

Evolution Equations for Weakly Nonlinear, Quasi-Planar Waves in Isotropic Dielectrics and Elastomers

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(ABSTRACT)

The propagation of waves through nonlinear media is of interest here, namely as it pertains to two specific examples, a nonlinear dielectric and a hyperelastic solid. In both cases, we examine the propagation of two-dimensional, weakly nonlinear, quasi-planar waves. It is found that such systems will have a nonlinearity that is intrinsically cubic, and therefore, a classical Zablolotskaya-Khokhlov equation cannot give an accurate description of the wave evolution. To determine the general evolution equation in such systems, a multi-timing technique developed by Kluwick and Cox (1998) and Cramer and Webb (1998) will be employed. The resultant evolution equations are both seen to involve only one new nonlinearity coefficient rather than the three coefficients found in other studies of cubically nonlinear systems. After determining the general evolution equation, inclusion of conduction, dispersion and dissipation effects can be easily incorporated.

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Table of Contents

Abstract.....	p. ii
Acknowledgements.....	p. iii
Chapter 1: Introduction.....	p. 1-6
Part I: General Considerations	
Chapter 2: Statement of Problem.....	p. 8-11
Chapter 3: Derivation of the General Evolution Equation.....	p. 12-19
Chapter 4: Summary.....	p. 20-21
Part II: The Nonlinear Dielectric	
Chapter 5: Introduction.....	p. 23
Chapter 6: Formulation-Nonlinear Dielectric.....	p. 24-26
Section 6.1: Discussion of the Relative Dielectric Constant.....	p. 26-29
Section 6.2: Derivation of the Final System of Equations.....	p. 29-33
Chapter 7: Derivation of the Evolution Equation.....	p. 34-38
Chapter 8: Inclusion of Conduction and Dispersion.....	p. 39-41
Chapter 9: Summary.....	p. 42
Part III: The Hyperelastic Solid	
Chapter 10: Introduction.....	p. 44
Chapter 11: Formulation-Hyperelastic Solids.....	p. 45-52
Chapter 12: Stress-Strain Relationships.....	p. 53-55
Chapter 13: Computation of Eigenvalues and Vectors.....	p. 56-60
Chapter 14: Nonlinearity Calculations for the Hyperelastic Solid.....	p. 61
Section 14.1: Determining the Order of Nonlinearity; P-mode.....	p. 61-63
Section 14.2: Determining the Order of Nonlinearity; S-mode.....	p. 63-65
Chapter 15: Evaluation of the Evolution Equation for the S-mode.....	p. 66-69

Chapter 16: Inclusion of Dissipation.....	p. 70-72
Chapter 17: Summary.....	p. 73-74
Chapter 18: Concluding Remarks.....	p. 75-76
Appendix A.....	p. 77
Appendix B.....	p. 78-79
Appendix C.....	p. 80-81
References.....	p. 82
Vita.....	p. 83

Chapter 1: Introduction

One of the most important models of wave propagation is that of linearized plane waves. However, this model is rarely found in actual physical systems. Waves will get distorted by inhomogeneities in the medium, reflections from the boundaries, or the finite size of the generating source. Huygen's principle requires that such waves propagate normal to themselves. As a result, such distorted wavefronts continue to distort and modify the wave amplitude due to focussing and de-focussing effects. Here, we will refer to wave distortion effects as either diffraction or refraction effects.

Nearly all physical systems are nonlinear, meaning, they function in such a way that any disturbances in the medium will cause changes in the way the medium responds to further disturbances. The main nonlinear effect of interest in studies of waves propagation is the dependence of the wave speed on the wave amplitude. Well known examples of nonlinear media include ordinary sound waves, where the speed of sound depends on the local pressure and temperature perturbations, and elastic waves in a stiffening or softening material.

If the effects of nonlinearity and diffraction are considered to be weak, they may be negligible over short propagation times and distances, and therefore may be neglected. However, these weak effects may become important over long propagation times. As an example, we consider a distorted wavefront having a radius of curvature, R . Significant amplification or attenuation will occur over propagation distances of order R . Thus, such modulations to the wave amplitude can be ignored over propagation distances which are much less than R , but are entirely non-negligible over propagation distances of order R or much greater. A second, more familiar, example of a very weak process which ultimately has a dramatic effect is that of a weakly damped oscillator. Over a few periods the effects of damping may not be noticeable. However, significant attenuation is always noticeable over time scales of the order of the inverse of the damping coefficient.

While the idealization of linearized plane waves is an important one for understanding the more elementary and even fundamental aspects of wave propagation in a particular medium, the effects of wave distortion and nonlinearity play a role in many problems of engineering and scientific interest. This is particularly true in applications

having large amplitudes and complicated geometries. An example treated later in this thesis is that of nonlinear optics, where lasers can generate relatively large amplitudes and beam spreading is a central issue. In understanding of the effects of nonlinearity, wave focussing and spreading is also of scientific interest. In configurations leading to wave convergence, linear focussing and the shift in wave speed due to nonlinearity can counterbalance each other. The final wave evolution will be determined by the competition between these two effects. Further discussion of the interaction of these effects can be found in Whitham (1974). Nonlinearity can also modify diverging waves. For example, the asymptotic decay rate for a weak cylindrically expanding shock wave is usually found to be approximately $r^{-3/4}$ rather than the well-known $r^{-1/2}$ law of the linear theory (Whitham, 1974). Here r is the distance of the cylindrical shock from its point of origin.

Exact analyses of multi-dimensional waves in nonlinear systems must be conducted on a case by case basis. However, the wave evolution can very often be described by a single partial differential equation of the following form,

$$(\psi_{\tau} + \Gamma\psi\psi_{\chi})_{\chi} + C\psi_{\eta\eta} = 0, \quad (1.1)$$

when the amplitudes are small and the distortions are weak and two-dimensional. The dependent variable $\psi = \psi(\chi, \eta, \tau)$ is a scalar shape function, χ is a spatial variable aligned with the primary direction of propagation, η is a spatial variable measured transverse to the direction of propagation, τ is a time-like variable, and C and Γ are order-one constants whose precise values will depend on the nature of the system under consideration. Equation (1.1) was first derived by Zabolotskaya and Khokhlov (1969) for the special case of nonlinear acoustics and will henceforth be called the classical Zabolotskaya-Khokhlov equation. Because (1.1) arises in many applications involving weak nonlinearity and quasi-plane waves, (1.1) is generally considered to be a canonical equation in the same way that Laplace's equation is the canonical equation for linear elliptical systems. In fact, one can derive (1.1) through the use of a multiple scales analysis similar to that presented in Part I of this thesis for fairly general systems; one recent derivation has been given by Kluwick and Cox (1998). One of the advantages of any canonical equation is that its solutions, once obtained, can be immediately applied to

all phenomena for which the canonical equation gives a reasonable approximation to the response of the system.

Another advantage of most canonical equations is that they tend to be simpler than the original set of equations governing the physical system of interest. As an example, the original set of equations governing the nonlinear acoustics problem considered by Zabolotskaya and Khokhlov (1969) comprised four first-order equations for the fluid density, entropy, and velocity components. The reduction to a single first-order equation has obvious advantages in both analytical and numerical studies. This benefit will become even more evident when we examine the sixth-order system governing the nonlinear waves in elastomers discussed in Part III of this thesis.

A final advantage of the reduction of systems to canonical equations is that the phenomena under study can be described with a minimum amount of information about the material response. In fact, canonical equations usually reveal precisely which aspects of the constitutive equations are relevant to the phenomenon of interest, at least in the context of the approximations being used. In the example of nonlinear acoustics, as governed by (1.1), coordinate scalings can be found for which $C=1/2$ and Γ is just the sign of the curvature of the isentrope in a pressure-volume diagram. Thus, all other details of the fluid medium are irrelevant with respect to the general evolution equation of the wave. Furthermore, the dependence of the details on other physical parameters can be determined explicitly. The latter fact is partially important in design situations where intuition and rules of thumb play a central role.

In many applications, the constant C reduces to $1/2$ in suitably chosen scaled variables χ , η , τ , and ψ . The remaining constant Γ is recognized as a measure of the nonlinearity of the system. Because ψ appears as a simple product with its derivative, the nonlinearity is seen to be quadratic. We will therefore refer to Γ as the quadratic nonlinearity coefficient.

The classical Zabolotskaya-Khokhlov equation (1.1) is known to provide a reasonable model for the wave evolution in many physical systems of interest. However, the computed value of Γ is known to vanish identically for some common physical systems. Examples include the Alfvén waves discussed by Cramer and Webb (1998), the

nonlinear dielectric considered in Part II of this thesis, and the shear waves treated in Part III. When this is the case, (1.1) reduces to

$$\psi_{\tau\chi} + C\psi\eta\eta = 0, \quad (1.2)$$

which implies that the nonlinearity is negligible over the time scales associated with (1.1). It is easily verified that (1.2) can be reduced to the ordinary linear wave equation approximated for quasi-plane waves. However, nonlinearity must ultimately play a role in the wave equation. Typically, these nonlinear effects will become noticeable at times which are much larger than the time scales used to derive (1.1). Mathematically, the source of these terms are the higher-order nonlinear terms correctly neglected in the derivation of the classical ($\Gamma=O(1)$) version of the Zabalotskaya-Khokhlov equation. In order to study the effects of nonlinearity and diffraction in systems having $\Gamma=0$, we must first determine the situations in which these two effects occur at the same rate; that is, we must determine the ranges of amplitude and wavefront curvature for which nonlinear effects and diffraction are comparable when $\Gamma=0$. Once this is done, we then need to approximate our original equations to determine the modified form of (1.1) valid for $\Gamma=0$. Both tasks have already been carried out by Kluwick and Cox (1998) and Cramer and Webb (1998). Both studies gave the new length scales and applied a modified multiple scales technique to determine where the quadratic nonlinearity coefficient either vanished or was small. The principal difference between the two studies were that the first also considered propagation in an inhomogeneous medium and along curved rays, where the study by Cramer and Webb (1998) included the effects of weak relaxation, dispersion and dissipation, albeit for propagation in a uniform medium.

The derivations given by Kluwick and Cox (1998) and Cramer and Webb (1998) are valid for a general quasi-linear, hyperbolic system of N differential equations of the form

$$\underline{u}_t + \underline{A}\underline{u}_x + \underline{B}\underline{u}_y = \underline{0}, \quad (1.3)$$

where \underline{u} is an $N \times 1$ solution vector, \underline{A} and \underline{B} are $N \times N$ coefficient matrices, and x , y and t are the spatial and temporal independent variables. A more complete statement of the restrictions on (1.3) is given in Chapter 2 of this thesis. As demonstrated in Part I of this thesis, the generalization of (1.1) for the $\Gamma=0$ case contains five scalar coefficients instead

of the two coefficients seen in (1.1). As in the case of the classical equation (1.1), these coefficients are given in terms of the coefficient matrices \underline{A} and \underline{B} . For any specific physical application, extensive calculations will be required in order to determine the dependence of the coefficient of the modified Zabolotskaya-Khokhlov equation in terms of the physical variables. The primary goal of the present thesis is to examine two examples of engineering interest and to determine the modified Zabolotskaya-Khokhlov equation valid for these physical applications. The examples chosen involve waves in a nonlinear dielectric and a hyperelastic solid. The work presented here will provide two nontrivial illustrations of the use of the results of Kluwick and Cox (1998) and Cramer and Webb (1998). In fact, the example of the hyperelastic solid is the largest system to which these authors' general scheme has been applied. Simultaneously we also provide the specific form of the modified Zabolotskaya-Khokhlov equation governing small amplitude, quasi-plane waves in these media.

This work will also provide two examples in which the resultant evolution equation contains only one non-zero nonlinearity coefficient. In each case, these nonlinearity coefficients will be seen to be identical to those obtained in the one-dimensional theory of Cramer and Sen (1992).

As already suggested, this thesis is divided in three parts. In Part I, we develop the general evolution equation, following the multi-timing scheme presented by Cramer and Webb (1998). This is done for a two-dimensional, quasi-planar wave that may exhibit dissipation, dispersion or relaxation effects. We then consider two direct applications for this multi-timing scheme; first, the nonlinear dielectric is presented in Part II. Maxwell's equations are given in Chapter 6 for a wave exhibiting the restrictions explained here, and these equations are arranged in a matrix form in Chapter 6.2, which makes them easy to manipulate within the multi-timing scheme presented in Part I. The evolution equation is derived in Chapter 7, as well as its relation to the physical variables of the system. Conduction and dispersion are also considered separately; the evolution equations for dispersive and conducting media are given in Chapter 8. Part III is a more involved application of the multi-timing scheme. The equilibrium and mass equations for a hyperelastic solid are given in Chapter 11, and the stress tensor is analyzed in Chapter 12. The resultant 6x6 system is then put into a matrix form, similar to that presented in

Part II, and the scalar coefficients needed to describe the evolution equation are determined in Chapter 15. In Chapter 16, dissipation effects are considered for the case where viscosity is present in the system. The scheme presented in Part I is closely followed and referred to throughout the body of this work as a way to determine the evolution equations for such systems as the nonlinear dielectric and the hyperelastic solid, and also as a way to relate the physical parameters of the system to each other.

Part I:

General Considerations

Chapter 2: Statement of Problem

In order to examine the propagation of waves in different media, we will begin with the following general system of partial differential equations,

$$\begin{aligned} \underline{u}_t + \underline{A}\underline{u}_x + \underline{B}\underline{u}_y = & \underline{C}(\underline{u} - \underline{u}^{(0)}) + \underline{D}^{(xx)}\underline{u}_{xx} + \underline{D}^{(xy)}\underline{u}_{xy} \\ & + \underline{D}^{(yy)}\underline{u}_{yy} + \underline{E}^{(xxx)}\underline{u}_{xxx} + \underline{E}^{(xxy)}\underline{u}_{xxy}, \\ & + \underline{E}^{(xyy)}\underline{u}_{xyy} + \underline{E}^{(yyy)}\underline{u}_{yyy} \end{aligned} \quad (2.1)$$

where the subscripts t , x , and y in the above equation are the time, x-direction and y-direction partial derivatives, respectively; $\underline{u} = \underline{u}(x,y,t)$ is an $N \times 1$ solution vector; $\underline{u}^{(0)}$ is an $N \times 1$ vector which denotes the value of the solution vector \underline{u} at the undisturbed state, which will be constant in all that follows; \underline{A} , \underline{B} , \underline{C} , $\underline{D}^{(xx)}$, $\underline{D}^{(xy)}$, $\underline{D}^{(yy)}$, $\underline{E}^{(xxx)}$, $\underline{E}^{(xxy)}$, $\underline{E}^{(xyy)}$ and $\underline{E}^{(yyy)}$ are all $N \times N$ matrices which will be described in terms of their physical effects on different systems later in this section.

Given the definitions above, it is now useful to note that we will take the system to correspond to a uniform medium. As a result, the coefficient matrices in Equation (2.1) will be taken to be independent of t , x , and y . We will also require that the system be quasi-linear, which requires that matrices \underline{A} and \underline{B} be functions of the solution vector \underline{u} only.

In applications involving propagation in the x- (or y-) direction only, the matrix \underline{A} (or \underline{B}) would be referred to as the speed matrix. Because our main interest is in waves that propagate primarily in the x-direction, the matrix \underline{A} will henceforth be referred to as the speed matrix.

The term proportional to $\underline{u} - \underline{u}^{(0)}$ in Equation (2.1) appears in many physical systems exhibiting relaxation. Examples of relaxing systems are the weakly conducting dielectric discussed in Part II of this work as well as the relaxation exhibited by Maxwell fluids. The terms involving the second derivatives of x and y seen on the right-hand side of Equation (2.1) are typically found in physical systems having dissipation. Such dissipation is usually due to some kind of internal friction. Examples include the viscous effects found in Newtonian fluids and the Kelvin-Voigt solid discussed in Part III of this

work. The terms on the right-hand side of Equation (2.1) which involve the third derivatives of x and y are found in systems having weak dispersion. A system has dispersion if the wave speed depends on the length or frequency of the wave. In optics, dispersion is responsible for the separation of visible light into its component colors by a prism. Weak dispersion will be included in the analysis of the electromagnetic system to be discussed in Part II of this work. Another example of a weakly dispersive system is that governing ordinary water waves. In all that follows we will refer to the terms proportional to $\underline{u} - \underline{u}^{(0)}$, the second and third derivatives of x, y as relaxation, dissipation and dispersion terms.

As pointed out in the previous section, we seek the evolution equation governing \underline{u} when the disturbances (amplitudes) are small, the transverse variations in y are weak and the effects of relaxation, dissipation and dispersion are also weak. If we linearize (2.1), neglect variations in the y -direction, and neglect relaxation, dissipation and dispersion, we find that (2.1) can be written as

$$\underline{u}_t + \underline{A}^{(0)} \underline{u}_x = \underline{0}, \quad (2.2)$$

where

$$\underline{A}^{(0)} \equiv \underline{A}(\underline{u}^{(0)}). \quad (2.3)$$

In order that this lowest order problem have wave-like solutions, only systems having an $\underline{A}^{(0)}$ matrix that has real eigenvalues will be discussed. The system is then said to be hyperbolic, similar to that presented by Cramer and Webb (1998). In such hyperbolic systems it is easy to show that wave-like solutions to (2.2) are

$$\underline{u} = \underline{u}^{(0)} + \underline{r} Y(x - \lambda t), \quad (2.4)$$

where λ is an eigenvalue of the matrix $\underline{A}^{(0)}$ and \underline{r} is the associated eigenvector of $\underline{A}^{(0)}$, which means that

$$\left(\underline{A}^{(0)} - \lambda \underline{I} \right) \underline{r} = \underline{0}, \quad (2.5)$$

where \underline{I} is the identity matrix. The function Y will be determined by any initial and boundary conditions specified in the problem. The form of (2.4) is recognized as a disturbance that travels to the right at a speed λ without a change in shape. The form of

the solution (2.4) also reveals one of the motivations for referring to $\underline{\underline{A}}$ as the speed matrix.

The condition that the transverse variations are weak can be written as

$$\frac{\partial}{\partial y} \ll \frac{\partial}{\partial x}. \quad (2.6)$$

This will be made more precise later in this chapter. A consistency condition on this quasi-plane wave approximation will be shown to be

$$\underline{\underline{l}}^T \underline{\underline{B}}^{(0)} \underline{\underline{r}} = 0, \quad (2.7)$$

where $\underline{\underline{B}}^{(0)} \equiv \underline{\underline{B}}(\underline{\underline{u}}^{(0)})$ which is the value of $\underline{\underline{B}}$ evaluated at the undisturbed state, and

where $\underline{\underline{l}}^T$ denotes the transpose of the left-hand eigenvector of matrix $\underline{\underline{A}}^{(0)}$. That is, $\underline{\underline{l}}$ is an $N \times 1$ vector that satisfies the following,

$$\left(\underline{\underline{A}}^{(0)} - \lambda \underline{\underline{I}} \right)^T \underline{\underline{l}} = \underline{\underline{0}}, \quad (2.8a)$$

or, equivalently,

$$\underline{\underline{l}}^T \left(\underline{\underline{A}}^{(0)} - \lambda \underline{\underline{I}} \right) = \underline{\underline{0}}. \quad (2.8b)$$

For the remainder of this thesis, the superscript T will denote the transpose. The necessity for the condition posed in Equation (2.7) will be demonstrated and discussed in further detail in the following chapter.

The requirement that the relaxation, dissipation and dispersion be small will require that the appropriate nondimensional form of the matrices $\underline{\underline{C}}$, $\underline{\underline{D}}^{(xx)}$, $\underline{\underline{D}}^{(xy)}$, etc. be small. If we represent any one of these matrices by $\underline{\underline{M}}$, then the quantity which must be small in each case is proportional to the scalar

$$\frac{\underline{\underline{l}}^T \underline{\underline{M}} \underline{\underline{r}}}{\underline{\underline{l}} \cdot \underline{\underline{r}}}. \quad (2.9)$$

Because each matrix has different units, the matrices $\underline{\underline{M}}$ will be multiplied by different factors of λ and L (where L is the x-length scale of the disturbance) in (2.9) to preserve nondimensionality. The precise size restriction on the aforementioned matrices will be

given later in this section, but note that the matrices $\underline{\underline{C}}, \underline{\underline{D}}^{(xx)}, \underline{\underline{D}}^{(xy)}$, etc. can be approximated by their lowest order values.

As discussed in the previous chapter, the new feature of the perturbation scheme developed by Kluwick and Cox (1998) and Cramer and Webb (1998) is that the quadratic nonlinearity, denoted as Γ in Chapter 1, is either small or will vanish. In all that follows, we will be considering the case only where this nonlinearity vanishes. The mathematical expression of this condition can be shown to be

$$\Gamma \equiv \frac{\underline{\underline{\ell}}^T \underline{\underline{G}} \underline{\underline{r}}}{\underline{\underline{\ell}} \cdot \underline{\underline{r}}} = 0, \quad (2.10)$$

where

$$\underline{\underline{G}} \equiv \left. \frac{\partial \underline{\underline{A}}}{\partial u_k} \right|_{\underline{\underline{u}}(0)} r_k \quad (2.11)$$

for $k=1,2,\dots,N$, and the Einstein summation convention is used.

In order to simplify the analysis, certain smallness assumptions will be made. In order to make the sizing precise, we will define a sizing factor Δ , which will be small in this case. This factor will be used to size the needed variables in a manner consistent with the work of Cramer and Webb (1998). We will therefore write,

$$\underline{\underline{u}} - \underline{\underline{u}}^{(0)} = O(\Delta), \quad (2.12a)$$

$$\frac{\partial}{\partial X} = O\left(\frac{1}{L}\right), \quad (2.12b)$$

$$\frac{\partial}{\partial y} = O\left(\frac{\Delta}{L}\right) \text{ and} \quad (2.12b)$$

$$\frac{\partial}{\partial t} = O\left(\frac{\lambda \Delta^2}{L}\right), \quad (2.12b)$$

where $X=x-\lambda t$.

Now that the assumptions which govern the perturbation method to be outlined in the next section have been explained, as well as restrictions on the system such that it will fit in with the type of wave effects that we want to observe, the formulation of the evolution equation can be given. This formulation will be outlined in the next section.

