Then by (29), $H(s(0)) = H(s(N)) = 0$. Variation of the previous argument shows that $s(0)$ and $s(N)$ are the sole minimizers of $H$.

*B.2. Maximizers.*

Recall that if $N$ is even, $s^*$ denotes those states where zeros and ones alternate. If $N$ is odd, $s^-$ denotes the states with a pattern $\ldots 101010010101 \ldots$, i.e. with a neighboring pair of zeros surrounded by alternating components, and $s^+$ denotes the states exhibiting a pattern $\ldots 010101101010 \ldots$, i.e. a neighboring pair of ones surrounded by alternating components.

Suppose $N$ is even. Clearly, every state $s$ with less than $N/2$ ones satisfies $H(s) < N\beta = H(s^*)$. So let us consider states $s$ with $k \geq N/2$ ones and $N-k$ zeros. Then $2k\beta - Q(N)w = 2k\beta - Nw$ is a lower bound for $H(s)$. Since $k \geq N/2$, $N-k \leq N/2$ and in order to maximize $H(s)$, we should place the $N-k$ zeros "optimally", i.e. such that each 0 has a 1 to its left and and a 1 to its right. Then each 0 reduces the "negative contribution" $Nw$ above by $2w$ and the "optimal" arrangement yields

$$H(s) = 2k\beta - Nw + 2(N-k)w = Nw + 2k(\beta - w).$$

If $\beta > w$, then $s(N)$ is the maximizer of $H$.
If $\beta < w$, then the $s^*$ are the maximixers of $H$.
If $\beta = w$, then all $s$ with $k \geq N/2$ ones and zeros and ones "optimally" arranged are the maximizers.

Suppose $N$ is odd. Clearly, every state $s$ with less than $(N-1)/2$ ones satisfies $H(s) < (N-1)\beta = H(s^-)$. Moreover, $H(s^+) = H(s^-) + 2\beta - w$. Once again, let us consider states $s$ with $k \geq N/2$ ones and $N-k$ zeros "optimally" arranged so that $H(s) = Nw + 2k(\beta - w)$.
If $\beta > w$, then $s(N)$ is the maximizer of $H$.
If $\beta = w$, then all $s$ with $k \geq (N+1)/2$ ones and zeros and ones "optimally" arranged are maximizers.
If $w/2 < \beta < w$, then the states $s^+$ are the maximizers of $H$.
If $w/2 = \beta$, then the states of the form $s^-$ and $s^+$ are the maximizers of $H$.
If $w/2 > \beta$, then the states $s^-$ are the maximizers.

## C. Local Interaction: $2k$ Neighbors

Let us end with the analysis of coordination games ($w > \beta > 0$) where the players are located on a circle and each player has $k$ neighbors to the left and $k$ neighbors to the right. An argument similar to the one used in B.1 shows that $H$ is maximized at $s(0)$ if $w/2 > 0$; at $s(N)$ if $w/2 > \beta > 0$; at $s(0)$ and $s(N)$ if $w/2 = \beta > 0$.