Chapter 3: Derivation of the General Evolution Equation

At this point, we can begin with the general equation for \underline{u} as described by Equation (2.1). In order to incorporate the quasi-plane assumptions (2.12) explicitly, the following transformation of variables is performed:

$$\chi \equiv \frac{x - \lambda t}{L}; \quad \eta \equiv \Delta \frac{y}{L}; \quad \tau \equiv \Delta^2 \frac{\lambda t}{L}; \quad (3.1)$$

where L is again a length scale measuring the variations in the x -direction and λ is an eigenvalue of $\underline{\underline{A}}^{(0)}$. When (3.1) are substituted into (2.1), we find:

$$\begin{aligned} \Delta^2 \lambda \underline{u}_\tau + (\underline{\underline{A}} - \lambda \underline{\underline{I}}) \underline{u}_\chi + \Delta \underline{\underline{B}} \underline{u}_\eta &= LC \left(\underline{u} - \underline{u}^{(0)} \right) + \frac{1}{L} \underline{\underline{D}}^{(xx)} \underline{u} \chi \chi \\ &+ \frac{\Delta}{L} \underline{\underline{D}}^{(xy)} \underline{u} \chi \eta + \frac{\Delta^2}{L} \underline{\underline{D}}^{(yy)} \underline{u} \eta \eta + \frac{1}{L^2} \underline{\underline{E}}^{(xxx)} \underline{u} \chi \chi \chi + \frac{\Delta}{L^2} \underline{\underline{E}}^{(xxy)} \underline{u} \chi \chi \eta \\ &+ \frac{\Delta^2}{L^2} \underline{\underline{E}}^{(xyy)} \underline{u} \chi \eta \eta + \frac{\Delta^3}{L^2} \underline{\underline{E}}^{(yyy)} \underline{u} \eta \eta \eta. \end{aligned} \quad (3.2)$$

This equation has a form which incorporates the basic assumptions set forth in Chapter 2, which are that the wave will vary slowly over time and vary slowly in the y -direction. At this point we can perform a perturbation about the undisturbed state, $\underline{u}^{(0)}$. The first step in this perturbation procedure is to expand the nonlinear $\underline{\underline{A}}$ and $\underline{\underline{B}}$ matrices in a Taylor series. When this is done we find that

$$\begin{aligned} \underline{\underline{A}}(\underline{u}) &= \underline{\underline{A}}^{(0)} + \frac{\partial \underline{\underline{A}}}{\partial u_k} \bigg|_{\underline{u}^{(0)}} \left(u_k - u_k^{(0)} \right) \\ &+ \frac{1}{2} \frac{\partial^2 \underline{\underline{A}}}{\partial u_k \partial u_\ell} \bigg|_{\underline{u}^{(0)}} \left(u_k - u_k^{(0)} \right) \left(u_\ell - u_\ell^{(0)} \right) + O(\Delta^3), \end{aligned} \quad (3.3a)$$

$$\underline{\underline{B}}(\underline{u}) = \underline{\underline{B}}^{(0)} + \frac{\partial \underline{\underline{B}}}{\partial u_\ell} \bigg|_{\underline{u}^{(0)}} \left(u_\ell - u_\ell^{(0)} \right) + O(\Delta^2), \quad (3.3b)$$

where we have again employed the Einstein summation convention for $k, l = 1, 2, \dots, N$, and u_k , etc. are the components or the \underline{u} vector, whereas $u_k^{(0)}$ are the components of $\underline{u}^{(0)}$. Since we will be interested in only the first three perturbation equations in order to derive the evolution equation, expressions on the order of Δ^4 and higher can be ignored from the

general equation. For this reason, expressions in the Taylor expansion of $\underline{\underline{A}}$ higher than order Δ^3 and in the Taylor expansion of $\underline{\underline{B}}$ higher than order Δ^2 can be ignored. At this point, we have anticipated the number of terms needed for accurate completion of the evolution equation analysis, and can ignore terms higher than those listed above.

In order to complete the expansion about $\underline{u}^{(0)}$, we must also expand the solution vector \underline{u} for small Δ as follows,

$$\underline{u} = \underline{u}^{(0)} + \Delta \underline{u}^{(1)} + \Delta^2 \underline{u}^{(2)} + \Delta^3 \underline{u}^{(3)} + O(\Delta^4). \quad (3.4)$$

We can then substitute this equation for \underline{u} into Equation (3.2) along with the expansion for the $\underline{\underline{A}}$ and $\underline{\underline{B}}$ matrices as given in Equations (3.3). Upon performing these substitutions, and then grouping terms as coefficients of the Δ parameter to different powers, we arrive at the following equation:

$$\begin{aligned} & \left(\underline{\underline{A}}^{(0)} - \lambda \underline{\underline{I}} \right) \underline{u} \chi + \Delta^2 \left[\frac{\partial \underline{\underline{A}}}{\partial u_k} \Big|_{\underline{u}^{(0)}} \underline{u} \chi u_k^{(1)} + \underline{\underline{B}}^{(0)} \underline{u} \eta^{(1)} \right] + \Delta^3 \left[\lambda \underline{u} \tau^{(1)} + \frac{\partial \underline{\underline{A}}}{\partial u_k} \Big|_{\underline{u}^{(0)}} \left(\underline{u}_x^{(2)} u_k^{(1)} + \underline{u}_x^{(1)} u_k^{(2)} \right) \right. \\ & \left. + \frac{1}{2} \frac{\partial \underline{\underline{A}}}{\partial u_k \partial u_\ell} \Big|_{\underline{u}^{(0)}} u_k^{(1)} u_l^{(1)} \underline{u} \tau^{(1)} + \underline{\underline{B}}^{(0)} \underline{u} \eta^{(2)} + \frac{\partial \underline{\underline{B}}}{\partial u_\ell} \Big|_{\underline{u}^{(0)}} u_l^{(1)} \underline{u} \eta^{(1)} \right] = \lambda \Delta^3 \hat{\underline{\underline{C}}} \underline{u}^{(1)} + \lambda \Delta^3 \hat{\underline{\underline{D}}}^{(xx)} \underline{u}_{xx}^{(1)} \quad (3.5) \\ & + \lambda \Delta^4 \hat{\underline{\underline{D}}}^{(yx)} \underline{u}_{\chi\eta}^{(1)} + \lambda \Delta^5 \hat{\underline{\underline{D}}}^{(yy)} \underline{u}_{\eta\eta}^{(1)} + \lambda \Delta^3 \hat{\underline{\underline{E}}}^{(xxx)} \underline{u}_{\chi\chi\chi}^{(1)} + \lambda \Delta^4 \hat{\underline{\underline{E}}}^{(xxy)} \underline{u}_{\chi\chi\eta}^{(1)} + \lambda \Delta^5 \hat{\underline{\underline{E}}}^{(xyy)} \underline{u}_{\chi\eta\eta}^{(1)} \\ & + \lambda \Delta^6 \hat{\underline{\underline{E}}}^{(yyy)} \underline{u}_{\eta\eta\eta}^{(1)} + O(\Delta^7), \end{aligned}$$

where we have defined the scaled matrices $\hat{\underline{\underline{C}}}$, $\hat{\underline{\underline{D}}}^{(xx)}$, $\hat{\underline{\underline{D}}}^{(xy)}$, etc. as follows,

$$\hat{\underline{\underline{C}}} \equiv \frac{LC}{\lambda \Delta^2}, \quad (3.6a)$$

$$\hat{\underline{\underline{D}}}^{(xx)} = \frac{\underline{\underline{D}}^{(xx)}}{L \lambda \Delta^2}, \quad \hat{\underline{\underline{D}}}^{(xy)} = \frac{\underline{\underline{D}}^{(xy)}}{L \lambda \Delta^2}, \quad \hat{\underline{\underline{D}}}^{(yy)} = \frac{\underline{\underline{D}}^{(yy)}}{L \lambda \Delta^2}, \quad \text{and} \quad (3.6b)$$

$$\hat{\underline{\underline{E}}}^{(xxx)} = \frac{\underline{\underline{E}}^{(xxx)}}{L^2 \lambda \Delta^2}, \quad \hat{\underline{\underline{E}}}^{(xxy)} = \frac{\underline{\underline{E}}^{(xxy)}}{L^2 \lambda \Delta^2}, \quad \hat{\underline{\underline{E}}}^{(xyy)} = \frac{\underline{\underline{E}}^{(xyy)}}{L^2 \lambda \Delta^2}, \quad \hat{\underline{\underline{E}}}^{(yyy)} = \frac{\underline{\underline{E}}^{(yyy)}}{L^2 \lambda \Delta^2}. \quad (3.6c)$$

It is useful to note here that the x-derivative of the \underline{u} vector is actually of the form

$$\underline{u}_\chi = \underline{u}_\chi^{(0)} + \Delta \underline{u}_\chi^{(1)} + \Delta^2 \underline{u}_\chi^{(2)} + \Delta^3 \underline{u}_\chi^{(3)} + \dots \quad (3.7)$$

but have been left in their unexpanded form on the left-hand side of in Equation (3.5) just to save space.

We will now arrive at the following three equations, each obtained by equating the coefficients of different powers of Δ in (3.5).

$$\left(\underline{\underline{A}}^{(0)} - \lambda \underline{\underline{I}} \right) \underline{\underline{u}}_x^{(1)} = 0, \quad (3.8a)$$

$$\left(\underline{\underline{A}}^{(0)} - \lambda \underline{\underline{I}} \right) \underline{\underline{u}}_x^{(2)} + \frac{\partial \underline{\underline{A}}}{\partial u_k} \Big|_{\underline{\underline{u}}^{(0)}} u_k^{(1)} \underline{\underline{u}}_x^{(1)} + \underline{\underline{B}}^{(0)} \underline{\underline{u}}_\eta^{(1)} = 0, \text{ and} \quad (3.8b)$$

$$\begin{aligned} & \left(\underline{\underline{A}}^{(0)} - \lambda \underline{\underline{I}} \right) \underline{\underline{u}}_x^{(3)} + \lambda \underline{\underline{u}}_\tau^{(1)} + \frac{\partial \underline{\underline{A}}}{\partial u_k} \Big|_{\underline{\underline{u}}^{(0)}} \left(\underline{\underline{u}}_x^{(2)} u_k^{(1)} + u_k^{(2)} \underline{\underline{u}}_x^{(1)} \right) + \\ & \frac{1}{2} \frac{\partial \underline{\underline{A}}}{\partial u_k \partial u_\ell} \Big|_{\underline{\underline{u}}^{(0)}} u_k^{(1)} u_\ell^{(1)} \underline{\underline{u}}_x^{(1)} + \underline{\underline{B}}^{(0)} \underline{\underline{u}}_\eta^{(2)} + \frac{\partial \underline{\underline{B}}}{\partial u_\ell} \Big|_{\underline{\underline{u}}^{(0)}} u_\ell^{(1)} \underline{\underline{u}}_\eta^{(1)} \\ & = \lambda \hat{\underline{\underline{C}}} \underline{\underline{u}}^{(1)} + \lambda \hat{\underline{\underline{D}}}^{(xx)} \underline{\underline{u}}_{xx}^{(1)} + \lambda \hat{\underline{\underline{E}}}^{(xxx)} \underline{\underline{u}}_{xxx}^{(1)}. \end{aligned} \quad (3.8c)$$

It is apparent in the third equation that only the $\hat{\underline{\underline{C}}}$, $\hat{\underline{\underline{D}}}^{(xx)}$ and $\hat{\underline{\underline{E}}}^{(xxx)}$ matrices will contribute at this order of magnitude. This comes from the scaling of the matrices as shown in Equations (3.6). Once this scaling is performed and the non-dimensional forms of the $\underline{\underline{C}}$, $\underline{\underline{D}}^{(xx)}$, $\underline{\underline{D}}^{(xy)}$, etc. matrices are substituted back into (3.5), all matrices except $\hat{\underline{\underline{C}}}$, $\hat{\underline{\underline{D}}}^{(xx)}$ and $\hat{\underline{\underline{E}}}^{(xxx)}$ will be multiplied by an order of Δ^4 or higher, which is beyond the accuracy needed for this work.

At this point we will examine the first perturbation equation. Because this is a homogeneous equation, a nontrivial solution is obtained if and only if the $\underline{\underline{u}}$ vector is parallel to an eigenvector of $\underline{\underline{A}}^{(0)}$ and where λ is the corresponding eigenvalue of $\underline{\underline{A}}$. We can therefore write $\underline{\underline{u}}^{(1)}$ in the following form:

$$\underline{\underline{u}}^{(1)} = \underline{\underline{r}} U(\chi, \eta, \tau), \quad (3.9)$$

where $U = U(\chi, \eta, \tau)$, which will be referred to as a shape function. We can note at this point that we are looking for nontrivial solutions for the shape function, U . Then, upon substituting this definition for $\underline{\underline{u}}^{(1)}$ into the first perturbation equation, we get the following:

$$\left(\underline{\underline{A}}^{(0)} - \lambda \underline{\underline{I}}\right) \underline{r} U_x = \underline{0}. \quad (3.10)$$

We can discard the trivial solution $U_x = 0$ immediately, and are then left with

$$\left(\underline{\underline{A}}^{(0)} - \lambda \underline{\underline{I}}\right) \underline{r} = \underline{0}, \quad (3.11)$$

which verifies the above statements regarding \underline{r} and λ . We can then solve for λ from the following equation,

$$\det\left(\underline{\underline{A}}^{(0)} - \lambda \underline{\underline{I}}\right) = 0. \quad (3.12)$$

The solution to this equation yields N values for λ . Each eigenvalue can then be substituted back into Equation (3.11), and the full set of eigenvectors \underline{r} can be found by using elementary linear algebra. Although it is not part of the solution for $\underline{u}^{(1)}$, it will usually be convenient to solve for the left-hand eigenvector of $\underline{\underline{A}}^{(0)}$ at this stage as well. The equation for \underline{l} is given by Equation (2.8b).

We will now examine the second perturbation equation, (3.8b). First, in order to ensure that $\underline{u}^{(2)}$ will have a solution that is self-consistent and that the system has only cubic nonlinearity, we need to satisfy the following conditions as set forth in Chapter 2,

$$\underline{l}^T \underline{\underline{B}}^{(0)} \underline{r} = 0, \text{ and} \quad (3.13)$$

$$\underline{l}^T \underline{\underline{G}} \underline{r} = 0. \quad (3.14)$$

If these are satisfied for the system of interest, we can combine (3.9) with (3.8b) and rearrange, which yields

$$\left(\underline{\underline{A}}^{(0)} - \lambda \underline{\underline{I}}\right) \underline{u}_x^{(2)} = -\underline{\underline{G}} \underline{r} U U_x - \underline{\underline{B}}^{(0)} \underline{r} U \eta, \quad (3.15)$$

where $\underline{\underline{G}}$ is given in (2.11). This is a set of N linear algebraic equations for $\underline{u}_x^{(2)}$ which, since the matrices $\underline{\underline{A}}^{(0)}$, $\underline{\underline{B}}^{(0)}$ and $\underline{\underline{G}}$ are all independent of U , U_x , and U_η , will give a solution for $\underline{u}_x^{(2)}$ that is the sum of one term proportional to $U U_x$, one proportional to U_η , and a homogeneous part. Thus, we can assume the form of the solutions as

$$\underline{u}_x^{(2)} = \underline{\gamma} U U_x + \underline{\delta} U \eta + \text{homogeneous part}, \quad (3.16)$$

where the homogeneous part of Equation (3.16) signifies the part of this solution that would be parallel to the right-hand eigenvector \underline{r} . It will be shown that the precise form

of this homogeneous solution will have no effect on the result of primary interest, namely, the evolution equation for U . At this point we can integrate the above to get

$$\underline{u}^{(2)} = \underline{\gamma} \frac{U^2}{2} + \underline{\delta} \int U_n d\chi + F(\eta, \tau), \quad (3.17)$$

where $F(\eta, \tau)$ signifies an integration function which is normally determined by the initial and boundary conditions. We will require that $\underline{u}^{(2)}$ and U go to zero as χ approaches infinity. As a result, the integration function in Equation (3.17) will be recognized to vanish as well. Thus, (3.17) can be written

$$\underline{u}^{(2)} = \underline{\gamma} \frac{U^2}{2} + \underline{\delta} V + \text{homogeneous}, \quad (3.18)$$

where we employ the definition

$$V \equiv \int U \eta d\chi, \text{ or } U \eta = V \chi. \quad (3.19)$$

Now that a solution for $\underline{u}^{(2)}$ has been found, it can be substituted into Equation (3.15) which will give the following,

$$\left\{ \left(\underline{A}^{(0)} - \lambda \underline{I} \right) \underline{\gamma} + \underline{G} \underline{r} \right\} \frac{U^2}{2} + \left\{ \left(\underline{A}^{(0)} - \lambda \underline{I} \right) \underline{\delta} + \underline{B}^{(0)} \underline{r} \right\} V = 0. \quad (3.20)$$

In order to have a nontrivial solution for the U and V functions, we will set the coefficients of U^2 and V in Equation (3.20) equal to zero separately, which results in the following algebraic equations for $\underline{\delta}$ and $\underline{\gamma}$,

$$\left(\underline{A}^{(0)} - \lambda \underline{I} \right) \underline{\gamma} = -\underline{G} \underline{r}, \text{ and} \quad (3.21)$$

$$\left(\underline{A}^{(0)} - \lambda \underline{I} \right) \underline{\delta} = -\underline{B}^{(0)} \underline{r}, \quad (3.22)$$

as seen in the one-dimensional analysis presented by Cramer and Sen (1992). Solving (3.21) and (3.22) will give solutions for the $\underline{\delta}$ and $\underline{\gamma}$ vectors, both of which will be needed in solving for the evolution equation coefficients in the following discussion. Now that all of the needed vectors have been solved for, we can examine the third perturbation equation, (3.8c). Since we have now found a solution for $\underline{u}^{(1)}$ as given in Equation (3.9), we can substitute it into Equation (3.8c), which can now be written:

$$\begin{aligned}
\left(\underline{\underline{A}}^{(0)} - \lambda \underline{\underline{I}} \right) \underline{\underline{u}} \chi^{(3)} = & -\lambda \underline{\underline{r}} U \tau - \frac{\partial \underline{\underline{A}}}{\partial u_k} \Big|_{\underline{\underline{u}}(0)} r_k \underline{\underline{u}} \chi^{(2)} U - \frac{\partial \underline{\underline{A}}}{\partial u_k} \Big|_{\underline{\underline{u}}(0)} u_k^{(2)} \underline{\underline{r}} U \chi \\
& - \frac{1}{2} \frac{\partial \underline{\underline{A}}}{\partial u_k \partial u_\ell} \Big|_{\underline{\underline{u}}(0)} r_k r_\ell \underline{\underline{r}} U^2 U \chi - \underline{\underline{B}}^{(0)} \underline{\underline{u}} \eta^{(2)} - \frac{\partial \underline{\underline{B}}}{\partial u_\ell} \Big|_{\underline{\underline{u}}(0)} r_\ell \underline{\underline{r}} U U \eta \quad (3.23) \\
& + \lambda \underline{\underline{C}} \underline{\underline{r}} U + \lambda \underline{\underline{D}}^{(xx)} \underline{\underline{r}} U \chi \chi + \lambda \underline{\underline{E}}^{(xxx)} \underline{\underline{r}} U \chi \chi \chi.
\end{aligned}$$

In order to have a self-consistent solution for $\underline{\underline{u}}^{(3)}$, we will dot the left-hand eigenvector, $\underline{\underline{l}}^T$, with Equation (3.23). After performing this step, the resulting compatibility condition ensuring that $\underline{\underline{u}}^{(3)}$ is self-consistent will be:

$$\begin{aligned}
0 = & -\lambda \underline{\underline{l}}^T \cdot \underline{\underline{r}} U \tau - \underline{\underline{l}}^T \underline{\underline{G}} U \underline{\underline{u}} \chi^{(2)} - \underline{\underline{l}}^T \frac{\partial \underline{\underline{A}}}{\partial u_k} \Big|_{\underline{\underline{u}}(0)} u_k^{(2)} \underline{\underline{r}} U \chi - \underline{\underline{l}}^T \underline{\underline{H}} \underline{\underline{r}} U^2 U \chi \\
& - \underline{\underline{l}}^T \underline{\underline{B}}^{(0)} \underline{\underline{u}} \eta^{(2)} - \underline{\underline{l}}^T \frac{\partial \underline{\underline{B}}}{\partial u_\ell} \Big|_{\underline{\underline{u}}(0)} r_\ell \underline{\underline{r}} U U \eta + \lambda \underline{\underline{l}}^T \underline{\underline{C}} \underline{\underline{r}} U + \lambda \underline{\underline{l}}^T \underline{\underline{D}}^{(xx)} \underline{\underline{r}} U \chi \chi \quad (3.24) \\
& + \lambda \underline{\underline{l}}^T \underline{\underline{E}}^{(xxx)} \underline{\underline{r}} U \chi \chi \chi,
\end{aligned}$$

where the matrix $\underline{\underline{G}}$ was defined in Equation (2.11), and $\underline{\underline{H}}$ is defined as follows:

$$\underline{\underline{H}} = \frac{1}{2} \frac{\partial^2 \underline{\underline{A}}}{\partial u_k \partial u_\ell} \Big|_{\underline{\underline{u}}(0)} r_k r_\ell. \quad (3.25)$$

The compatibility equation (3.24) imposes a constraint on the shape of the function $U(\chi, \eta, \tau)$. We may now simplify this equation by substituting (3.18) into (3.24). Then we will divide the resulting simplified equation through by $\lambda \underline{\underline{l}} \cdot \underline{\underline{r}}$ and collect terms based on the coefficients of derivatives of the shape functions U and V . This will obtain the following evolution equation:

$$\begin{aligned}
0 = U_\tau + & \left[\frac{\underline{\ell}^T \underline{G} \underline{\delta}}{\lambda \underline{\ell} \cdot \underline{r}} + \frac{\underline{\ell}^T \underline{B}^{(0)} \underline{\gamma}}{\lambda \underline{\ell} \cdot \underline{r}} + \frac{\underline{\ell}^T \frac{\partial \underline{B}}{\partial u_\ell} \Big|_{\underline{u}^{(0)}} r_{\ell \underline{r}}}{\lambda \underline{\ell} \cdot \underline{r}} \right] U U_\eta + \frac{\underline{\ell}^T \frac{\partial \underline{A}}{\partial u_k} \Big|_{\underline{u}^{(0)}} \delta_{k \underline{r}}}{\lambda \underline{\ell} \cdot \underline{r}} V U_\chi \\
& + \frac{\underline{\ell}^T \underline{B}^{(0)} \underline{\delta}}{\lambda \underline{\ell} \cdot \underline{r}} V \eta + \left[\frac{\underline{\ell}^T \underline{G} \underline{\gamma}}{\lambda \underline{\ell} \cdot \underline{r}} + \frac{\underline{\ell}^T \frac{\partial \underline{A}}{\partial u_k} \Big|_{\underline{u}^{(0)}} \gamma_{k \underline{r}}}{2 \lambda \underline{\ell} \cdot \underline{r}} + \frac{\underline{\ell}^T \underline{H} \underline{r}}{\lambda \underline{\ell} \cdot \underline{r}} \right] U^2 U_\chi \\
& - \frac{\underline{\ell}^T \hat{\underline{C}}^{(xx)} \underline{r}}{\underline{\ell} \cdot \underline{r}} - \frac{\underline{\ell}^T \hat{\underline{D}}^{(xx)} \underline{r}}{\underline{\ell} \cdot \underline{r}} - \frac{\underline{\ell}^T \hat{\underline{E}}^{(xxx)} \underline{r}}{\underline{\ell} \cdot \underline{r}},
\end{aligned} \tag{3.26}$$

where we have used Equation (3.19), and thereby have replaced all V_χ functions in (3.26) by U_η . At this point, it is convenient to rewrite (3.26) using the following definitions:

$$A \equiv \frac{1}{\lambda \underline{\ell} \cdot \underline{r}} \left(\frac{\underline{\ell}^T \underline{B}^{(0)} \underline{\gamma} + \underline{\ell}^T \underline{G} \underline{\delta} + \underline{\ell}^T \frac{\partial \underline{B}}{\partial u_k} \Big|_{\underline{u}^{(0)}} r_{k \underline{r}} \right), \tag{3.27a}$$

$$B \equiv \frac{1}{\lambda \underline{\ell} \cdot \underline{r}} \left(\frac{\underline{\ell}^T \frac{\partial \underline{A}}{\partial u_k} \Big|_{\underline{u}^{(0)}} \delta_{k \underline{r}} \right), \tag{3.27b}$$

$$C \equiv \frac{1}{\lambda \underline{\ell} \cdot \underline{r}} \left(\underline{\ell}^T \underline{B}^{(0)} \underline{\delta} \right), \tag{3.27c}$$

$$\Lambda \equiv \frac{2}{\lambda \underline{\ell} \cdot \underline{r}} \left(\underline{\ell}^T \underline{H} \underline{r} + \underline{\ell}^T \underline{G} \underline{\gamma} + \frac{1}{2} \underline{\ell}^T \frac{\partial \underline{A}}{\partial u_k} \Big|_{\underline{u}^{(0)}} \gamma_{k \underline{r}} \right), \tag{3.27d}$$

$$\Delta_C \equiv \underline{\ell}^T \frac{\hat{\underline{C}} \underline{r}}{\underline{\ell} \cdot \underline{r}}, \tag{3.27e}$$

$$\Delta_D \equiv \underline{\ell}^T \frac{\hat{\underline{D}}^{(xx)} \underline{r}}{\underline{\ell} \cdot \underline{r}}, \text{ and} \tag{3.27f}$$

$$\Delta_E \equiv \underline{\ell}^T \frac{\hat{\underline{E}}^{(xxx)} \underline{r}}{\underline{\ell} \cdot \underline{r}}. \tag{3.27g}$$

We then arrive at a condensed version of the evolution equation, which is

$$U_{\tau} + AUU_{\eta} + BVU_{\chi} + CV_{\eta} + \frac{\Lambda}{2}U^2U_{\chi} = \Delta_C U + \Delta_D U_{\chi\chi} + \Delta_E U_{\chi\chi\chi}. \quad (3.28)$$

Thus, the solution for U is governed by the evolution equations (3.28) and (3.19). Once (3.28) and (3.19) are solved, the original solution vector will be given by

$$\underline{u} = \underline{u}^{(0)} + \Delta \underline{r} U + \Delta^2 \left(\underline{\gamma} \frac{U^2}{2} + \underline{\delta} V \right) + O(\Delta^3), \quad (3.29)$$

where the solutions for the vectors $\underline{u}^{(1)}$ and $\underline{u}^{(2)}$ have been employed.

At this point we can note that the solution to \underline{u} is amply described if we use only the particular parts of the solutions for $\underline{\delta}$ and $\underline{\gamma}$ and disregard the homogeneous parts. These homogeneous parts are vectors that are parallel to the right-hand eigenvector, \underline{r} , which are embedded in the solutions for $\underline{\delta}$ and $\underline{\gamma}$. These can be disregarded since the scalars given in (3.27) require that these homogeneous solutions be multiplied by \underline{G} or $\underline{B}^{(0)}$, depending on which scalar is being considered. These homogeneous solutions, when multiplied by such vectors, will vanish, as described by the conditions set forth in (3.14), as follows

$$\underline{l}^T \underline{G} (\underline{\delta}_{\text{particular}} + \underline{\delta}_{\text{homogeneous}}) = \underline{l}^T \underline{G} \underline{\delta} + \underline{l}^T \underline{G} \underline{r} = \underline{l}^T \underline{G} \underline{\delta}. \quad (3.30)$$

Along the same lines, the homogeneous part of the solution for $\underline{u}^{(2)}$ has also been left out in (3.29), as it too has been described in the $\underline{u}^{(1)}$ solution and is therefore not needed.

Chapter 4: Summary

Now that the process for determining the evolution equation for a system that obeys the governing assumptions set forth in Chapter 2 has been outlined, we can examine the results. Looking at the evolution equation presented in (3.28), we can specifically discuss the scalar terms presented. The scalars Δ_C , Δ_D and Δ_E are the coefficients for the relaxation, dissipation and dispersion effects, respectively. The A , B and Λ terms represent nonlinear effects found in the system. If we were to ignore all these terms, Equations (3.28) and (3.18) become

$$U_\tau + CV_\eta = 0 \text{ and} \quad (4.1)$$

$$U_\eta = V_\chi, \quad (4.2)$$

which can be shown to be reducible to the quasi-plane wave approximation for a linearized system. Thus, the approach described here is at least consistent with the results of the well-known linear theory.

If, on the other hand, we ignore all transverse variations, i.e.

$$\frac{\partial}{\partial y} = 0, \quad (4.3)$$

then (3.18) requires that V is a constant, and (3.28) reduces to

$$U_\tau + \frac{\Lambda}{2} U^2 U_\chi = \Delta_C U + \Delta_D U_{\chi\chi} + \Delta_E U_{\chi\chi\chi}, \quad (4.4)$$

which is in complete agreement with the one-dimensional, $\Gamma=0$ theory developed by Cramer and Sen (1992).

Inspection of (3.27c) through (3.27g) reveals that Λ , Δ_C , Δ_D and Δ_E are all independent of $\underline{\delta}$ and therefore are identical to the scalars obtained in a one-dimensional, i.e., $\frac{\partial}{\partial y} = 0$ theory. Results for these scalars obtained in an analysis of the one-

dimensional theory can simply be carried over to the quasi-plane problems considered here. Furthermore, C is independent of any terms associated with the system nonlinearity. Thus, the results for C , and even Δ_C , Δ_D and Δ_E , are consistent with the linear theory that has been considered in previous work. The only terms which do not

have a counterpart in such simpler theories are the nonlinearity coefficients A and B . These terms represent a coupling between the nonlinearity and the transverse variations and must be computed even if results from one-dimensional or linear theories are already known.

At this point, we have an approximation for the solution vector \underline{u} which is valid out to times of order $L/\lambda\Delta^2$, which is the time at which nonlinearity and diffraction become dominant. We have thusly reduced, via approximation, the system of N first order equations (2.1) to a set of two equations, (3.28) and (3.18).

The point of the preceding chapters has been to present a detailed derivation of the modified Zabolotskaya-Khokhlov equation derived originally by Kluwick and Cox (1998) and Cramer and Webb (1998). In the remainder of this thesis we will apply these results to two systems of engineering interest, those being waves in a nonlinear dielectric and waves in a hyperelastic solid.

Part II:

The Nonlinear Dielectric

Chapter 5: Introduction

In Part II of this thesis, we will discuss Maxwell's equations as they pertain to the propagation of waves in a dielectric system. In order to be consistent with the theory presented in Chapters 2 and 3, we will make restrictions on the system so that it will be isotropic, constant in time, uniform, and has waves that propagate in two dimensions only. Also, we will consider that the system will have zero disturbance.

Maxwell's equations for a weakly conducting dielectric, also called an insulator, are presented in Chapter 6 of this part. There we also reduce these equations to their two-dimensional form. Nonlinearity and dispersion are introduced in Section 6.1 through the use of the dielectric constant. In Section 6.2, we make the assumption of small disturbances and long waves (or weak dispersion), which allows us to derive the final form of Maxwell's equations. The form obtained will then be seen to be exactly that of (2.1). In Chapter 7 we then use the results of Part I to derive the evolution equation governing weakly nonlinear, quasi-plane waves in nondispersive, perfectly insulating media. The effects of dispersion and conduction are then treated separately in Chapter 8.

Chapter 6: Formulation - Nonlinear Dielectrics

The propagation of waves through a nonlinear medium is governed by Maxwell's equations, which can be written (Schwartz, 1990):

$$\nabla \cdot \underline{D} = \rho, \quad (6.1a)$$

$$\nabla \cdot \underline{B} = 0, \quad (6.1b)$$

$$\nabla \times \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t}, \quad (6.1c)$$

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}, \quad (6.1d)$$

where \underline{E} is the electric field vector, \underline{D} is the dielectric displacement vector, \underline{H} is the magnetic field vector, \underline{B} is the magnetic induction vector, ρ is a scalar quantity representing the charge density, \underline{J} is the current density vector, and ∇ is the gradient operator. Equations (6.1a, b) are known as Gauss' Law for electric fields and magnetic fields, respectively. An alternate term for Equation (6.1b) is the condition of no magnetic monopoles. Ampere's Circuital Law is given in Equation (6.1c), and Faraday's Law is Equation (6.1d). These are all given in mks (meters, kilograms, seconds) units. In other units there would be the inclusion of a $4\pi\epsilon_0$ term.

The following constitutive relations are required in order to close the system (6.1):

$$\underline{D} = \epsilon_0 \kappa_e \underline{E}, \quad (6.2a)$$

$$\underline{B} = \mu_0 \kappa_m \underline{H}, \quad (6.2b)$$

$$\underline{J} = \sigma \underline{E}, \quad (6.2c)$$

where ϵ_0 is defined as the free-space dielectric constant, the constant μ_0 is the free-space magnetic permeability, κ_e is the relative dielectric constant, κ_m is the relative permeability, and σ is the conductivity. Equations (6.2) hold only for an isotropic material. Another point of note is that Equation (6.2c) is the vector form of Ohm's law.

We will assume that the medium is uniform and constant in time. We therefore take κ_e , κ_m and σ to be independent of both time and space. Further, we regard the medium to be neutral; the latter condition requires that we set ρ equal to zero. Because

the medium is regarded as non-magnetic, we set the parameter κ_m equal to one. These equations may then be simplified to the following form:

$$\nabla \cdot \underline{D} = 0, \quad (6.3a)$$

$$\nabla \cdot \underline{H} = 0, \quad (6.3b)$$

$$\nabla_x \underline{H} = \frac{\sigma}{\epsilon_0 \kappa_e} \underline{D} + \frac{\partial \underline{D}}{\partial t}, \quad (6.3c)$$

$$\nabla_x \underline{E} = -\mu_0 \frac{\partial \underline{H}}{\partial t}, \quad (6.3d)$$

exactly. Maxwell's equations have now been written in terms of the dielectric displacement, \underline{D} , and the magnetic field strength, \underline{H} , with the exception of (6.3d).

In this thesis, we will only consider the two-dimensional fields propagating in the dielectric. We therefore choose our coordinate system to be such that all variations in the z-direction go to zero. From here forth, then, the vectors will be defined as follows,

$$\underline{H} = (0, 0, H_z) = (0, 0, H), \quad (6.4a)$$

$$\underline{D} = (D_x, D_y, 0), \quad (6.4b)$$

where the H_z component of \underline{H} will simply be called H for notation purposes.

Noting the two-dimensional assumption that is being used for this case, the following equation can be determined by substituting the \underline{D} vector as shown in Equation (6.4b) into (6.3a) and again noting that all partial derivatives with respect to the z-component to go zero,

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} = 0. \quad (6.5)$$

In the same way, Equation (6.3b) combined with (6.4a) will yield the following,

$$\frac{\partial H}{\partial z} = 0, \quad (6.6)$$

which is satisfied automatically. Equation (6.5) will be used at a later time.

If we now substitute Equation (6.4a) into (6.3c) and take the curl of \underline{H} , the following equations for the x, y, and z components of Equation (6.3c) reduce to

$$\frac{\partial H}{\partial y} = \frac{\sigma}{\epsilon_0 \kappa_e} D_x + \frac{\partial D_x}{\partial t}, \quad (6.7a)$$

$$-\frac{\partial H}{\partial x} = \frac{\sigma}{\epsilon_0 \kappa_e} D_y + \frac{\partial D_x}{\partial t}, \quad (6.7b)$$

$$0 = \frac{\partial D_z}{\epsilon_0 \kappa_e} + \frac{\partial D_z}{\partial t}. \quad (6.7c)$$

Equation (6.7c) is automatically satisfied by requiring that the component D_z be equal to zero. We note that exactly the same result is found in the linear theory of electromagnetics. In other words, this is to say that the \underline{D} and \underline{H} vectors are orthogonal for plane waves propagating in an isotropic system.

At this point, we collect and summarize the two-dimensional form of Maxwell's equations. The resultant nontrivial equations can be written:

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} = 0, \quad (6.8a)$$

$$\frac{\partial D_x}{\partial t} - \frac{\partial H}{\partial y} = -\frac{\sigma}{\epsilon_0 \kappa_e} D_x, \quad (6.8b)$$

$$\frac{\partial D_y}{\partial t} + \frac{\partial H}{\partial x} = -\frac{\sigma}{\epsilon_0 \kappa_e} D_y, \quad (6.8c)$$

$$\mu_0 \left(\frac{\partial H}{\partial t} + \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) = 0. \quad (6.8d)$$

Equations (6.8a), (b), and (c) were given previously in Equations (6.5), (6.7a) and (6.7b), respectively. Equation (6.8d), however, is simply an expansion of Equation (6.3d) given the two-dimensional assumption. The system (6.8) is completed by the inclusion of the constitutive relation (6.2a). Because the effects of nonlinearity and dispersion are incorporated through κ_e , we have provided a detailed discussion of this material constant in the following subsections.

Section 6.1: Discussion of the Relative Dielectric Constant

In most introductory treatments of electromagnetism, the dielectric constant κ_e is taken to be a constant. However, as the wave frequency increases it is well known that κ_e can deviate from its static (zero frequency) value. Physically, this deviation is due to the

fact that the constituent molecules require a non-zero time to “polarize.” (Young, 1968) From a phenomenological standpoint, this frequency dependence of κ_e and therefore the speed of light is responsible for the separation of white light into its constituent colors by a prism. The standard model for this effect restricts attention to plane or quasi-plane wave trains and regards κ_e to be a function of either the frequency or the vector wavenumber, \underline{k} . For our purposes it is convenient to use the latter, such that we will write

$$\kappa_e = \kappa_e(\underline{k}). \quad (6.1.1)$$

We note that the medium has already been taken to be isotropic. As a result, κ_e must depend on \underline{k} only through its scalar invariant, $|\underline{k}|^2$. Thus, we will take

$$\kappa_e = \kappa_e(|\underline{k}|^2) \quad (6.1.2)$$

as the condition for a linear dispersive medium.

An example of a frequency dependent dielectric constant is given by Slater and Frank (1947) simply as

$$\kappa_e = 1 + \sum_i \frac{N_i e^2}{m \epsilon_0} \frac{1}{\omega_i^2 - \omega^2} \quad (6.1.3)$$

where i is summed over the number of interacting electrons, and (6.1.3) is given in mks units. In (6.1.3), N_i is the number of electrons per unit volume having a natural frequency of vibration ω_i , ω is the wave frequency, e is the electron charge, and m is the electron mass. When we are dealing with low frequencies, i.e. $\omega \ll \omega_i$, the dielectric constant can be considered to be its static value,

$$\kappa_e \approx 1 + \frac{e^2}{m \epsilon_0} \sum_i \frac{N_i}{\omega_i^2}. \quad (6.1.4)$$

However, at higher frequencies, we may approximate (6.1.3) as

$$\kappa_e \approx 1 + \frac{e^2}{m \epsilon_0} \sum_i \frac{N_i}{\omega_i^2} + \frac{\omega^2 e^2}{m \epsilon_0} \sum_i \frac{N_i}{\omega_i^4} + O(\omega^4), \quad (6.1.5)$$

which reveals that κ_e increases gradually with frequency. At frequencies corresponding to resonance ($\omega_i \approx \omega$), κ_e will vary rapidly. Under these conditions, very small damping not included in (6.1.3) is no longer negligible and results in absorption. At frequencies

well above any natural frequency, $\kappa_e < 1$ and decreases with increasing frequency. Here we only consider wave frequencies well below the lowest resonant frequency. As a result, $\kappa_e \approx \kappa_e(\omega = 0)$ and the dispersion that does occur will be weak.

An example of the dependence of κ_e on the vector wave number \underline{k} can be constructed through the use of (6.1.5). If we note that $\kappa_e \approx \text{constant}$, we may anticipate that the speed of light, c_0 , will also be approximately constant. Thus, the frequency and wave number can be related $\omega^2 \approx c_0^2 |\underline{k}|^2$ and (6.1.5) becomes

$$\kappa_e \approx 1 + \frac{e^2}{m\epsilon_0} \sum_i \frac{N_i}{\omega_i^2} + |\underline{k}|^2 \frac{c_0^2 e^2}{m\epsilon_0} \sum_i \frac{N_i}{\omega_i^4} + O(\omega^4). \quad (6.1.6)$$

A generalized version of (6.1.6) will be employed in Section 6.2.

From the point of view of continuum mechanics, Equation (6.1.2) is not in the form of a true constitutive relation due to the fact that the material properties appear to depend on the imposed disturbance. Furthermore, the classical discussion says nothing about nonperiodic or nonplanar waves. The appropriate interpretation of (6.1.2) is that \underline{D} depends not only on \underline{E} but on its spatial derivatives, i.e.,

$$\underline{D} = \underline{D}(\underline{E}, \nabla \underline{E}, \nabla \nabla \underline{E}, \dots), \quad (6.1.7)$$

where each dependency is taken to be consistent with a linear isotropic material.

Nonetheless, (6.1.2) is convenient for our purposes and we will use it in all that follows.

In Section 6.2 we will approximate (6.1.2) for the case of long waves, such that

$$|\underline{k}| \rightarrow 0. \quad (6.1.8)$$

At that stage we will convert our expression to one involving spatial derivatives by replacing \underline{k} with $-i \nabla$, where i is the square root of negative one. Such a transform is clearly heuristic in nature but is commonly used in many areas of classical physics.

In addition to dispersion, we will also allow for the presence of nonlinear effects, i.e., we allow for a variation of κ_e with the wave amplitude. As a result, we will take

$$\kappa_e = \kappa_e \left(|\underline{E}|^2, |\underline{k}|^2 \right) \quad (6.1.9)$$

as the final form of the dielectric constant. The use of $|\underline{E}|^2$ instead of \underline{E} is again due to the fact that we are only considering isotropic materials.

In the following section, we will first consider the case where the dielectric constant varies with the both the wavenumber \underline{k} and the \underline{E} field. This will allow us to express the electric field \underline{E} as a function of the derivatives of the dielectric constant with respect to both \underline{k} and \underline{E} and the dielectric displacement vector \underline{D} . Then we will have expressed the system completely in terms of the \underline{D} and \underline{H} fields, and in accordance with the results obtained in Chapter 3, we will be able to find an expression for the evolution equation.

Section 6.2: Derivation of the Final System of Equations

We will now explicitly incorporate our assumption of small disturbances and weak dispersion as presented in Chapter 2. These conditions will permit us to further simplify the two-dimensional form of Maxwell's equations derived earlier in this chapter. In particular, these conditions will permit us to replace the electric field vector \underline{E} by the dielectric displacement vector \underline{D} in Faraday's law (6.8d) in an explicit manner.

At this point we consider the expression for κ_e as given in (6.1.9). We note that the conditions of small disturbances ($\underline{E} \rightarrow 0$) and long waves ($\underline{k} \rightarrow 0$) permits us to expand (6.1.9) in the following Taylor series,

$$\begin{aligned} \kappa_e(|\underline{E}|^2, |\underline{k}|^2) = & \kappa_{e00} + \dot{\kappa}_e(0,0) |\underline{k}|^2 + \kappa'_e(0,0) |\underline{E}|^2 \\ & + O(|\underline{k}|^4, |\underline{k}|^2 |\underline{E}|^2, |\underline{E}|^4), \end{aligned} \quad (6.2.1)$$

where

$$\kappa_{e00} \equiv \kappa_e(0,0), \quad \dot{\kappa}_e \equiv \frac{\partial \kappa_e}{\partial |\underline{k}|^2}, \quad \kappa'_e \equiv \frac{\partial \kappa_e}{\partial |\underline{E}|^2}. \quad (6.2.2)$$

The constitutive relation (6.2a) may now be approximated

$$\underline{D} = \varepsilon_0 \kappa_{e00} \left[1 + \frac{\dot{\kappa}_e}{\kappa_e} \bigg|_{0,0} \underline{k} \cdot \underline{k} + \frac{\kappa'_e}{\kappa_e} \bigg|_{0,0} \underline{E} \cdot \underline{E} + O(|\underline{k}|^4, |\underline{k}|^2 |\underline{E}|^2, |\underline{E}|^4) \right] \underline{E}, \quad (6.2.3)$$

where we have written $|\underline{k}|^2 = \underline{k} \cdot \underline{k}$ and $|\underline{E}|^2 = \underline{E} \cdot \underline{E}$ for convenience. If we note that (6.2.3) implies that

$$\underline{E} \equiv \frac{\underline{D}}{\varepsilon_0 \kappa_{e00}} \quad (6.2.4)$$

to lowest order, we may now invert (6.2.3) to obtain

$$\underline{E} = \frac{\underline{D}}{\varepsilon_0 \kappa_{e00}} \left[1 - \frac{\kappa'_e}{\kappa_e} \bigg|_{0,0} \frac{\underline{k} \cdot \underline{k} - \frac{\kappa'_e}{\kappa_e} \bigg|_{0,0}}{\varepsilon_0^2 \kappa_{e00}^2} \frac{\underline{D} \cdot \underline{D}}{\varepsilon_0^2 \kappa_{e00}^2} + \mathcal{O}(|\underline{k}|^4, |\underline{k}|^2 |\underline{E}|^2, |\underline{E}|^4) \right]. \quad (6.2.5)$$

The use of the vector wavenumber \underline{k} has been both a traditional and a convenient way to represent the effect of dispersion in a dielectric. As pointed out in Section 6.1, any observed dependence on \underline{k} must be interpreted as resulting from a constitutive relation of the form (6.1.7). Now that we have the approximation (6.2.3) or (6.2.5), it is a relatively simple matter to deduce the dependence of \underline{D} on the gradients of \underline{E} or, conversely, the dependence of \underline{E} on \underline{D} and its gradients.

We begin by noting that the inclusion of \underline{k} originated with the restriction to a plane wave solution, e.g.,

$$\underline{D} = \underline{D}_0 e^{(k_x x + k_y y - \omega t)}, \quad (6.2.6)$$

where \underline{D}_0 is a constant amplitude vector, k_x and k_y are the x- and y-components of \underline{k} , ω is the frequency and $i \equiv \sqrt{-1}$. Differentiation of (6.2.6) yields

$$\frac{\partial^2 \underline{D}}{\partial x^2} = -k_x^2 \underline{D}, \text{ and} \quad (6.2.7a)$$

$$\frac{\partial^2 \underline{D}}{\partial y^2} = -k_y^2 \underline{D}, \quad (6.2.7b)$$

from which we conclude that

$$(k_x^2 + k_y^2) \underline{D} = -\nabla^2 \underline{D}, \quad (6.2.8)$$

where ∇^2 is the two-dimensional Laplacian operator. Because the wave is two-dimensional, $\underline{k} \cdot \underline{k} = k_x^2 + k_y^2$ and we conclude that the second term on the right-hand side of (6.2.5) can be replaced by a term proportional to $\nabla^2 \underline{D}$. Thus, we may write

$$\underline{E} = \frac{\underline{D}}{\varepsilon_0 \kappa_{e00}} \left[1 - \frac{\kappa'_e}{\kappa_e} \bigg|_{0,0} \frac{\underline{D} \cdot \underline{D}}{\varepsilon_0^2 \kappa_{e00}^2} + \frac{1}{\varepsilon_0 \kappa_{e00}} \frac{\kappa'_e}{\kappa_e} \bigg|_{0,0} \nabla^2 \underline{D} \right] + \frac{\underline{D}}{\varepsilon_0 \kappa_{e00}} \left[\mathcal{O}(|\underline{D}| |\underline{k}|^4, |\underline{D}|^3 |\underline{k}|^2, |\underline{D}|^5) \right]. \quad (6.2.9)$$

The approach to construct (6.2.9) is commonly employed in the study of linear dispersive waves. As pointed out in Section 6.1, the proper way to view (6.2.9) is that this is the correct approximation to the actual constitutive relation for long waves (weak gradients) and small disturbances. The original form (6.1.6) is simply a traditional form for the representation.

The approximate constitutive relation (6.2.9) now permits us to replace the electric field vector \underline{E} by \underline{D} in (6.8d). When the indicated differentiations are performed and we use (6.8a) to replace

$$\frac{\partial D_x}{\partial x} \text{ by } -\frac{\partial D_y}{\partial y}$$

wherever it occurs, we obtain the following simplification of (6.8d)

$$\begin{aligned} & \frac{\partial H}{\partial t} + c_0^2 \left[1 - \frac{D_x^2 + 3D_y^2}{\epsilon_0^2 \kappa_{e00}^2} \frac{\kappa'_e}{\kappa_e} \Big|_{0,0} \right] \frac{\partial D_y}{\partial x} - c_0^2 \left[1 - \frac{3D_x^2 + D_y^2}{\epsilon_0^2 \kappa_{e00}^2} \frac{\kappa'_e}{\kappa_e} \Big|_{0,0} \right] \frac{\partial D_x}{\partial y} \\ & + 4 \frac{c_0^2}{\epsilon_0^2 \kappa_{e00}^2} \frac{\kappa'_e}{\kappa_e} \Big|_{0,0} D_x D_y \frac{\partial D_y}{\partial y} = \\ & - c_0^2 \frac{\kappa'_e}{\kappa_e} \Big|_{0,0} \left[\frac{\partial^3 D_y}{\partial x^3} + 2 \frac{\partial^3 D_y}{\partial x \partial y^2} - \frac{\partial^3 D_x}{\partial y^3} \right] + \text{H.O.T.} \end{aligned} \quad (6.2.10)$$

where

$$c_0 \equiv \frac{1}{\sqrt{\mu_0 \epsilon_0 \kappa_{e00}}}, \quad (6.2.11)$$

and H.O.T. represents terms of higher order than the smallest of those shown here.

Equation (6.2.11) is recognized as the linearized, long-wave speed of light in the dielectric. If we now combine (6.2.10) with (6.8b) and (6.8c), we obtain a set of three equations for three variables, D_x , D_y and H . This system can be written in the matrix form

$$\begin{aligned} \underline{u}_t + \underline{A} \underline{u}_x + \underline{B} \underline{u}_y = \underline{C} \left(\underline{u} - \underline{u}^{(0)} \right) + \underline{E}^{(xxx)} \underline{u}_{xxx} + \underline{E}^{(xxy)} \underline{u}_{xxy} \\ + \underline{E}^{(xyy)} \underline{u}_{xyy} + \underline{E}^{(yyy)} \underline{u}_{yyy} \end{aligned} \quad (6.2.12)$$

which is essentially the same as (2.1) with $\underline{D}^{(xx)} = \underline{D}^{(xy)} = \underline{D}^{(yy)} = 0$. If we write the solution vector as

$$\underline{u} = \begin{Bmatrix} D_x \\ D_y \\ H \end{Bmatrix} \quad (6.2.13)$$

and $\underline{u}^{(0)} = \underline{0}$, we find that the three-by-three matrices in (6.2.12) are:

$$\underline{\underline{A}} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & c_0^2 \left[1 - \frac{D_x^2 + 3D_y^2}{\varepsilon_0^2 \kappa_{e00}^2} \frac{\kappa'_e}{\kappa_e} \right]_{0,0} & 0 \end{bmatrix}, \quad (6.2.14)$$

$$\underline{\underline{B}} \equiv \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -c_0^2 \left[1 + \frac{\kappa'_e}{\kappa_e} \right]_{0,0} \frac{3D_x^2 + D_y^2}{\varepsilon_0^2 \kappa_{e00}^2} & 4 \frac{c_0^2}{\varepsilon_0^2 \kappa_{e00}^2} \frac{\kappa'_e}{\kappa_e} \Big|_{0,0} D_x D_y & 0 \end{bmatrix}, \quad (6.2.15)$$

$$\underline{\underline{C}} \equiv \begin{bmatrix} -\frac{\sigma}{\varepsilon_0 \kappa_{e00}} & 0 & 0 \\ 0 & -\frac{\sigma}{\varepsilon_0 \kappa_{e00}} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (6.2.16)$$

$$\underline{\underline{E}}^{(xxx)} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -c_0^2 \frac{\dot{\kappa}_e}{\kappa_e} \Big|_{0,0} & 0 \end{bmatrix}, \quad (6.2.17)$$

$$\underline{\underline{E}}^{(xy)} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (6.2.18)$$

$$\underline{\underline{E}}^{(xyy)} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2c_0^2 \frac{\dot{\kappa}_e}{\kappa_e} \Big|_{0,0} & 0 \end{bmatrix}, \quad (6.2.19)$$

$$\underline{\underline{E}}^{(yyy)} \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_0^2 \frac{\dot{\kappa}_e}{\kappa_e} \Big|_{0,0} & 0 \end{bmatrix}, \quad (6.2.20)$$

where we have recognized that only the lowest order terms in $\underline{\underline{C}}, \underline{\underline{E}}^{(xxx)}, \underline{\underline{E}}^{(xxy)}$ etc. are required.

Chapter 7: Derivation of the Evolution Equation

In Chapter 6 we presented the equations governing nonlinear wave propagation in a dielectric that included effects of weak nonlinearity, conduction and dispersion. These equations were then rearranged into a matrix form similar to that discussed in Chapters 2 and 3. All assumptions from Chapter 2 will also hold here. We can begin with the governing equation, given by Equation (6.2.12), which includes both dispersion and conduction effects. Discussion of the eigenvalues, eigenvectors, and other needed vectors is independent of the particular case we are observing. From here, both cases where conduction and dispersion are present will be considered separately.

At this point, we can return to the discussion of the perturbation equations as given in Chapter 3. We begin by evaluating the eigenvalues as defined by (3.12). First we need to evaluate $\underline{\underline{A}}^{(0)}$, which is the value of the matrix $\underline{\underline{A}}$, as given in Equation (6.2.14), at its undisturbed state. If we set $\underline{D} = \underline{H} = 0$ in (6.2.14), we find that

$$\underline{\underline{A}}^{(0)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & c_0^2 & 0 \end{bmatrix}. \quad (7.1)$$

When $\underline{\underline{A}}^{(0)}$ is substituted in (3.12), the result for the eigenvalues is,

$$\lambda_{1,2,3} = c_0, -c_0, 0. \quad (7.2)$$

The case where λ is zero is a non-propagating mode and will be ignored in all that follows. The first two eigenvalues correspond to the left- and right-running electromagnetic waves traveling at the speed of light of the dielectric. We then use the first two eigenvalues to solve for the eigenvectors, \underline{r} and \underline{l} , as given by Equations (3.11) and (2.8b), respectively. The results are

$$\underline{r} = r_2 \begin{Bmatrix} 0 \\ 1 \\ \lambda \end{Bmatrix} \text{ and } \underline{l} = l_3 \begin{Bmatrix} 0 \\ \lambda \\ 1 \end{Bmatrix}. \quad (7.3)$$

The r_2 and l_3 factors in the eigenvectors are constants which are necessarily nonzero but are otherwise arbitrary. In (7.3), the eigenvalue may be taken to be $\pm c_0$.

Now we can perform the check that allows us to see if the system in fact will not have quadratic nonlinearities and is also self-consistent. The consistency for quasi-planar waves is given in Equation (2.7). The $\underline{\underline{B}}^{(0)}$ matrix corresponding to (6.2.15) is

$$\underline{\underline{B}}^{(0)} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -c_0^2 & 0 & 0 \end{bmatrix}. \quad (7.4)$$

Because \underline{r} and \underline{l} have already been determined, we can substitute them into (2.7) along with (7.4), and we will see that the result for $\underline{l}^T \underline{\underline{B}}^{(0)} \underline{r}$ is identically zero. At this point, we know that the self-consistency condition set forth in (2.7) is satisfied by our system.

The second check is to demonstrate that the nonlinearity of (6.2.12) is cubic rather than quadratic. To do this, we first need to compute the 3x3 matrix $\underline{\underline{G}}$ as defined in Equation (2.11). When this is done for (6.2.14), we find that $\underline{\underline{G}}$ is identically zero. As a result, the condition (2.10) is satisfied identically and the multiple scale scheme developed in Chapters 2 and 3 must be employed.

Our result that $\underline{\underline{G}}$ is identically zero is due to the fact that the elements of the $\underline{\underline{A}}$ matrix are either constant or are proportional to the squares of D_x and D_y . As a result, the nonzero derivatives of $\underline{\underline{A}}$ are proportional to D_x or D_y . Because we must evaluate the derivative of $\underline{\underline{A}}$ at its undisturbed state, all the derivatives of $\underline{\underline{A}}$ vanish. This is an unusual situation when calculating $\underline{\underline{G}}$, a fact which will become more evident when we examine the hyperelastic solid in Part III.

From Equations (3.21) and (3.22), we can solve for the $\underline{\delta}$ and $\underline{\gamma}$ vectors that will be needed in determining the coefficients in the evolution equation. Substituting the matrices given in (7.1) and (7.4) and the vectors in (7.3) into (3.21) and (3.22), and solving for the $\underline{\delta}$ and $\underline{\gamma}$ vectors, gives the following results,

$$\underline{\delta} = \left\{ \begin{array}{c} -r_2 \\ \delta_2 \\ \lambda \delta_2 \end{array} \right\} \text{ and } \underline{\gamma} = \gamma_2 \left\{ \begin{array}{c} 0 \\ 1 \\ \lambda \end{array} \right\}. \quad (7.5)$$

We can rewrite the $\underline{\delta}$ vector as follows,

$$\underline{\delta} = \begin{Bmatrix} -r_2 \\ 0 \\ 0 \end{Bmatrix} + \delta_2 \begin{Bmatrix} 0 \\ 1 \\ \lambda \end{Bmatrix}, \quad (7.6)$$

which shows that we can set δ_2 equal to zero without a loss of generality in the system. This is true because the part of the $\underline{\delta}$ vector that is multiplied by δ_2 is parallel to the right-hand eigenvector, \underline{r} , therefore making it part of the homogeneous solution for the \underline{u} vector. As pointed out in Chapter 3, any portion of either $\underline{\delta}$ or $\underline{\gamma}$ parallel with \underline{r} will have no influence on the final coefficients of the evolution equation. The same reasoning holds true for the $\underline{\gamma}$ vector as well, which is entirely parallel to the \underline{r} vector (i.e. has no particular part) and can therefore be set equal to zero without any loss of generality in the system. In retrospect, we realize that the reason the $\underline{\gamma}$ vector is strictly parallel to \underline{r} is that $\underline{G} = \underline{0}$ for this problem. As a result, the right-hand side of (3.21) vanishes identically, yielding a homogeneous equation for $\underline{\gamma}$.

The next step is to evaluate the scalar coefficients in the evolution equation (3.28). All vectors and matrices needed to accomplish this task have been given in this chapter, except for the \underline{H} matrix defined by (3.25). Upon evaluating this matrix, we obtain

$$\underline{H} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -3 \frac{\kappa'_e}{\kappa_e} \Big|_{0,0} & \frac{r_2^2}{\kappa_{e00}^3 \mu_0 \epsilon_0^3} \end{bmatrix}. \quad (7.7)$$

We may now compute the values of the scalars A, B, C, Λ as given in Equations (3.27a-d). After straightforward but tedious calculations, we find that these are given by:

$$A = 0, \quad (7.8)$$

$$B = 0, \quad (7.9)$$

$$C = \frac{1}{2}, \text{ and} \quad (7.10)$$

$$\Lambda = -3 \frac{\kappa'_e}{\kappa_e} \Big|_{0,0} \frac{r_2^2}{\kappa_{e00}^2 \epsilon_0^2}. \quad (7.11)$$

The reason that the scalar A and B terms vanish is that each term within the definitions for A and B themselves go to zero.

If we substitute (7.8) through (7.11) into the evolution equation (3.28) and ignore that presence of conduction and dispersion, the resultant evolution equation is

$$U \tau - \frac{3 \kappa'_e}{2 \kappa_e} \Big|_{0,0} - \frac{r_2^2}{\kappa_{e00}^2 \epsilon_0^2} U^2 \chi + \frac{1}{2} V \eta = 0. \quad (7.12)$$

As a check, we note that the one-dimensional version of (7.12) is in complete agreement with the results of Cramer and Sen (1992).

Now that we have determined the nondimensional form for the evolution equation (7.12), we can recast this in terms of the physical variables of the system by solving for the specific form of the \underline{u} vector given in (6.2.13). Combining the \underline{r} , $\underline{\delta}$ and $\underline{\chi}$ vectors as given in (7.3), (7.6) and (7.5) with the solutions for $\underline{u}^{(1)}$ and $\underline{u}^{(2)}$ given in (3.9) and (3.18) respectively, we can then solve for \underline{u} in (3.4), which gives the following,

$$\underline{u} = \begin{Bmatrix} D_x \\ D_y \\ H \end{Bmatrix} = \Delta \begin{Bmatrix} 0 \\ r_2 \\ \lambda r_2 \end{Bmatrix} U + \Delta^2 \begin{Bmatrix} -r_2 \\ 0 \\ 0 \end{Bmatrix} V + O(\Delta^3). \quad (7.13)$$

As pointed out earlier, the $\underline{\chi}$ vector is parallel to the \underline{r} vector and therefore can be ignored for the present purposes. For the same reason, we are also ignoring the homogeneous part of $\underline{\delta}$. This will give the following relationships between the components of \underline{u} and the shape functions U and V :

$$D_x = -\Delta^2 r_2 V + O(\Delta^3), \quad (7.14a)$$

$$D_y = \Delta r_2 U + O(\Delta^3), \quad (7.14b)$$

$$H = \Delta \lambda r_2 U + O(\Delta^3). \quad (7.14c)$$

Inspection of (7.14) reveals that the nonlinearity does not generate any new perturbations in \underline{D} and \underline{H} , whereas the effects of diffraction induce an $O(\Delta^2)$ perturbation in D_x . The latter would have been expected even in the simplest situations described by geometrical optics. In such a case the \underline{E} and \underline{D} vectors are always orthogonal to the direction of propagation. The presence of weak transverse variations modifies the direction of propagation which must therefore tilt the \underline{E} and \underline{D} vectors to produce small perturbations

in D_x . From a more mathematical point of view, one can also see that (6.8b) will always induce D_x perturbations whenever the magnetic field has variations in the y-direction.

We may also combine (7.14) with (7.12) to recast the evolution equation completely in terms of physical variables. To do this, we multiply (7.12) by $\Delta^2 r_2$ and use (7.14) and (3.1) to yield

$$\frac{\partial D_y}{\partial t} \mp \frac{c_0}{2} \frac{\partial D_x}{\partial y} = \pm \frac{3}{2} \frac{\kappa_e'}{\kappa_e} \bigg|_{0,0} \frac{c_0}{\kappa_e^2 \epsilon_0^2} D_y^2 \frac{\partial D_y}{\partial X}; \quad (7.15)$$

where (6.2.11) and (7.2) have been employed, and $X \equiv x - \lambda t$.

We can also recast the compatibility condition given in Equation (6.5) in terms of the physical variables by using the condition set forth in (3.19), which is

$$U_\eta = V_\chi$$

If we take the χ derivative of Equation (7.14a) and the η derivative of (7.14b), and then rearrange using algebra, we will arrive at the following equation,

$$-\frac{1}{\Delta} (D_x)_\chi = (D_y)_\eta. \quad (7.16)$$

Now, if we use the variable transformations given in Equation (3.1) to convert (7.16), we can work backwards yield the following, in terms of x, y and t only,

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} = 0, \quad (7.17)$$

which is recognized as (6.5). Thus, the compatibility condition between U and V, given in (3.19), is completely consistent with the original governing equations. It also reflects the fact that our three-by-three system (6.2.12) is self-consistent only if we include the fourth Maxwell equation (6.8a).

Equations (7.15) and (7.17) comprise two equations for the two components of \underline{D} . Once solved, the magnetic field vector is obtained by eliminating $U\Delta r_2$ between (7.14b) and (7.14c). The result is that $H \approx \pm c_o D_y$, which is identical to the plane wave result. Since we have now given a general outline for the solution of the \underline{u} vector as well as the evolution equation for the dielectric, we will now move on to cases where there are conduction and dispersion effects present in the system. The evolution equation for the systems with such characteristics will be presented in the following chapter.

Chapter 8: Inclusion of Conduction and Dispersion

In this section we determine the modification to the evolution equation (7.12) required when conduction and dispersion are non-negligible effects. These cases must be considered separately, as will be explained further in this chapter. First, we can consider the case where conduction is present in the system, and will then proceed on to the discussion of dispersion effects.

If we include only conduction, the system (6.2.12) reduces to,

$$\underline{u}_t + \underline{A} \underline{u}_x + \underline{B} \underline{u}_y = \underline{C} (\underline{u} - \underline{u}^{(0)}). \quad (8.1)$$

In order to non-dimensionalize the \underline{C} matrix, we perform a transformation as given in Equation (3.6a). This will give a $\hat{\underline{C}}$ matrix as

$$\hat{\underline{C}} = \begin{bmatrix} -\hat{\sigma} & 0 & 0 \\ 0 & -\hat{\sigma} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (8.2)$$

where the scaled conductivity, $\hat{\sigma}$, is defined,

$$\hat{\sigma} = \frac{\sigma L}{\epsilon_0 \kappa_{e00} \lambda \Delta^2} = O(1). \quad (8.3)$$

The fact that $\hat{\sigma}$ is $O(1)$ is equivalent to our assumption, made in Chapter 2, that a suitably defined \underline{C} matrix is of order Δ^2 . That is, it is equivalent to requiring that the effects of relaxation modulate the pulse at the same rate as nonlinearity and diffraction. It is also easily verified that $\hat{\sigma}$ is a non-dimensional quantity.

We have already found the eigenvectors and eigenvalues for the general system as shown in Chapter 7. As shown in Equation (3.28), the addition of conduction to the system will give one additional term on the right-hand side of the evolution equation proportional to Δ_c . The latter quantity is defined in general by (3.27e). Upon evaluating this coefficient and substituting it back into the evolution equation, we find that

$$\Delta_c = -\frac{\hat{\sigma}}{2}, \text{ which yields the following extension of (7.12),}$$

$$U_\tau - \frac{3 \kappa'_e}{2 \kappa_e} \Big|_{0,0} - \frac{r_2^2}{\kappa_{e00}^2 \epsilon_0^2} U^2 U_x + \frac{1}{2} V_\eta = -\frac{\hat{\sigma}}{2} U, \quad (8.4)$$

which is the evolution equation for a wave affected by conduction in a nonlinear dielectric. This evolution equation can be cast in physical variables as an extension of (7.15). This will give an extra term on the right-hand side, as seen in the following,

$$\frac{\partial D_y}{\partial t} \mp \frac{c_0}{2} \frac{\partial D_x}{\partial y} = \pm \frac{3 \kappa'_e}{2 \kappa_e} \Big|_{0,0} - \frac{c_0}{\kappa_{e00}^2 \epsilon_0^2} D_y^2 \frac{\partial D_y}{\partial X} - \frac{\sigma}{2 \epsilon_0 \kappa_{e00}} D_y. \quad (8.5)$$

We see that the inclusion of conduction here is in terms of its unscaled quantity, σ .

Now that we have considered the case where conduction is present in the system, we can similarly solve for the evolution equation with dispersion included. We must ignore the presence of conduction in this case, since conduction will be noticeable at short wavelengths in weakly conducting insulators. On the other hand, the weak dispersion of interest here is seen in the presence of long waves.

The same general procedure for finding the evolution equation is employed as described in Chapter 3. If we refer back to Chapter 3, we can recall that only the $\underline{\underline{E}}^{(xxx)}$ matrix will contribute once the matrices are scaled properly. This dispersion matrix is given in Equation (6.2.17). The eigenvalues and eigenvectors will remain the same as those presented in Chapter 7. The coefficient of the term that will determine dispersion in the evolution equation is given in Equation (3.27g) once $\underline{\underline{E}}^{(xxx)}$ has been scaled according to (3.6c). We will give the $\hat{\underline{\underline{E}}}^{(xxx)}$ as follows,

$$\hat{\underline{\underline{E}}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\hat{\beta} & 0 \end{bmatrix}, \quad (8.6)$$

where

$$\hat{\beta} = \frac{c_0^2}{L^2 \lambda \Delta^2} \frac{\kappa'_e}{\kappa_e} \Big|_{0,0} = O(1). \quad (8.7)$$

Upon evaluating the Δ_E coefficient and substituting in into the general evolution equation, (3.28), the following is the result,

$$U_\tau - \frac{3}{2} \frac{\kappa'_e}{\kappa_e} \Big|_{0,0} - \frac{r_2^2}{\kappa_{e00}^2 \epsilon_0^2} U^2 U_\chi + \frac{1}{2} V_\eta = -\frac{\hat{\beta}}{2\lambda} U_{\chi\chi\chi}. \quad (8.8)$$

Given the evolution equation in (8.8), we can again recast this equation in terms of the physical variables of the system, which will give

$$\frac{\partial D_y}{\partial t} \mp \frac{c_0}{2} \frac{\partial D_x}{\partial y} = \pm \frac{3}{2} \frac{\kappa'_e}{\kappa_e} \Big|_{0,0} - \frac{c_0}{\kappa_{e00}^2 \epsilon_0^2} D_y^2 \frac{\partial D_y}{\partial X} \mp \frac{c_0}{2} \frac{\kappa'_e}{\kappa_e} \Big|_{0,0} \frac{\partial^3 D_y}{\partial X^3}, \quad (8.9)$$

where the $\hat{\beta}$ term has been substituted from (8.7) to give (8.9).

Chapter 9: Summary

In Part II we demonstrated that the nonlinearity the dielectric exhibits is cubic rather than quadratic in nature, due to the fact that the $\Gamma=0$ condition was satisfied. This result could have been anticipated once we observe that the terms present in the \underline{A} and \underline{B} matrices are all second order in terms of the components of \underline{D} , and then are multiplied by a derivative of \underline{D} , which can be seen in Equation (6.2.10). Thus, the original system of equations involves only cubically nonlinear terms. The result that $\Gamma=0$ is therefore not surprising.

We also found that the nonlinearity coefficients (3.27a)-(3.27b) were identically zero. As a result, only one non-zero coefficient related to the nonlinearity remains. As already discussed in Part I, that last coefficient (Λ) is identical to that computed in the one-dimensional theory. The fact that A and B are zero is in contrast with the result found by Cramer and Webb (1998) for the case of quasi-plane Alfvén waves, where it was found that A and B are non-zero. As a result, the evolution equations for Alfvén waves and waves in a nonlinear dielectric are fundamentally different. Furthermore, our results serve to point out the fact that the nonlinearity coefficients A and B may or may not be zero depending on the particular physical system.

The evolution equation for negligible conduction and negligible dispersion are given by (7.15) and (7.17) in physical variables. The solution for H is given by (7.18). Other variables such as \underline{B} and \underline{E} can be found through the use of the constitutive relations (6.2b) and (6.2.5).

The effects of weakly conducting and weakly dispersive waves have also been incorporated. The equation governing waves in weakly conducting dielectrics is valid for relatively short waves and is given by (8.5) and (7.17) in physical variables. The dispersion is weak when the waves are long but finite. The resultant evolution equation is given by (8.7) and (7.17).

The case of the nonlinear dielectric was seen to be relatively simple. In Part III of this thesis we examine the more complicated problem of the hyperelastic solid.

Part III:

The Hyperelastic Solid

Chapter 10: Introduction

In Part II of this thesis, we showed how the perturbation scheme described in Chapter 3 could be used to derive the evolution equation governing waves in a nonlinear dielectric. In this part, we will provide a second, more complicated example by examining waves in a hyperelastic solid. This physical system is the most complicated application of the multi-timing scheme studied yet. An advantage of studying the hyperelastic solid is that it supports two distinct, nontrivial modes; these are nonlinear versions of the well-known shear and longitudinal modes found in the theory of linear, isotropic elasticity.

Similar to the case for the nonlinear dielectric, we will consider here only two-dimensional motions from a zero undisturbed state; that is, the undisturbed state will be taken to be strain-free. In Chapter 11, we discuss the equilibrium and mass equations for a general solid, and simplify these equations for the special case of two-dimensional motions. In Chapter 12 we then complete the formulation by introducing the constitutive relation for the hyperelastic solid. In the remaining chapters of Part III we will apply the multi-timing scheme described in Part I of this thesis. We will first verify that the nonlinearity of the longitudinal mode is quadratic whereas the nonlinearity of the shear mode is cubic. We then restrict our attention to the latter non-classical case and derive the modified Zabolotskaya-Khokhlov governing equation for quasi-plane shear waves.

Chapter 11: Formulation- Hyperelastic Solids

As discussed in Chapter 10, we begin by considering the equilibrium equation. The equilibrium equation (Spencer, 1980) which governs the motion of particles in the hyperelastic solid is

$$\rho \ddot{u}_i = \frac{\partial \sigma_{ij}}{\partial x_j}, \quad (11.1)$$

where u_i are the three components of the position vector, with the dot notation denoting the second derivative with respect to time, ρ is the density of the material, σ_{ij} are the components of the stress tensor, and x_j are the Eulerian coordinates of a material particle. The subscripts on σ_{ij} , u_i and x_j will range from 1 to 3 and the Einstein summation convention will again be used.

The equation expressing conservation of mass can be written as follows (Spencer, 1980),

$$\rho = \frac{\rho_0}{F}, \quad (11.2)$$

where ρ_0 is the density at the undeformed configuration, and F is the determinant of the deformation matrix, F_{ij} . The latter matrix is defined as

$$F_{ij} = \frac{\partial x_i}{\partial X_j}, \quad (11.3)$$

where X_i is the Cartesian coordinate of a material particle in the undeformed configuration; normally, we would refer to X_i as the Lagrangian coordinates of the particle. We will cast the deformation matrix in a more useable form by using the following definition,

$$u_i (X_p, t) = x_i - X_i, \quad (11.4)$$

where u_i shows an element of the displacement vector as given above. Combining (11.3) and (11.4) will give the following,

$$F_{ij} = \delta_{ij} + \left. \frac{\partial u_i}{\partial X_j} \right|_t, \quad (11.5)$$

where δ_{ij} is the Kronecker delta. This definition for F_{ij} will be used later, but was convenient to write at this time for clarification.

Returning to Equation (11.2), we can use ρ as described in this equation and substitute it into (11.1). The result is the following,

$$\rho_0 \ddot{u}_i = F \frac{\partial \sigma_{ji}}{\partial x_j}. \quad (11.6)$$

We can then expand the partial derivative by noting that

$$\frac{\partial X_p}{\partial x_j} \frac{\partial \sigma_{ji}}{\partial X_p} = \frac{\partial \sigma_{ji}}{\partial x_j} \quad (11.7)$$

from simply using the chain rule. We then arrive at the following equation, which is a combination of (11.6) and (11.7),

$$\rho_0 \ddot{u}_i = F \frac{\partial X_p}{\partial x_j} \frac{\partial \sigma_{ji}}{\partial X_p}. \quad (11.8)$$

As in the case of the dielectric of Part II, we will simplify (11.8) by considering only two-dimensional motions. The motion mapping $x_i = x_i(X_p, t)$ therefore reduces to

$$x_1 = X_1 + u_1(X_1, X_2, t), \quad (11.9a)$$

$$x_2 = X_2 + u_2(X_1, X_2, t), \quad (11.9b)$$

$$x_3 = X_3. \quad (11.9c)$$

From (11.9c) we can see that there will be no displacements in the z-direction. This will imply that all partial derivatives with respect to X_3 will go to zero, with the exception of the partial derivative of x_3 with respect to X_3 , which is obviously one. We may also show that (11.9) implies that there will be no stress changes in the X_3 direction, meaning that all derivatives of σ_{ij} with respect to X_3 will also be zero.

Now that the restriction to two-dimensional motions has been introduced, we can simplify Equation (11.8) and cast it in the matrix form identical to that used in Chapter 2. The first step in the simplification process is to express F in terms of the permutation symbol, ε_{ijk} , where the indices will again range from 1 to 3. This can be done as follows,

$$\varepsilon_{ijk} \det(F_{ij}) = \varepsilon_{lmn} F_{il} F_{jm} F_{kn}, \quad (11.10)$$

which is just the definition of a determinant in indicial form. (Hunter,1976) Using Equation (11.3), we can now write (11.10) as

$$\varepsilon_{ijk} \det(F_{ij}) = \varepsilon_{pmn} \frac{\partial x_i}{\partial X_p} \frac{\partial x_j}{\partial X_m} \frac{\partial x_k}{\partial X_n}. \quad (11.11)$$

If we then multiply both sides of (11.11) with

$$\frac{\partial X_a}{\partial x_i}, \quad (11.12)$$

and employ the Kronecker delta, expressed as

$$\delta_{ab} = \frac{\partial X_a}{\partial X_b} = \frac{\partial X_a}{\partial x_i} \frac{\partial x_i}{\partial X_b}, \quad (11.13)$$

where the indices on the Kronecker delta each range from 1 to 3, we will now have Equation (11.11) in the following form,

$$\frac{\partial X_a}{\partial x_i} \varepsilon_{ijk} F = \varepsilon_{amn} \frac{\partial x_j}{\partial X_m} \frac{\partial x_k}{\partial X_n}, \quad (11.14)$$

where F is the determinant of F_{ij} . In (11.14) we have employed the fact that the Kronecker delta, when multiplied by the permutation symbol, simply changes the indicial notation, as in the following (Spencer, 1980)

$$\varepsilon_{lmn} \delta_{al} = \varepsilon_{amn}. \quad (11.15)$$

Upon multiplying (11.14) with ε_{pij} and noting the following facts (Spencer,1980),

$$\varepsilon_{pij} \varepsilon_{ijk} = 2\delta_{pi}, \quad \text{and} \quad (11.16a)$$

$$\delta_{pi} \frac{\partial X_a}{\partial x_i} = \frac{\partial X_a}{\partial x_p}, \quad (11.16b)$$

we now have the following,

$$\frac{\partial X_a}{\partial x_p} F = \frac{1}{2} \varepsilon_{pij} \varepsilon_{amn} \frac{\partial x_j}{\partial X_m} \frac{\partial x_k}{\partial X_n}. \quad (11.17)$$

This can now be substituted into the original form of the equilibrium equation, (11.8).

We note that the dummy variables in (11.17) have been changed in order to perform this step. The expanded form of the equilibrium equation in terms of the permutation operator will therefore be written as follows,

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \frac{1}{2} \varepsilon^{jkm} \varepsilon^{prs} \frac{\partial x_k}{\partial X_r} \frac{\partial x_m}{\partial X_s} \frac{\partial \sigma_{ji}}{\partial X_p}. \quad (11.18)$$

From here, we can sum on the permutation indicies $j, k, m, p, r,$ and s . Again, the range will be from 1 to 3, since we are dealing with a three dimensional coordinate system. This will give us the correct signs in the following expression, as determined by the permutation operations. However, this is a time consuming process, so the actual summation process will not be shown here. Upon performing the summation, the result is as follows,

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \frac{1}{2} \left\{ \frac{\partial \sigma_{1i}}{\partial X_1} \left[\frac{\partial x_2}{\partial X_2} - \frac{\partial x_2}{\partial X_1} \right] + \frac{\partial \sigma_{2i}}{\partial X_1} \left[\frac{\partial x_1}{\partial X_1} - \frac{\partial x_1}{\partial X_2} \right] \right. \\ \left. + \frac{\partial \sigma_{1i}}{\partial X_2} \left[\frac{\partial x_2}{\partial X_2} - \frac{\partial x_2}{\partial X_1} \right] + \frac{\partial \sigma_{2i}}{\partial X_2} \left[\frac{\partial x_1}{\partial X_1} - \frac{\partial x_1}{\partial X_2} \right] \right\}. \quad (11.19)$$

We will now make the following definition, which is

$$e_{ij} \equiv \frac{\partial u_i}{\partial X_j}. \quad (11.20)$$

We see, from the two-dimensional assumption as well as (11.9), that $e_{3i}=e_{i3}=0$. Despite the notation, this definition should not be confused with the definition for strain. Also, it is important to note that e_{12} and e_{21} are not equivalent. Equation (11.20) will be beneficial in keeping the notation concise, and will be used in the remainder of this work. If we rearrange (11.4), we can show that

$$x_i = u_i + X_i, \quad (11.21)$$

which further gives the following, when substituting Equation (11.20) into (11.21);

$$\frac{\partial x_2}{\partial X_2} = 1 + \frac{\partial u_2}{\partial X_2} = 1 + e_{22}, \quad (11.22a)$$

$$\frac{\partial x_2}{\partial X_1} = \frac{\partial u_2}{\partial X_1} = e_{21}, \quad (11.22b)$$

$$\frac{\partial x_1}{\partial X_1} = 1 + \frac{\partial u_1}{\partial X_1} = 1 + e_{11}, \quad (11.22c)$$

$$\frac{\partial x_1}{\partial X_2} = \frac{\partial u_1}{\partial X_2} = e_{12}, \quad (11.22d)$$

$$\frac{\partial x_1}{\partial X_3} = \frac{\partial u_1}{\partial X_3} = e_{13}, \quad (11.22e)$$

$$\frac{\partial x_2}{\partial X_3} = \frac{\partial u_2}{\partial X_3} = e_{23}, \quad (11.22f)$$

$$\frac{\partial x_3}{\partial X_3} = 1, \quad (11.22g)$$

$$\frac{\partial x_3}{\partial X_1} = \frac{\partial u_3}{\partial X_1} = e_{31}, \quad (11.22h)$$

$$\frac{\partial x_3}{\partial X_2} = \frac{\partial u_3}{\partial X_2} = e_{32}. \quad (11.22i)$$

The equilibrium equation (11.19) can now be written as

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{1i}}{\partial X_1} (1 + e_{22}) - \frac{\partial \sigma_{2i}}{\partial X_1} (e_{12}) + \frac{\partial \sigma_{2i}}{\partial X_2} (1 + e_{11}) - \frac{\partial \sigma_{1i}}{\partial X_2} (e_{21}). \quad (11.23)$$

At this point, we will want to get all derivatives of σ_{ij} in terms of e_{ij} as given in (11.20), since we will be defining the solution vector of (2.1) as follows,

$$\underline{u} = \begin{Bmatrix} e_{11} \\ e_{12} \\ e_{21} \\ e_{22} \\ v_1 \\ v_2 \end{Bmatrix}. \quad (11.24)$$

Here, we will employ the following definitions,

$$v_1 \equiv \frac{\partial u_1}{\partial t}, \quad (11.25a)$$

$$v_2 \equiv \frac{\partial u_2}{\partial t}, \quad (11.25b)$$

where the u_1 and u_2 are displacements, and not components of the solution vector. Using the chain rule again, all partial derivatives of σ_{ij} with respect to X_i are split up as follows,

$$\begin{aligned} \frac{\partial \sigma_{pi}}{\partial X_q} &= \frac{\partial \sigma_{pi}}{\partial e_{11}} \frac{\partial e_{11}}{\partial X_q} + \frac{\partial \sigma_{pi}}{\partial e_{12}} \frac{\partial e_{12}}{\partial X_q} + \frac{\partial \sigma_{pi}}{\partial e_{21}} \frac{\partial e_{21}}{\partial X_q} + \frac{\partial \sigma_{pi}}{\partial e_{22}} \frac{\partial e_{22}}{\partial X_q} \\ &+ \frac{\partial \sigma_{pi}}{\partial v_1} \frac{\partial v_1}{\partial X_q} + \frac{\partial \sigma_{pi}}{\partial v_2} \frac{\partial v_2}{\partial X_q}, \end{aligned} \quad (11.26)$$

where all partial derivatives with respect to v_1 and v_2 go to zero since the stress tensor has no dependence on these velocity terms. This expansion is performed for each of the partial derivatives of σ_{ij} in (11.23), resulting in an equivalent equation for the equilibrium equation as presented in Equation (11.1) in terms of the index i . These versions of σ_{ij} obtained from (11.26) can then be substituted into Equation (11.23) to give the two equilibrium equations that govern the system.

Using the definition of the particle velocity as given by Equations (11.25), we can construct the following compatibility conditions:

$$\frac{\partial e_{11}}{\partial t} = \frac{\partial}{\partial X_1} \frac{\partial u_1}{\partial t} = \frac{\partial v_1}{\partial X_1}, \quad (11.27a)$$

$$\frac{\partial e_{12}}{\partial t} = \frac{\partial v_1}{\partial X_2}, \quad (11.27b)$$

$$\frac{\partial e_{21}}{\partial t} = \frac{\partial v_2}{\partial X_1}, \quad (11.27c)$$

$$\frac{\partial e_{22}}{\partial t} = \frac{\partial v_2}{\partial X_2}. \quad (11.27d)$$

Since the vector \underline{u} includes velocity terms, we note that displacements and velocities are simply related, as follows,

$$\frac{\partial^2 u_1}{\partial t^2} = \frac{\partial}{\partial t} \left(\frac{\partial u_1}{\partial t} \right) = \frac{\partial v_1}{\partial t}, \quad (11.28a)$$

$$\frac{\partial^2 u_2}{\partial t^2} = \frac{\partial v_2}{\partial t}, \quad (11.28b)$$

where the v_1 and v_2 terms are given in Equations (11.25). We can use this velocity definition in Equation (11.23) to obtain the following matrix form for the system,

$$\underline{u}_t + \underline{A} \underline{u}_{x_1} + \underline{B} \underline{u}_{x_2} = \underline{0}, \quad (11.29)$$

where we have divided through by ρ_0 in (11.29) and incorporated the division inside of the $\underline{\underline{A}}$ and $\underline{\underline{B}}$ matrices. The system will then become, in full form,

$$\begin{aligned}
 \left. \begin{array}{c} e_{11} \\ e_{12} \\ e_{21} \\ e_{22} \\ v_1 \\ v_2 \end{array} \right\}_t + \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ A_{51} & 0 & A_{53} & 0 & 0 & 0 \\ A_{61} & 0 & A_{63} & 0 & 0 & 0 \end{array} \right] \left. \begin{array}{c} e_{11} \\ e_{12} \\ e_{21} \\ e_{22} \\ v_1 \\ v_2 \end{array} \right\}_{X_1} \\
 + \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ B_{51} & B_{52} & B_{53} & B_{54} & 0 & 0 \\ B_{61} & B_{62} & B_{63} & B_{64} & 0 & 0 \end{array} \right] \left. \begin{array}{c} e_{11} \\ e_{12} \\ e_{21} \\ e_{22} \\ v_1 \\ v_2 \end{array} \right\}_{X_2} = 0, \tag{11.30}
 \end{aligned}$$

where the following have been employed in order to make the $\underline{\underline{A}}$ matrix as sparse as possible,

$$\frac{\partial e_{12}}{\partial X_1} = \frac{\partial e_{11}}{\partial X_2}, \tag{11.31a}$$

$$\frac{\partial e_{21}}{\partial X_1} = \frac{\partial e_{21}}{\partial X_2}. \tag{11.31b}$$

The relations given in (11.31) arise from the definition (11.20) and have permitted us to shift certain elements of the $\underline{\underline{A}}$ matrix to the $\underline{\underline{B}}$ matrix. This explains why the $\underline{\underline{B}}$ matrix is more full than the $\underline{\underline{A}}$ in Equation (11.30). The elements of $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are given as follows;

$$A_{51} = -\frac{1}{\rho_0} \left[(1 + e_{22}) \frac{\partial \sigma_{11}}{\partial e_{11}} - e_{12} \frac{\partial \sigma_{21}}{\partial e_{11}} \right], \tag{11.32a}$$

$$A_{53} = -\frac{1}{\rho_0} \left[(1 + e_{22}) \frac{\partial \sigma_{11}}{\partial e_{21}} - e_{12} \frac{\partial \sigma_{21}}{\partial e_{21}} \right], \tag{11.32b}$$

$$A_{61} = -\frac{1}{\rho_0} \left[(1 + e_{22}) \frac{\partial \sigma_{12}}{\partial e_{11}} - e_{12} \frac{\partial \sigma_{22}}{\partial e_{11}} \right], \tag{11.32c}$$

$$A_{63} = -\frac{1}{\rho_0} \left[(1+e_{22}) \frac{\partial \sigma_{12}}{\partial e_{21}} - e_{12} \frac{\partial \sigma_{22}}{\partial e_{21}} \right], \quad (11.32d)$$

$$B_{51} = -\frac{1}{\rho_0} \left[(1+e_{22}) \frac{\partial \sigma_{11}}{\partial e_{12}} - e_{12} \frac{\partial \sigma_{21}}{\partial e_{12}} + (1+e_{11}) \frac{\partial \sigma_{21}}{\partial e_{11}} - e_{21} \frac{\partial \sigma_{11}}{\partial e_{11}} \right], \quad (11.32e)$$

$$B_{52} = -\frac{1}{\rho_0} \left[(1+e_{11}) \frac{\partial \sigma_{21}}{\partial e_{12}} - e_{12} \frac{\partial \sigma_{11}}{\partial e_{12}} \right], \quad (11.32f)$$

$$B_{53} = -\frac{1}{\rho_0} \left[(1+e_{22}) \frac{\partial \sigma_{11}}{\partial e_{22}} - e_{12} \frac{\partial \sigma_{21}}{\partial e_{22}} + (1+e_{11}) \frac{\partial \sigma_{21}}{\partial e_{21}} - e_{21} \frac{\partial \sigma_{11}}{\partial e_{21}} \right], \quad (11.32g)$$

$$B_{54} = -\frac{1}{\rho_0} \left[(1+e_{11}) \frac{\partial \sigma_{21}}{\partial e_{22}} - e_{21} \frac{\partial \sigma_{11}}{\partial e_{22}} \right], \quad (11.32h)$$

$$B_{61} = -\frac{1}{\rho_0} \left[(1+e_{22}) \frac{\partial \sigma_{12}}{\partial e_{12}} - e_{12} \frac{\partial \sigma_{22}}{\partial e_{12}} + (1+e_{11}) \frac{\partial \sigma_{22}}{\partial e_{11}} - e_{21} \frac{\partial \sigma_{12}}{\partial e_{11}} \right], \quad (11.32i)$$

$$B_{62} = -\frac{1}{\rho_0} \left[(1+e_{11}) \frac{\partial \sigma_{22}}{\partial e_{12}} - e_{21} \frac{\partial \sigma_{12}}{\partial e_{12}} \right], \quad (11.32j)$$

$$B_{63} = -\frac{1}{\rho_0} \left[(1+e_{22}) \frac{\partial \sigma_{12}}{\partial e_{22}} - e_{12} \frac{\partial \sigma_{22}}{\partial e_{22}} + (1+e_{11}) \frac{\partial \sigma_{22}}{\partial e_{21}} - e_{21} \frac{\partial \sigma_{12}}{\partial e_{21}} \right], \quad (11.32k)$$

$$B_{64} = -\frac{1}{\rho_0} \left[(1+e_{11}) \frac{\partial \sigma_{22}}{\partial e_{22}} - e_{21} \frac{\partial \sigma_{12}}{\partial e_{22}} \right]. \quad (11.32l)$$

Now the system that governs the motion of the hyperelastic solid has been described in matrix form. Equation (11.30) is exact for a two-dimensional system as we have described here, since no smallness assumptions have yet been introduced. In the next chapter, we will write out the constitutive relation for a hyperelastic solid and write out σ_{ij} explicitly in terms of e_{ij} for two-dimensional motions.

Chapter 12: Stress-Strain Relations

In this chapter, we complete the description of the solid by specifying the stress-strain relation. In particular, this will permit us to write out $\sigma_{ij}=\sigma_{ij}(e_{lm})$ explicitly. As a result, the elements of the coefficient matrices seen in Equations (11.32) can be written out explicitly.

The stress tensor of a hyperelastic solid can be given as the following, (Hunter,1976)

$$\sigma_{ij}=\alpha_0\delta_{ij}+\alpha_1B_{ij}+\alpha_2B_{il}B_{lj}, \quad (12.1)$$

where B_{ij} is the left-Cauchy Green tensor, which can be written in terms of the deformation matrix as

$$B_{ij}\equiv F_{il}F_{jl}, \quad (12.2)$$

which is a symmetric matrix. The α_0 , α_1 and α_2 coefficients are functions of the invariants of the of B_{ij} , called I, II, and III. For the two-dimensional deformations defined in Chapter 11, these invariants can be expressed as follows

$$I=B_{ii}=B_{11}+B_{22}+1, \quad (12.3a)$$

$$\begin{aligned} II &= \frac{1}{2} \left[I^2 - (B_{ij}B_{ji}) \right] \\ &= B_{11}+B_{22}+B_{11}B_{22}-B_{12}^2, \end{aligned} \quad (12.3b)$$

$$III = \det(B_{ij})=B_{11}B_{22}-B_{12}^2, \quad (12.3c)$$

where the B_{ij} matrix, as given in (12.2), is

$$\underline{\underline{B}} = \begin{bmatrix} (1+e_{11})^2+e_{12}^2 & e_{21}(1+e_{11})+e_{12}(1+e_{22}) & 0 \\ e_{21}(1+e_{11})+e_{12}(1+e_{22}) & (1+e_{22})^2+e_{21}^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (12.4)$$

for the two-dimensional motions of interest here. In the undeformed configuration, $F_{ij}=\delta_{ij}$, $B_{ij}=\delta_{ij}$ so that σ_{ij} goes to zero. This will then require that the following equation be satisfied,

$$\alpha_0+\alpha_1+\alpha_2=0 \quad (12.5)$$

in the undeformed configuration.

It will be convenient to define the invariants in terms of their deviations from the undisturbed state. We will therefore define a new matrix,

$$\underline{\tilde{B}} = \underline{B} - \underline{I}, \quad (12.6)$$

where the non-zero components of \tilde{B}_{ij} are given as

$$\tilde{B}_{11} = 2e_{11} + e_{11}^2 + e_{12}^2, \quad (12.7b)$$

$$\tilde{B}_{22} = 2e_{22} + e_{22}^2 + e_{21}^2, \quad (12.7b)$$

$$\tilde{B}_{12} = \tilde{B}_{21} = (1 + e_{11})e_{21} + (1 + e_{22})e_{12}. \quad (12.7c)$$

It is easily verified that the \tilde{B}_{ij} matrix vanishes when the disturbance vanishes. The invariants, as given in (12.3), can now be rewritten as

$$I = 3 + \tilde{B}_{11} + \tilde{B}_{22}, \quad (12.8a)$$

$$II = 3 + 2(\tilde{B}_{11} + \tilde{B}_{22}) + \tilde{B}_{11}\tilde{B}_{22} - (\tilde{B}_{12})^2, \quad (12.8b)$$

$$III = 1 + \tilde{B}_{11} + \tilde{B}_{22} + \tilde{B}_{11}\tilde{B}_{22} - (\tilde{B}_{12})^2. \quad (12.8c)$$

Clearly, I and II go to the constant value of 3 and III goes to 1 as the strain vanishes. It will therefore be convenient to define the following,

$$\tilde{I} = I - 3 = \tilde{B}_{11} + \tilde{B}_{22}, \quad (12.9a)$$

$$\tilde{II} = II - 3 = 2(\tilde{B}_{11} + \tilde{B}_{22}) + \tilde{B}_{11}\tilde{B}_{22} - (\tilde{B}_{12})^2, \quad (12.9b)$$

$$\tilde{III} = III - 1 = \tilde{B}_{11} + \tilde{B}_{22} + \tilde{B}_{11}\tilde{B}_{22} - (\tilde{B}_{12})^2. \quad (12.9c)$$

We may now recast the stress tensor, (12.1), in terms of the new variables (12.6),

$$\sigma_{ij} = \tilde{\alpha}_0 \delta_{ij} + \tilde{\alpha}_1 \tilde{B}_{ij} + \tilde{\alpha}_2 \tilde{B}_{ie} \tilde{B}_{ej}, \quad (12.10)$$

where the notation has been made more compact by using the following variables,

$$\tilde{\alpha}_0 = \tilde{\alpha}_0(\tilde{I}, \tilde{II}, \tilde{III}) \equiv \alpha_0 + \alpha_1 + \alpha_2, \quad (12.11a)$$

$$\tilde{\alpha}_1 = \tilde{\alpha}_1(\tilde{I}, \tilde{II}, \tilde{III}) \equiv \alpha_1 + 2\alpha_2, \quad (12.11b)$$

$$\tilde{\alpha}_2 = \tilde{\alpha}_2(\tilde{I}, \tilde{II}, \tilde{III}) \equiv \alpha_2. \quad (12.11b)$$

The coefficients $\tilde{\alpha}_0$, $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are material functions whose precise form depends on the material described.

We close this chapter by expressing the stress tensor as given in (12.10) in terms of the e_{ij} components by using the \tilde{B}_{ij} values given in (12.7). After straightforward calculation we find that the non-zero components of the stress tensor can be written,

$$\begin{aligned} \sigma_{11} = & \tilde{\alpha}_0 + \tilde{\alpha}_1 (2e_{11} + e_{11}^2 + e_{12}^2) + \tilde{\alpha}_2 (4e_{11}^2 + e_{21}^2 + 2e_{12}e_{21} + e_{12}^2 + 4e_{11}^3 \\ & + 4e_{11}e_{12}^2 + 2e_{22}e_{12}e_{21} + 2e_{11}e_{12}e_{21} + e_{11}^4 + 2e_{11}^2e_{12}^2 + e_{12}^4) \\ & + e_{11}^2e_{21}^2 + e_{22}^2e_{12}^2 + 2e_{22}e_{11}e_{12}e_{21}), \end{aligned} \quad (12.12a)$$

$$\begin{aligned} \sigma_{22} = & \tilde{\alpha}_0 + \tilde{\alpha}_1 (2e_{22} + e_{22}^2 + e_{21}^2) \\ & + \tilde{\alpha}_2 \{ 4e_{22}^2 + e_{21}^2 + e_{12}^2 + 2e_{21}e_{12} + 4e_{22}^3 + 4e_{22}e_{21}^2 \\ & + 2[(e_{11} + e_{22})e_{12}e_{21} + e_{11}e_{21}^2 + e_{22}e_{12}^2] + e_{22}^4 + 2e_{22}^2e_{21}^2 \\ & + e_{21}^4 + e_{11}^2e_{21}^2 + 2e_{22}e_{11}e_{12}e_{21} + e_{22}^2e_{12}^2 \}, \end{aligned} \quad (12.12b)$$

$$\begin{aligned} \sigma_{12} = \sigma_{21} = & \tilde{\alpha}_1 (e_{21} + e_{12} + e_{11}e_{21} + e_{22}e_{12}) \\ & + \tilde{\alpha}_2 [2e_{12}(e_{11} + e_{22}) + 2e_{21}(e_{11} + e_{22}) + e_{12}^3 + e_{21}^3 + 3e_{22}^2e_{12} \\ & + 3e_{11}^2e_{21} + e_{12}e_{11}^2 + e_{21}e_{22}^2 + e_{12}^2e_{21} + e_{12}e_{21}^2 \\ & + 2e_{22}e_{11}(e_{21} + e_{12}) + e_{11}^3e_{21} + e_{22}^3e_{12} \\ & + e_{11}e_{21}(e_{22} + e_{12}) + e_{22}e_{12}(e_{11} + e_{21}) \\ & + e_{11}e_{21}^3 + e_{22}e_{12}^3]. \end{aligned} \quad (12.12c)$$

Our formulation of the mathematical system governing the motion of the nonlinear solid is now complete. The complete 6x6 system is given by (11.30) and (11.32) combined with (12.12). In the following chapter we will first analyze this system in order to determine the nature of the nonlinearity and then derive the modified Zablotskaya-Khokhlov equation for any modes having $\Gamma=0$.

Chapter 13: Computation of Eigenvalues and Vectors

We will begin by taking the needed derivatives of the stress tensor in order to describe all of the matrix components of $\underline{\underline{A}}$ and $\underline{\underline{B}}$. However, we will omit writing out these expanded derivatives, since in order to get the eigenvalues and needed vectors, we need to only consider the matrices at their lowest order. As shown in Chapter 2, these lowest-order matrices will be denoted $\underline{\underline{A}}^{(0)}$ and $\underline{\underline{B}}^{(0)}$. We will show that step here, and after that, calculate the vectors.

In Equation (12.10), we described the stress tensor in terms of $\tilde{\alpha}_0, \tilde{\alpha}_1, \tilde{\alpha}_2$ and \tilde{B}_{ij} . We want to take derivatives of this tensor with respect to e_{ij} in order to fully describe the $\underline{\underline{A}}^{(0)}$ and $\underline{\underline{B}}^{(0)}$ matrices, which are lowest-order versions of those given in (11.30). Also, we will consider that these derivatives will be taken at the state where all e_{ij} components are set equal to zero, as undisturbed. After taking the needed derivatives, the components can be shown to be:

$$A_{51}^{(0)} = -\frac{1}{\rho_0} \left(\left. \frac{\partial \tilde{\alpha}_0}{\partial e_{11}} \right|_0 + 2\tilde{\alpha}_1 \Big|_0 \right), \quad (13.1a)$$

$$A_{63}^{(0)} = -\frac{1}{\rho_0} \tilde{\alpha}_1 \Big|_0, \quad (13.1b)$$

$$B_{52}^{(0)} = -\frac{1}{\rho_0} \tilde{\alpha}_1 \Big|_0, \quad (13.1c)$$

$$B_{53}^{(0)} = -\frac{1}{\rho_0} \left(\left. \frac{\partial \tilde{\alpha}_0}{\partial e_{22}} \right|_0 + \tilde{\alpha}_1 \Big|_0 \right), \quad (13.1d)$$

$$B_{61}^{(0)} = -\frac{1}{\rho_0} \left(\left. \frac{\partial \tilde{\alpha}_0}{\partial e_{11}} \right|_0 + \tilde{\alpha}_1 \Big|_0 \right), \quad (13.1e)$$

$$B_{64}^{(0)} = -\frac{1}{\rho_0} \left(\left. \frac{\partial \tilde{\alpha}_0}{\partial e_{22}} \right|_0 + 2\tilde{\alpha}_1 \Big|_0 \right), \quad (13.1f)$$

where all other elements of $\underline{\underline{A}}^{(0)}$ and $\underline{\underline{B}}^{(0)}$ are zero, aside from $A_{15}^{(0)}$, $A_{36}^{(0)}$, $B_{25}^{(0)}$ and $B_{46}^{(0)}$ which are all equal to -1 .

By definition of the $\tilde{\alpha}_0, \tilde{\alpha}_1, \tilde{\alpha}_2$ coefficients, they will depend on e_{ij} through the invariants, (12.9). In this way, we can use the chain rule to take the derivatives above, as in the following example,

$$\frac{\partial \tilde{\alpha}_0}{\partial e_{11}} = \frac{\partial \tilde{\alpha}_0}{\partial \tilde{I}} \frac{\partial \tilde{I}}{\partial e_{11}} + \frac{\partial \tilde{\alpha}_0}{\partial \tilde{II}} \frac{\partial \tilde{II}}{\partial e_{11}} + \frac{\partial \tilde{\alpha}_0}{\partial \tilde{III}} \frac{\partial \tilde{III}}{\partial e_{11}}. \quad (13.2)$$

It is important to note that all derivatives of the invariants and of the $\tilde{\alpha}_i$ variables are evaluated at the undisturbed state from here forth, unless otherwise noted. Similar to (13.2), we can take the derivative of $\tilde{\alpha}_0$ with respect to e_{22} as well. The details of these derivatives have been recorded in Appendix A. Here we simply list the final results,

$$\frac{\partial \tilde{\alpha}_0}{\partial e_{11}} = 2\tilde{\beta}_0 = \frac{\partial \tilde{\alpha}_0}{\partial e_{22}}, \quad (13.3)$$

where the new variable, $\tilde{\beta}_i$, is defined

$$\tilde{\beta}_i \equiv \frac{\partial \tilde{\alpha}_i}{\partial \tilde{I}} + 2 \frac{\partial \tilde{\alpha}_i}{\partial \tilde{II}} + \frac{\partial \tilde{\alpha}_i}{\partial \tilde{III}}, \quad (13.4)$$

again, where all derivatives are taken at the undisturbed state. Using Equation (13.4) for $i=0$, we can rewrite the $A_{51}^{(0)}$, $B_{53}^{(0)}$, $B_{61}^{(0)}$ and $B_{64}^{(0)}$ components to the following:

$$A_{51}^{(0)} = -\frac{2}{\rho_0} \left(\tilde{\alpha}_1|_0 + \tilde{\beta}_0 \right); \quad (13.5a)$$

$$B_{53}^{(0)} = -\frac{1}{\rho_0} \left(\tilde{\alpha}_1|_0 + 2\tilde{\beta}_0 \right); \quad (13.5b)$$

$$B_{61}^{(0)} = -\frac{1}{\rho_0} \left(\tilde{\alpha}_1|_0 + 2\tilde{\beta}_0 \right); \quad (13.5c)$$

$$B_{64}^{(0)} = -\frac{2}{\rho_0} \left(\tilde{\alpha}_1|_0 + \tilde{\beta}_0 \right). \quad (13.5d)$$

The entire system can now be rewritten using the new variables presented in the above equations. When this is done, the $\underline{\underline{A}}^{(0)}$ and $\underline{\underline{B}}^{(0)}$ matrices can be written as

$$\underline{\underline{A}}^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -c_p^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -c_s^2 & 0 & 0 & 0 \end{bmatrix}, \underline{\underline{B}}^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -c_s^2 & c_s^2 - c_p^2 & 0 & 0 & 0 \\ c_s^2 - c_p^2 & 0 & 0 & 0 & -c_p^2 & 0 \end{bmatrix}, \quad (13.6)$$

where

$$c_p = \sqrt{\frac{2}{\rho_0} \left(\tilde{\beta}_0 + \tilde{\alpha}_1 \Big|_0 \right)}, \quad (13.7a)$$

$$c_s = \sqrt{\frac{\tilde{\alpha}_1 \Big|_0}{\rho_0}}, \quad (13.7b)$$

which are the speeds of the pressure and shear waves, respectively.

We may now substitute (13.6) into (3.11) to determine the eigenvalues of $\underline{\underline{A}}^{(0)}$.

The eigenvalues were found to be,

$$\lambda_p = \pm c_p, \quad \lambda_s = \pm c_s. \quad (13.8)$$

Equation (13.8) does not include the zero eigenvalues, which are also solutions but represent non-propagating modes and will henceforth be ignored. The eigenvector corresponding to the first set of eigenvalues is

$$\underline{r}_p = r_1 \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -\lambda_p \\ 0 \end{bmatrix} \right\}. \quad (13.9)$$

Similarly, for the second set of eigenvalues, we find

$$\underline{r}_s = r_3 \left\{ \begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -\lambda_s \end{array} \right\}. \quad (13.10)$$

We can see here that there are clearly two cases emerging, one for the motion of a shear wave (denoted by a subscript s) and the other for a pressure wave (denoted by a subscript p). These modes are named as such since they produce strains in the solid that propagate in a shearing manner or in a longitudinal manner, respectively. For the rest of this discussion, we will call these cases the s-mode and the p-mode. Note that r_1 and r_3 are different and specific to the mode that we are considering.

Similar to the solution for the right-hand eigenvector, we can obtain the left-hand eigenvector, by substituting (13.6) into (2.8b). This will give the following vectors, one for each of the modes in question,

$$\underline{\ell}_p = \ell_5 \left\{ \begin{array}{c} -\lambda_p \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{array} \right\} \text{ and } \underline{\ell}_s = \ell_6 \left\{ \begin{array}{c} 0 \\ 0 \\ -\lambda_s \\ 0 \\ 0 \\ 1 \end{array} \right\}. \quad (13.11)$$

Before progressing, we must verify that the condition (2.7) is satisfied. Since we have already defined the $\underline{\underline{B}}^{(0)}$ matrix in Equation (13.6), we will use this along the \underline{r} and $\underline{\ell}$ vectors found for each of the modes, and substitute them into Equation (2.7). We will see that, for each mode, the result is zero.

Since (3.22) will have self-consistent solutions in each case, we may proceed further and solve for the $\underline{\delta}$ vectors. Using Equation (3.22), along with the $\underline{\underline{A}}^{(0)}$ and $\underline{\underline{B}}^{(0)}$ matrices in (13.6) and the \underline{r} vectors for each separate case, we get the following results for each vector,

$$\underline{\delta}_p = \begin{Bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ -\lambda_p \end{Bmatrix} r_1 \quad \text{and} \quad \underline{\delta}_s = \begin{Bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ \lambda_s \\ 0 \end{Bmatrix} r_3. \quad (13.12)$$

In (13.12) we have only included the particular solutions for these vectors and have dropped the homogeneous parts, which is valid as discussed in Chapter 3.

Now that all of the needed vectors associated with the linear theory have been solved for in this chapter, we can proceed on to determining whether the modes in question satisfy the condition given in Equation (2.10), namely that there are no quadratic nonlinearities present in the system. Since this process requires many calculations, it will be discussed in the following chapter. It is in the next chapter that we will determine which modes will be suitable for the process outlined in Chapters 2 and 3.

Chapter 14: Calculations of the Quadratic Nonlinearity for the Hyperelastic Solid

In the previous chapter, we found that there are two propagating modes of wave motion that arise in the case of the hyperelastic solid. Both modes have eigenvectors that satisfy the self-consistency condition for $\underline{u}^{(2)}$ set forth in Chapter 2. At this point, we determine the type of nonlinearity exhibited by each mode. If quadratic nonlinearities do exist, we would have to consider the classical theory in order to provide the evolution equation. However, if quadratic nonlinearities are negligible or do not exist, we can proceed to using the method outlined in Chapter 3 to find the evolution equation.

The general form of the quadratic nonlinearity coefficient is given in Equation (2.10). In order to determine this coefficient, we must first determine the $\underline{\underline{G}}$ matrix for each mode, and then substitute this matrix into (3.14). The process of determining this $\underline{\underline{G}}$ matrix, and subsequently solving for Γ , will be done in this chapter. This will be done for each mode separately in order to avoid confusion.

Section 14.1: Determining the Order of Nonlinearity; P-Mode

In this section, we will begin by determining the components of the $\underline{\underline{G}}$ for the p-mode. We will henceforth drop the subscripts from each mode, since they will henceforth be considered separately. Our calculations will be simplified by first substituting (13.9) in the definition (2.11) to obtain:

$$\underline{\underline{G}} = \frac{\partial \underline{\underline{A}}}{\partial u_1} \Big|_0 r_1 + \frac{\partial \underline{\underline{A}}}{\partial u_5} \Big|_0 r_5, \quad (14.1.1)$$

where u_1 and u_5 denote the first and fifth components of the \underline{u} solution vector. The above result can be simplified further by noting that $\underline{\underline{A}}$ does not depend on $u_5 \equiv v_1$, which means that $\underline{\underline{G}}$ can be simplified to

$$\underline{\underline{G}} = r_1 \frac{\partial \underline{\underline{A}}}{\partial e_{11}} \Big|_0, \quad (14.1.2)$$

where we have used the fact that $u_I \equiv e_{11}$.

Inspection of (11.32) reveals that there are only four non-constant components of matrix $\underline{\underline{A}}$. As a result, the $\underline{\underline{G}}$ matrix can be written,

$$\underline{\underline{G}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ G_{51} & 0 & G_{53} & 0 & 0 & 0 \\ G_{61} & 0 & G_{63} & 0 & 0 & 0 \end{bmatrix}. \quad (14.1.3)$$

We can now solve for Γ in terms of the $\underline{\underline{G}}$ components, which will result in the following,

$$\Gamma = \frac{G_{51}}{2\lambda}. \quad (14.1.4)$$

Substituting (14.1.2) into (14.1.4) for $\underline{\underline{G}}$ results in the following,

$$\Gamma = -\frac{1}{2\lambda} \left[r_1 \left. \frac{\partial A_{51}}{\partial e_{11}} \right|_0 \right]. \quad (14.1.5)$$

Thus, in order to determine whether this mode satisfies the $\Gamma=0$ condition, we need only to take one derivative of the $\underline{\underline{A}}$ matrix, which simplifies the calculations considerably.

In order to determine Γ , we must take the derivative of A_{51} as given by (11.32a).

When the requisite differentiation is carried out we find that,

$$\left. \frac{\partial A_{51}}{\partial e_{11}} \right|_0 = \frac{1}{\rho_0} \left[\left. \frac{\partial^2 \sigma_{21}}{\partial e_{11}^2} \right|_0 (e_{12}) - \left. \frac{\partial^2 \sigma_{11}}{\partial e_{11}^2} \right|_0 (1 + e_{22}) \right]. \quad (14.1.6)$$

Since we are considering that this derivative is taken at the undisturbed, or $\underline{\underline{u}}=0$, state, any terms multiplied by $O(e_{ij})$ go to zero and we can thusly reduce this equation further to the following result,

$$\left. \frac{\partial A_{51}}{\partial e_{11}} \right|_0 = -\frac{1}{\rho_0} \left. \frac{\partial^2 \sigma_{11}}{\partial e_{11}^2} \right|_0. \quad (14.1.7)$$

We will now take the second derivative of (12.12a) with respect to e_{11} . When the resultant expression is evaluated at the undisturbed state, we find that

$$\left. \frac{\partial^2 \sigma_{11}}{\partial e_{11}^2} \right|_0 = \frac{\partial^2 \tilde{\alpha}_0}{\partial e_{11}^2} + 4 \frac{\partial \tilde{\alpha}_1}{\partial e_{11}} + 2\tilde{\alpha}_1 + 8\tilde{\alpha}_2, \quad (14.1.8)$$

where all quantities on the right-hand side are evaluated at the undisturbed state. The derivatives for the $\tilde{\alpha}_i$ variables can then be expanded in terms of the invariants using the chain rule, as previously shown in (13.2). Details of these derivatives are given in Appendix B. Upon performing the derivatives, we get

$$\begin{aligned} \left. \frac{\partial^2 \sigma_{11}}{\partial e_{11}^2} \right|_0 &= 2\tilde{\beta}_0 + 8 \left\{ 2 \frac{\partial^2 \tilde{\alpha}_0}{\partial \tilde{I} \partial \tilde{I}} + 2 \frac{\partial^2 \tilde{\alpha}_0}{\partial \tilde{I} \partial \tilde{I}^2} + \frac{\partial^2 \tilde{\alpha}_0}{\partial \tilde{I}^2} \right\} + \\ &4 \left\{ \frac{\partial^2 \tilde{\alpha}_0}{\partial \tilde{I}^2} + 4 \frac{\partial^2 \tilde{\alpha}_0}{\partial \tilde{I}^2} + \frac{\partial^2 \tilde{\alpha}_0}{\partial \tilde{I}^2} \right\} + 8\tilde{\beta}_1 + 2\tilde{\alpha}_1 \Big|_0 + 8\tilde{\alpha}_2 \Big|_0 \end{aligned} \quad (14.1.9)$$

which, in general, is non-zero. All terms in this derivative are evaluated at the undisturbed state, $\underline{u}=\underline{0}$. This can be substituted into (14.1.5) along with (14.1.7), which gives a non-zero answer for the Γ variable. Since we have found that Γ does not equal zero, we know that the evolution of the p-mode will be governed by the classical Zabalotskaya-Khokhlov equation, (1.1).

Section 14.2: Determining the Order of Nonlinearity; S-Mode

Similar to the process presented in Section 14.1, we can begin computing the $\underline{\underline{G}}$ matrix by using the right-hand eigenvector given in Equation (13.10). This \underline{r} vector has again only two non-zero components, one of them being the r_6 component. Since the $\underline{\underline{A}}$ matrix has no dependence on $u_6 \equiv v_2$, this derivative will go to zero, and we can simplify the $\underline{\underline{G}}$ matrix to the following

$$\underline{\underline{G}} = \left. \frac{\partial \underline{\underline{A}}}{\partial e_{21}} \right|_0 \underline{r}_3. \quad (14.2.1)$$

Again, we note that the $\underline{\underline{A}}$ matrix has only four non-constant components, which means $\underline{\underline{G}}$ will have only four components that are non-zero. These are G_{51} , G_{53} , G_{61} and G_{63} .

We will record the expressions for each of these non-zero components because we foresee the need to use them in subsequent chapters.

Since we are taking the derivatives and then evaluating them at the undisturbed (zero) state, the components of $\underline{\underline{G}}$ can be written as follows:

$$G_{51} = \left. \frac{\partial A_{51}}{\partial e_{21}} \right|_0 r_3 = - \left. \frac{r_3}{\rho_0} \frac{\partial^2 \sigma_{11}}{\partial e_{21} \partial e_{11}} \right|_0, \quad (14.2.2a)$$

$$G_{53} = \left. \frac{\partial A_{53}}{\partial e_{21}} \right|_0 r_3 = - \left. \frac{r_3}{\rho_0} \frac{\partial^2 \sigma_{11}}{\partial e_{21}^2} \right|_0, \quad (14.2.2b)$$

$$G_{61} = \left. \frac{\partial A_{61}}{\partial e_{21}} \right|_0 r_3 = - \left. \frac{r_3}{\rho_0} \frac{\partial^2 \sigma_{12}}{\partial e_{11} \partial e_{21}} \right|_0, \quad (14.2.2c)$$

$$G_{63} = \left. \frac{\partial A_{63}}{\partial e_{21}} \right|_0 r_3 = - \left. \frac{r_3}{\rho_0} \frac{\partial^2 \sigma_{12}}{\partial e_{21}^2} \right|_0, \quad (14.2.2d)$$

where the expressions for the A_{51} , A_{53} , A_{61} and A_{63} components are given in Equation (11.32a)-(11.32d). The second derivatives of the σ_{ij} terms in Equations (14.2.2) can be found by using the chain rule, in a similar way to that described in Equation (13.11). Details of these calculations are given in Appendix C. The results are as follows:

$$\left. \frac{\partial^2 \sigma_{11}}{\partial e_{21} \partial e_{11}} \right|_0 = 0, \quad (14.2.3a)$$

$$\left. \frac{\partial^2 \sigma_{11}}{\partial e_{21}^2} \right|_0 = 2 \left. \frac{\partial \tilde{\alpha}_0}{\partial \tilde{I}} \right|_0 + 2 \left. \frac{\partial \tilde{\alpha}_0}{\partial \tilde{II}} \right|_0 + 2 \tilde{\alpha}_2 \Big|_0, \quad (14.2.3b)$$

$$\left. \frac{\partial^2 \sigma_{12}}{\partial e_{11} \partial e_{21}} \right|_0 = 2 \left. \frac{\partial \tilde{\alpha}_1}{\partial \tilde{I}} \right|_0 + 4 \left. \frac{\partial \tilde{\alpha}_1}{\partial \tilde{II}} \right|_0 + 2 \left. \frac{\partial \tilde{\alpha}_1}{\partial \tilde{III}} \right|_0 + \tilde{\alpha}_1 \Big|_0 + 2 \tilde{\alpha}_2 \Big|_0, \quad (14.2.3c)$$

$$\left. \frac{\partial^2 \sigma_{12}}{\partial e_{21}^2} \right|_0 = 0, \quad (14.2.3d)$$

where the derivatives of the $\tilde{\alpha}_i$ variables are again taken at the undisturbed state. Upon substituting derivatives given in (14.2.3) into Equations (14.2.2), we can record each of the evaluated components of the $\underline{\underline{G}}$ matrix. These are

$$G_{51} = 0, \quad (14.2.4.a)$$

$$G_{53} = -\frac{r_3}{\rho_0} \left[2 \frac{\partial \tilde{\alpha}_0}{\partial \tilde{I}} + 2 \frac{\partial \tilde{\alpha}_0}{\partial \tilde{I}^2} + 2 \tilde{\alpha}_2 \Big|_0 \right], \quad (14.2.4b)$$

$$G_{61} = -\frac{r_3}{\rho_0} \left[2 \frac{\partial \tilde{\alpha}_1}{\partial \tilde{I}} + 4 \frac{\partial \tilde{\alpha}_1}{\partial \tilde{I}^2} + 2 \frac{\partial \tilde{\alpha}_1}{\partial \tilde{I}^3} + \tilde{\alpha}_1 \Big|_0 + 2 \tilde{\alpha}_2 \Big|_0 \right], \quad (14.2.4c)$$

$$G_{63} = 0, \quad (14.2.4d)$$

again, where the derivatives in (14.2.4) are taken at the undisturbed state.

We now have solved for all of the components in the matrix $\underline{\underline{G}}$ in terms of the $\tilde{\alpha}_i$ coefficients and the invariants of $\underline{\underline{B}}$. We can use this matrix in Equation (3.14) along with the right- and left-hand eigenvectors from Equations (13.10) and (13.11), respectively. This will give a solution for Γ equal to zero, which implies that the system nonlinearity is cubic, at most. As a result, the evolution of quasi-plane, weakly non-linear shear waves will be governed by the modified Zabolotskaya-Khokhlov equation derived in Part I of this thesis. In the remainder of Part III we will focus on this non-classical case and determine the precise form of the nonlinearity and diffraction coefficients.

Chapter 15: Evaluation of the Evolution Equation for the S-mode

In Chapter 14, we found that the s-mode alone will give us a system that will have no quadratic nonlinearities. The evolution equation therefore needs to be derived by the scheme of Part I. Since all needed matrices and vectors have already been evaluated, except for $\underline{\gamma}$ and \underline{H} , these will be evaluated first. We will then use the vectors and matrices developed here and in the previous two chapters in order to evaluate the scalar constants (3.27).

Since we defined all the components of the \underline{G} matrix in the previous chapter, we can begin by substituting the \underline{G} matrix, along with the $\underline{A}^{(0)}$ matrix from Equation (13.6) and the \underline{r} vector from (13.10), in Equation (3.21) so that we can solve for $\underline{\gamma}$. This will give the following solution

$$\underline{\gamma} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ \lambda \\ 0 \end{bmatrix} \begin{cases} G_{53}r_3 \\ c_s^2 - c_p^2 \end{cases}, \quad (15.1)$$

where the homogeneous part has been dropped, meaning (15.1) gives the particular part only. Here, c_s and c_p are given in Equations (13.7) and G_{53} is given by (14.2.2b) or (14.2.4b). For notation purposes, we will henceforth call

$$\gamma_1 = \frac{G_{53}r_3}{c_s^2 - c_p^2}. \quad (15.2)$$

The \underline{H} matrix, as given in Equation (3.25), can be evaluated in a similar manner to the \underline{G} matrix. Our calculations can be simplified by first noting that the product of $\underline{\ell}^T$ and \underline{r} with \underline{H} , which can be expanded out to

$$\underline{\ell}^T \underline{H} \underline{r} = r_3 \ell_6 H_{63}, \quad (15.3)$$

where we have used (13.10) and (13.11). Thus, so we have to only evaluate the H_{63} component of the $\underline{\underline{H}}$ matrix. If we use the definition (3.25) and (13.10)-(13.11), we find that H_{63} can be written

$$H_{63} = \frac{1}{2} \frac{\partial^2 A_{63}}{\partial u_k \partial u_l} \Big|_0 r_k r_l = \frac{1}{2} \left[\frac{\partial^2 A_{63}}{\partial u_3^2} \Big|_0 r_3^2 + \frac{\partial^2 A_{63}}{\partial u_3 \partial u_6} \Big|_0 r_3 r_6 + \frac{\partial^2 A_{63}}{\partial u_6^2} \Big|_0 r_6^2 \right], \quad (15.4)$$

which reduces to

$$H_{63} = \frac{r_3^2}{2} \frac{\partial^2 A_{63}}{\partial e_{21}^2} \Big|_0. \quad (15.5)$$

since $\underline{\underline{A}}$ has no dependence on $u_6 \equiv v_2$. In terms of σ_{ij} (15.5) becomes

$$H_{63} = -\frac{r_3^2}{2\rho_0} \frac{\partial^3 \sigma_{12}}{\partial e_{21}^3} \Big|_0, \quad (15.6)$$

where, from the chain rule, the third derivative can be written,

$$\frac{\partial^3 \sigma_{12}}{\partial e_{21}^3} \Big|_0 = 3 \frac{\partial^2 \tilde{\alpha}_1}{\partial e_{21}^2} + 6\tilde{\alpha}_2 \Big|_0, \quad (15.7)$$

where the derivative of the $\tilde{\alpha}_1$ term in (15.7) is taken at the undisturbed state. The second derivative in (15.7) can be taken as shown by the chain rule in (B.2). This gives the following,

$$\frac{\partial^2 \tilde{\alpha}_1}{\partial e_{21}^2} = 2 \frac{\partial \tilde{\alpha}_1}{\partial \tilde{I}} + 2 \frac{\partial \tilde{\alpha}_1}{\partial \tilde{II}}, \quad (15.8)$$

where, again, the derivatives of the $\tilde{\alpha}_1$ terms with respect to the invariants are taken at the undisturbed state. This results in the following for the H_{63} component,

$$H_{63} = -\frac{3r_3^2}{\rho_0} \left\{ \frac{\partial \tilde{\alpha}_1}{\partial \tilde{I}} + \frac{\partial \tilde{\alpha}_1}{\partial \tilde{II}} + \tilde{\alpha}_2 \Big|_0 \right\}, \quad (15.9)$$

where the derivatives are all evaluated at the undisturbed state.

The scalar components can be now be evaluated from Equations (3.27a) through (3.27d). The calculations needed to perform this step are straightforward but tedious, and will not be written out explicitly here. The scalars are as follows:

$$A=0, \quad (15.10)$$

$$B=0, \quad (15.11)$$

$$C=\frac{1}{2}, \quad (15.12)$$

$$\Lambda = \frac{r_3^2}{2\rho_0 c_s^2} \left[\frac{\partial^3 \sigma_{12}}{\partial e_{21}^3} \Big|_0 + \frac{3}{\rho_0 (c_s^2 - c_p^2)} \frac{\partial^2 \sigma_{12}}{\partial e_{11} \partial e_{21}} \Big|_0 \frac{\partial^2 \sigma_{11}}{\partial e_{21}^2} \Big|_0 \right], \quad (15.13)$$

where the expressions for the σ_{ij} derivatives are given in terms of the $\tilde{\alpha}_{ij}$ in Appendix C.

The fact that A and B go to zero is due to the fact that all terms within the definition of these scalars themselves go to zero. These can be substituted into the form of the evolution equation as given in Equation (3.28), ignoring dissipation, dispersion and relaxation effects. The outcome is

$$U_\tau + \frac{\Lambda}{2} U^2 U_\chi + \frac{1}{2} V_\eta = 0, \quad (15.14)$$

where Λ is given in (15.13) and V satisfies (3.18).

Given the evolution equation in (15.14), we can now recast this equation in terms of the physical variables in the system, as shown for the nonlinear dielectric in Chapter 7. To do this, we will substitute the vectors found for the s-mode in (13.9)-(13.12) into the equation for the \underline{u} solution vector presented in (3.4). The result is,

$$\underline{u} = \begin{Bmatrix} e_{11} \\ e_{12} \\ e_{21} \\ e_{22} \\ v_1 \\ v_2 \end{Bmatrix} = \Delta \begin{Bmatrix} 0 \\ 0 \\ r_3 \\ 0 \\ 0 \\ -\lambda_s r_3 \end{Bmatrix} \left\{ U + \Delta^2 \right\} \begin{Bmatrix} -r_3 \\ 0 \\ 0 \\ r_3 \\ \lambda_s r_3 \\ 0 \end{Bmatrix} \left\{ V + \Delta^2 \right\} \begin{Bmatrix} -\gamma_1 \\ 0 \\ 0 \\ 0 \\ \lambda_s \gamma_1 \\ 0 \end{Bmatrix} \left\{ \frac{U^2}{2} + O(\Delta^3) \right\}, \quad (15.15)$$

where only the particular parts of the solutions for $\underline{\delta}$ and $\underline{\gamma}$ have been included. This will give the following solutions for the physical parameters of the system:

$$e_{11} = -\Delta^2 (r_3 V + \gamma_1 \frac{U^2}{2}) + O(\Delta^3), \quad (15.16a)$$

$$e_{12} = O(\Delta^3), \quad (15.16b)$$

$$e_{21} = \Delta r_3 U + O(\Delta^3), \quad (15.16c)$$

$$e_{22} = \Delta^2 r_3 V + O(\Delta^3), \quad (15.16d)$$

$$v_1 = \Delta^2 \lambda_s (r_3 V + \gamma_1 \frac{U^2}{2}) + O(\Delta^3), \quad (15.16e)$$

$$v_2 = -\Delta \lambda_s r_3 U + O(\Delta^3). \quad (15.16f)$$

Substituting e_{22} , v_1 and v_2 into the evolution equation (15.14), multiplying by factors of Δ and r_3 , and using the variable transformations given in (3.1), we get the following,

$$\frac{\partial e_{21}}{\partial t} \pm \frac{c_s}{2} \frac{\partial e_{22}}{\partial X_2} = \mp \frac{c_s}{2} \frac{\Lambda}{r_3^2} e_{21}^2 \frac{\partial e_{21}}{\partial X}, \quad (15.17)$$

where $X \equiv X_I - \lambda t$. This relates the physical variables of the system in a similar manner to that described in Part II.

From (15.14) we can also rederive the compatibility condition as given in (3.19). If we take the derivative of (15.16c) and (15.16d) with η and χ , respectively, and rearrange them working backwards from the transformation of variables presented in (3.1), we will arrive at

$$\frac{\partial e_{21}}{\partial X_2} = \frac{\partial e_{22}}{\partial X_1}. \quad (15.18)$$

If we use the definition (11.20), we can show that the compatibility condition (3.19) reduces to the condition

$$\frac{\partial^2 u_2}{\partial X_1 \partial X_2} = \frac{\partial^2 u_2}{\partial X_1 \partial X_2}. \quad (15.19)$$

which is recognized as a compatibility condition on the shear displacement $u_2(X_1, X_2, t)$.

Chapter 16: Dissipation in the Hyperelastic Solid

Dissipation may be incorporated by modifying our original stress tensor (12.1) to include Kelvin-Voigt viscosity terms. Such a model can be written,

$$\sigma_{ij} = \sigma_{ij}^{(e)} (B_{ij}) + \sigma_{ij}^{(v)} (\nabla \underline{v}), \quad (16.1)$$

where B_{ij} is the left-Cauchy Green tensor given in Equation (12.4) and should not be confused with the $\underline{\underline{B}}$ matrix in the general equation (3.28), $\sigma_{ij}^{(e)}$ is the elastic tensor identical to (12.1), and $\sigma_{ij}^{(v)}$ is the viscous part of the stress tensor. Again, we are using the Einstein summation convention with the range of 1 to 3 for $\sigma_{ij}^{(v)}$, $\sigma_{ij}^{(e)}$, and B_{ij} . The viscous part of the stress tensor will be written as a linear function of the gradient of the velocity, \underline{v} , as follows,

$$\sigma_{ij}^{(v)} = \lambda_v (v_{,l,l}) \delta_{ij} + \mu_v (v_{i,j} + v_{j,i}), \quad (16.2)$$

which can be approximated

$$\sigma_{ij}^{(v)} \approx \lambda_v^{(0)} v_{l,l} \delta_{ij} + \mu_v^{(0)} (v_{i,j} + v_{j,i}), \quad (16.3)$$

where $\lambda_v^{(0)}$ and $\mu_v^{(0)}$ are lowest order approximations for the viscosities, λ_v and μ_v , and where the comma notation used in conjunction with the subscripts denotes partial differentiation with respect to either x_i or X_i in this lowest-order expression.

If we substitute (16.1) into the equilibrium equation (11.1) and expand for two-dimensional motions in a similar manner to that presented in Chapter 12, we get

$$\begin{aligned} \rho_0 \frac{\partial v_i}{\partial t} - (1+e_{22}) \frac{\partial \sigma_{1i}^{(e)}}{\partial X_1} + e_{12} \frac{\partial \sigma_{2i}^{(e)}}{\partial X_1} + e_{21} \frac{\partial \sigma_{1i}^{(e)}}{\partial X_2} - (1+e_{11}) \frac{\partial \sigma_{2i}^{(e)}}{\partial X_2} \\ = (1+e_{22}) \frac{\partial \sigma_{1i}^{(v)}}{\partial X_1} - e_{12} \frac{\partial \sigma_{2i}^{(v)}}{\partial X_1} - e_{21} \frac{\partial \sigma_{1i}^{(v)}}{\partial X_2} + (1+e_{11}) \frac{\partial \sigma_{2i}^{(v)}}{\partial X_2}. \end{aligned} \quad (16.4)$$

Because $\sigma_{ij}^{(e)}$ is identical to (11.19), (16.4) is identical to (11.23), with the addition of the $\sigma_{ij}^{(v)}$ terms on the right-hand side. Therefore, the evaluation of the evolution equation is the same as given in the previous chapter, except that we must now account for the added viscous terms.

We recall from Chapter 3 that the only terms contributing to the final evolution equation are those involving only x (or, in this example, X_I) derivatives. Furthermore, only the lowest order viscous terms will be required. We therefore find that (16.4) can be written as:

$$\begin{aligned} \rho_0 \frac{\partial v_i}{\partial t} - (1 + e_{22}) \frac{\partial \sigma_{1i}^{(e)}}{\partial X_1} + e_{12} \frac{\partial \sigma_{2i}^{(e)}}{\partial X_1} + e_{21} \frac{\partial \sigma_{1i}^{(e)}}{\partial X_2} - (1 + e_{11}) \frac{\partial \sigma_{2i}^{(e)}}{\partial X_2} \\ = \frac{\partial \sigma_{1i}^{(v)}}{\partial v_{1,1}} \frac{\partial^2 v_1}{\partial X_1^2} + \frac{\partial \sigma_{1i}^{(v)}}{\partial v_{2,1}} \frac{\partial^2 v_2}{\partial X_1^2} + H.O.T. \end{aligned} \quad (16.5)$$

to the lowest order, where *H.O.T.* refers to higher order terms, and where the derivatives on the right-hand side of (16.4) have been substituted in from the following,

$$\frac{\partial \sigma_{1i}^{(v)}}{\partial X_1} = \frac{\partial \sigma_{1i}^{(v)}}{\partial v_{1,1}} v_{1,11} + \frac{\partial \sigma_{1i}^{(v)}}{\partial v_{1,2}} v_{1,21} + \frac{\partial \sigma_{1i}^{(v)}}{\partial v_{2,1}} v_{2,11} + \frac{\partial \sigma_{1i}^{(v)}}{\partial v_{2,2}} v_{2,22}, \quad (16.6)$$

where

$$v_{1,11} \equiv \frac{\partial^2 v_1}{\partial X_1^2}, \quad v_{1,21} \equiv \frac{\partial^2 v_1}{\partial X_1 \partial X_2}, \quad \text{etc.} \quad (16.7)$$

Since the $\underline{\underline{D}}^{(xx)}$ matrix needed to show dissipation effects is multiplied by the second derivative with respect to the X_I , only the $v_{2,11}$ and $v_{1,11}$ terms in (16.6) will be included when substituted into (16.5). It can be shown that the only derivatives of $\sigma_{ij}^{(v)}$ which contribute to the final evolution equation are

$$\frac{\partial \sigma_{11}^{(v)}}{\partial v_{1,1}} = \lambda_v^{(0)} + 2\mu_v^{(0)}, \quad (16.9a)$$

$$\frac{\partial \sigma_{11}^{(v)}}{\partial v_{2,1}} = 0, \quad (16.9b)$$

$$\frac{\partial \sigma_{12}^{(v)}}{\partial v_{1,1}} = 0, \quad (16.9c)$$

$$\frac{\partial \sigma_{12}^{(v)}}{\partial v_{2,1}} = \mu_v^{(0)}, \quad (16.9d)$$

which, when substituted back into (16.8), gives the following system

$$\underline{u}_t + \underline{A} \underline{u}_{X_1} + \underline{B} \underline{u}_{X_2} = \underline{D}^{(xx)} \underline{u}_{X_1 X_1} + H.O.T., \quad (16.10)$$

where the \underline{A} and \underline{B} components are given in Equation (11.32), and $\underline{D}^{(xx)}$ is

$$\underline{D}^{(xx)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\rho_0} (\lambda_v^{(0)} + 2\mu_v^{(0)}) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\rho_0} \mu_v^{(0)} \end{bmatrix}. \quad (16.11)$$

Now that we have determined the dissipation matrix which applies to our system, we can evaluate the needed scalar coefficient as given in (3.27f). First, (16.11) is nondimensionalized as given in (3.6b). The coefficient can then be given as

$$\Delta_D = \frac{\mu_v^{(0)}}{2L\lambda\Delta^2\rho_0} = O(1). \quad (16.12)$$

Using the evolution equation for the hyperelastic solid as given in Equation (15.14), we note that dissipation simply adds a term to this equation on the right-hand side, and we are left with the final evolution equation,

$$U_\tau + \frac{\Lambda}{2} U^2 U_\chi + \frac{1}{2} V_\eta = \Delta_D U \chi\chi, \quad (16.13)$$

where Λ is given by (15.3). We can again recast the evolution equation in terms of the physical variables. This results in (15.17) with the addition of a term on the right-hand side, which is as follows,

$$\frac{\partial e_{21}}{\partial t} \pm \frac{c_s}{2} \frac{\partial e_{22}}{\partial X_2} = \mp \frac{c_s}{2} \frac{\Lambda}{r_3^2} e_{21}^2 \frac{\partial e_{21}}{\partial X} + \frac{\mu_v^{(0)}}{2\rho_0} \frac{\partial^2 e_{21}}{\partial X^2}. \quad (16.14)$$

Chapter 17: Summary

The analysis of the hyperelastic system in Part III of this thesis utilizes the multi-timing scheme from the work of Kluwick and Cox (1998) and Cramer and Webb (1998) in order to determine the evolution equation for quasi-plane, weakly nonlinear shear waves. This is the largest system to date that uses this scheme, and the fact that the compatibility condition was an outcome of this process points to the fact that this scheme is appropriate for analysis of such a system, despite its complexity.

Through the use of the multi-timing scheme, it was found that two distinct modes arise in a hyperelastic system, one governing the motion of pressure (longitudinal) waves, the other the motion of shear waves. However, since the p-mode was found to have quadratic nonlinearities, it was not an appropriate candidate for the multi-timing scheme. In this thesis, only the case where $\Gamma=0$ was considered, which is a divergence from the classical Zabolotskaya-Khokhlov case. To determine the evolution equation for a system that includes quadratic nonlinearities, the classical approach must be used, which is completely valid but was not considered here.

Though the p-mode did present a case where quadratic nonlinearities are present, the s-mode was found to have $\Gamma=0$. In considering the s-mode, the multi-timing scheme was appropriate in order to determine a modified Zabolotskaya-Khokhlov equation. The analysis of this mode is an extension of the work done by Carman and Cramer (1992). This earlier work considered the case where Γ was small, in one dimension, though the Γ =small case does not affect the form of the nonlinear scalar Λ . The calculation of Λ in this thesis is consistent with Carman and Cramer (1992). Weak dissipation was also considered for this mode, as presented in Chapter 16. Another point of note is that (11.30), the general equation presented for the hyperelastic solid, is quadratic in terms of the physical e_{ij} variables. This differs from the nonlinear dielectric case, where the general equation was seen to be cubic in terms of the physical variables; therefore, it was a logical conclusion in this case that the nonlinearity be cubic. For the hyperelastic s-mode, however, despite the fact that the general equation was quadratic in terms of the physical variables, since it was shown that Γ vanishes, we know that these quadratic terms lead to cubic nonlinearities only.

When the scalar coefficients for the evolution equation were calculated for the s-mode, we found that both A and B vanished. This was not an obvious conclusion, considering the complexity of the system in question. The Alfvén wave system of Cramer and Webb (1998) showed a case where A and B did not vanish, which was considered to be a reasonable outcome for a straight-forward system. However, since the system here was more complicated, obtaining zero values for both A and B were results neither obvious nor expected. That leads to the fact that only the nonlinear (Λ) coefficient is required to determine the evolution equation for the s-mode, which was given by the one dimensional theory presented by Carman and Cramer (1992).

Finally, the evolution equation in terms of physical variables expressed the compatibility condition given in (3.19), but in terms of the displacement vector, u_i . This gives the result that the X_1 and X_2 derivatives of the u_2 displacement vector can be interchanged. The interchangeable nature of derivatives is consistent with simple rules of calculus, which verifies the scheme used.

Chapter 18: Summary

This thesis presented two examples of nonlinear wave phenomenon, a nonlinear dielectric and a hyperelastic solid, in order to determine the evolution equations in each case in terms of the generalized shape functions as well as the physical variables. This work was done as an extension of the theory of Kluwick and Cox (1998) and Cramer and Webb (1998). In the present study, only cases where the quadratic nonlinearity completely vanished were considered, whereas small quadratic nonlinearities were considered in previous works mentioned. As a result, the scheme presented in Part I was required for the derivation of the evolution equation.

In both Part II and Part III of this thesis, the case where $\Gamma=0$ exactly was considered. This condition was set forth in (2.10) as it relates to the \underline{G} matrix and the eigenvectors of the system. When the quadratic nonlinearity completely vanishes, as in this case, we can reduce the classical Zabolotskaya-Khokhlov equation (1.1) to the more simple, linearized form (1.2). However, this linearized equation for quasi-plane wave motion is inappropriate here, since higher-order nonlinearities were found to be present in the system. These nonlinearities were found to be cubic, so in order to observe their effect on the system in question, larger time scales must be considered.

In addition to the nonlinear coefficient Λ , two other coefficients were found to be present as a result of the multi-timing scheme. These scalar A and B terms were found to go to zero in both cases considered in this thesis, whereas these values were found to be non-zero in the case of Alfvén waves presented by Cramer and Webb (1998). Considering the complexity of the hyperelastic solid, this result is not expected, and cannot be predicted from the simpler, Alfvén wave case. This points to the fact that the A and B coefficients are system-specific, and that they arise not out of complexity in the system, but instead the actual physics and the structure of the governing equations.

The compatibility condition given in (3.18), namely that $U_\eta=V_\chi$, is seen to have a counterpart in terms of the physical variables of the system in both examples considered. The compatibility condition (3.18) arises naturally from the mathematics of the multi-timing scheme presented in Chapter 3. When the physical variables in the systems in question were applied to this compatibility condition, the result in both cases was that

system-specific conditions were met. In the case of the nonlinear dielectric, the relation given in (3.18) was able to return the two-dimensional form of Gauss' Law, which is consistent with the system, whereas in the hyperelastic solid case, we were able to redetermine a simple second-order derivative in terms of the u_i variables using the same relation. Thus, this compatibility condition given by (3.18) is an important part of the multi-timing scheme, and it allows us to determine self-consistency for the systems in question.

Another aspect of the multi-timing scheme presented is that derivatives of the $\underline{\underline{A}}$ matrix are needed in order to determine the scalar coefficients. However, not merely the $\underline{\underline{A}}$ matrix, but the first and second derivatives of the terms in this matrix were needed, which complicated the calculations. Therefore, the $\underline{\underline{A}}$ matrix was made as sparse as possible in both examples. This was done for the nonlinear dielectric through the use of the two-dimensional form of Gauss Law as given in (6.5), and for the hyperelastic solid through the use of (11.31). In this way, the calculation time was reduced without loss of accuracy of the original governing equations.

Finally, the inclusion of dispersion, dissipation and relaxation was considered for the examples presented here. The multi-timing scheme in Chapter 3 made the inclusion of such effects an easy addition to the general evolution equation. Conduction and dispersion effects were considered for the nonlinear dielectric, though not simultaneously due to the restrictions on the wavelength for each case, and dissipation was considered for the hyperelastic solid through the inclusion of viscosity in the stress tensor. Thus, the evolution equation for any system analyzed by this scheme can include these effects without having to reconsider the system from the beginning, which makes calculations simpler and more concise.

Appendix A

We will use the following partial derivatives in order to evaluate the components of the matrices A and B. These will also be used and referenced in the following appendices.

$$\frac{\partial \tilde{I}}{\partial e_{11}} = 2 + 2e_{11}, \quad (\text{A.1})$$

$$\frac{\partial \tilde{I}}{\partial e_{11}} = 4 + 4e_{11} + 4e_{22} + 2e_{22}^2 + 2e_{21}^2 + 4e_{22}e_{11} - 2e_{21}e_{12} + O(e_{ij}^3), \quad (\text{A.2})$$

$$\frac{\partial \tilde{I}I}{\partial e_{11}} = 2 + 2e_{11} + 4e_{22} + 2e_{22}^2 + 2e_{21}^2 + 4e_{22}e_{11} - 2e_{21}e_{12} + O(e_{ij}^3), \quad (\text{A.3})$$

$$\frac{\partial \tilde{I}}{\partial e_{22}} = 2 + 2e_{22}, \quad (\text{A.4})$$

$$\frac{\partial \tilde{I}}{\partial e_{22}} = 4 + 4e_{11} + 4e_{22} + 4e_{11}e_{22} + 2e_{11}^2 + 2e_{12}^2 - 2e_{21}e_{12} + O(e_{ij}^3), \quad (\text{A.5})$$

$$\frac{\partial \tilde{I}I}{\partial e_{22}} = 2 + 2e_{22} + 4e_{11} + 4e_{11}e_{22} + 2e_{11}^2 + 2e_{12}^2 - 2e_{21}e_{12} + O(e_{ij}^3). \quad (\text{A.6})$$

It is important to note that all derivatives are taken at the zero undeformed state in the above equations and all that follows, so the notation to this effect will be dropped. All derivatives are understood to be evaluated at this state unless otherwise specified. Upon applying these partial derivatives to the following gives us more specific derivatives for the α values when considering that they are evaluated at the $\underline{u=0}$ undisturbed state:

$$\frac{\partial \tilde{\alpha}_0}{\partial e_{11}} = \frac{\partial \tilde{\alpha}_0}{\partial \tilde{I}} \frac{\partial \tilde{I}}{\partial e_{11}} + \frac{\partial \tilde{\alpha}_0}{\partial \tilde{I}I} \frac{\partial \tilde{I}I}{\partial e_{11}} + \frac{\partial \tilde{\alpha}_0}{\partial \tilde{I}I} \frac{\partial \tilde{I}I}{\partial e_{11}} = 2 \left\{ \frac{\partial \tilde{\alpha}_0}{\partial \tilde{I}} + 2 \frac{\partial \tilde{\alpha}_0}{\partial \tilde{I}I} + \frac{\partial \tilde{\alpha}_0}{\partial \tilde{I}I} \right\}, \quad (\text{A.7})$$

$$\frac{\partial \tilde{\alpha}_0}{\partial e_{22}} = \frac{\partial \tilde{\alpha}_0}{\partial \tilde{I}} \frac{\partial \tilde{I}}{\partial e_{22}} + \frac{\partial \tilde{\alpha}_0}{\partial \tilde{I}I} \frac{\partial \tilde{I}I}{\partial e_{22}} + \frac{\partial \tilde{\alpha}_0}{\partial \tilde{I}I} \frac{\partial \tilde{I}I}{\partial e_{22}} = 2 \left\{ \frac{\partial \tilde{\alpha}_0}{\partial \tilde{I}} + 2 \frac{\partial \tilde{\alpha}_0}{\partial \tilde{I}I} + \frac{\partial \tilde{\alpha}_0}{\partial \tilde{I}I} \right\}. \quad (\text{A.8})$$

Appendix B

In order to determine whether the nonlinearity will vanish for the p-mode in the hyperelastic solid, only one derivative must be considered, as given in (14.1.5). The partial derivative of A_{5I} with respect to e_{11} is proportional to the following ,

$$\frac{\partial^2 \sigma_{11}}{\partial e_{11}^2} = \frac{\partial^2 \tilde{\alpha}_0}{\partial e_{11}^2} + 4 \frac{\partial \tilde{\alpha}_1}{\partial e_{11}} + 2\tilde{\alpha}_1 + 8\tilde{\alpha}_2, \quad (\text{B.1})$$

where all terms are evaluated at the undeformed state. We will look at the first two terms in this expression separately. To consider the second derivative of $\tilde{\alpha}_0$, we have to look at the chain rule,

$$\begin{aligned} \frac{\partial^2 \tilde{\alpha}_0}{\partial e_{11}^2} &= \frac{\partial \tilde{\alpha}_0}{\partial \tilde{I}} \frac{\partial^2 \tilde{I}}{\partial e_{11}^2} + \frac{\partial \tilde{\alpha}_0}{\partial \tilde{II}} \frac{\partial^2 \tilde{II}}{\partial e_{11}^2} + \frac{\partial \tilde{\alpha}_0}{\partial \tilde{III}} \frac{\partial^2 \tilde{III}}{\partial e_{11}^2} + \\ &\frac{\partial \tilde{I}}{\partial e_{11}} \left\{ \frac{\partial^2 \tilde{\alpha}_0}{\partial \tilde{I}^2} \frac{\partial \tilde{I}}{\partial e_{11}} + \frac{\partial^2 \tilde{\alpha}_0}{\partial \tilde{I} \partial \tilde{II}} \frac{\partial \tilde{II}}{\partial e_{11}} + \frac{\partial^2 \tilde{\alpha}_0}{\partial \tilde{I} \partial \tilde{III}} \frac{\partial \tilde{III}}{\partial e_{11}} \right\} + \\ &\frac{\partial \tilde{II}}{\partial e_{11}} \left\{ \frac{\partial^2 \tilde{\alpha}_0}{\partial \tilde{II} \partial \tilde{I}} \frac{\partial \tilde{I}}{\partial e_{11}} + \frac{\partial^2 \tilde{\alpha}_0}{\partial \tilde{II}^2} \frac{\partial \tilde{II}}{\partial e_{11}} + \frac{\partial^2 \tilde{\alpha}_0}{\partial \tilde{II} \partial \tilde{III}} \frac{\partial \tilde{III}}{\partial e_{11}} \right\} + \\ &\frac{\partial \tilde{III}}{\partial e_{11}} \left\{ \frac{\partial^2 \tilde{\alpha}_0}{\partial \tilde{III} \partial \tilde{I}} \frac{\partial \tilde{I}}{\partial e_{11}} + \frac{\partial^2 \tilde{\alpha}_0}{\partial \tilde{III} \partial \tilde{II}} \frac{\partial \tilde{II}}{\partial e_{11}} + \frac{\partial^2 \tilde{\alpha}_0}{\partial \tilde{III}^2} \frac{\partial \tilde{III}}{\partial e_{11}} \right\}. \end{aligned} \quad (\text{B.2})$$

The partial derivatives of the invariants with respect to e_{11} are given in Appendix A. The second derivatives are all constants when evaluated at the undeformed state, which are

$$\frac{\partial^2 \tilde{I}}{\partial e_{11}^2} = 2, \quad \frac{\partial^2 \tilde{II}}{\partial e_{11}^2} = 4, \quad \frac{\partial^2 \tilde{III}}{\partial e_{11}^2} = 2. \quad (\text{B.3})$$

When these are substituted into (B.2), the result is

$$\frac{\partial^2 \tilde{\alpha}_0}{\partial e_{11}^2} = 2\beta_0 + 8 \left\{ 2 \frac{\partial^2 \tilde{\alpha}_0}{\partial \tilde{I} \partial \tilde{II}} + 2 \frac{\partial^2 \tilde{\alpha}_0}{\partial \tilde{II} \partial \tilde{III}} + \frac{\partial^2 \tilde{\alpha}_0}{\partial \tilde{I} \partial \tilde{III}} \right\} + 4 \left\{ \frac{\partial^2 \tilde{\alpha}_0}{\partial \tilde{I}^2} + 4 \frac{\partial^2 \tilde{\alpha}_0}{\partial \tilde{II}^2} + \frac{\partial^2 \tilde{\alpha}_0}{\partial \tilde{III}^2} \right\}. \quad (\text{B.4})$$

The second term in (B.1) is also determined from the chain rule, and the result is

$$\frac{\partial \tilde{\alpha}_1}{\partial e_{11}} = 2 \frac{\partial \tilde{\alpha}_1}{\partial \tilde{I}} + 4 \frac{\partial \tilde{\alpha}_1}{\partial \tilde{II}} + 2 \frac{\partial \tilde{\alpha}_1}{\partial \tilde{III}}. \quad (\text{B.5})$$

It is upon substitution of (B.4) and (B.5) into (B.1) that we obtain the result for the second derivative of σ_{II} with respect to e_{II} as given in (14.1.9).

Appendix C

All terms in the $\underline{\underline{G}}$ matrix are evaluated in their full nonlinear form, since this matrix will be used to determine the scalars in the evolution equation of the s-mode. Again, because we are evaluating the derivatives of σ_{ij} at $\underline{u}=\underline{0}$, we can simplify the calculations needed in order to evaluate the components of $\underline{\underline{G}}$. First, we can consider G_{51} . In order to evaluate this component, we need the following derivatives of the invariants.

$$\frac{\partial \tilde{I}}{\partial e_{21}} = 2e_{21}, \quad (\text{C.1})$$

$$\frac{\partial \tilde{I}}{\partial e_{21}} = 2e_{21} - 2e_{12} + 4e_{11}e_{21} - 2e_{22}e_{12} - 2e_{11}e_{12}, \quad (\text{C.2})$$

$$\frac{\partial \tilde{I}^2}{\partial e_{21}} = -2e_{12} + 4e_{11}e_{21} - 2e_{22}e_{12} - 2e_{11}e_{12}. \quad (\text{C.3})$$

We then evaluate the following, found from expanding the derivatives by using the chain rule, to obtain

$$\left. \frac{\partial^2 \sigma_{11}}{\partial e_{21} \partial e_{11}} \right|_0 = \frac{\partial^2 \tilde{\alpha}_0}{\partial e_{21} \partial e_{11}} + 2 \frac{\partial \tilde{\alpha}_1}{\partial e_{21}}, \quad (\text{C.4})$$

where the derivatives on the right-hand side of (C.4) are taken at the undeformed state. Now the chain rule is again used to evaluate the terms in (C.4), in a similar manner to that presented in Appendix B. The results are

$$\frac{\partial^2 \tilde{\alpha}_0}{\partial e_{21} \partial e_{11}} = O(e_{ij}) = 0, \quad (\text{C.5})$$

$$\frac{\partial \tilde{\alpha}_1}{\partial e_{21}} = O(e_{ij}) = 0. \quad (\text{C.6})$$

All of the terms in these expansions are not shown here as they are all of order e , which means these terms will vanish since we are evaluating at the undeformed state.

We will now evaluate G_{53} . To do this, we first can show

$$\left. \frac{\partial^2 \sigma_{11}}{\partial e_{21}^2} \right| = \frac{\partial^2 \tilde{\alpha}_0}{\partial e_{21}^2} + 2\tilde{\alpha}_2, \quad (\text{C.7})$$

where the terms on the right-hand side are evaluated at the undisturbed state. The second derivative of the $\tilde{\alpha}_0$ term is

$$\frac{\partial^2 \tilde{\alpha}_0}{\partial e_{21}^2} = 2 \frac{\partial \tilde{\alpha}_0}{\partial \tilde{I}} + 2 \frac{\partial \tilde{\alpha}_0}{\partial \tilde{II}} + O(e_{ij}) \frac{\partial \tilde{\alpha}_0}{\partial \tilde{III}} + O(e_{ij}) \frac{\partial \tilde{I}}{\partial e_{21}} + O(e_{ij}) \frac{\partial \tilde{II}}{\partial e_{21}} + O(e_{ij}) \frac{\partial \tilde{III}}{\partial e_{21}}, (\text{C.8})$$

where the $O(e_{ij})$ terms will go to zero, and the derivatives are taken at the undeformed state. This gives the final derivative used for G_{53} as shown in Chapter 14.2.

Now component G_{61} . It can be shown that

$$\left. \frac{\partial^2 \sigma_{12}}{\partial e_{11} \partial e_{21}} \right|_0 = \frac{\partial \tilde{\alpha}_1}{\partial e_{11}} + \tilde{\alpha}_1 + 2\tilde{\alpha}_2, \quad (\text{C.9})$$

where all terms are evaluated at the undeformed state, and where the derivative can be written

$$\frac{\partial \tilde{\alpha}_1}{\partial e_{11}} = 2 \frac{\partial \tilde{\alpha}_1}{\partial \tilde{I}} + 4 \frac{\partial \tilde{\alpha}_1}{\partial \tilde{II}} + 2 \frac{\partial \tilde{\alpha}_1}{\partial \tilde{III}}, \quad (\text{C.10})$$

which gives the solution for the G_{61} component.

We now look at the final component, G_{63} . To determine this, we need the following,

$$\left. \frac{\partial^2 \sigma_{12}}{\partial e_{21}^2} \right|_0 = 2 \frac{\partial \tilde{\alpha}_1}{\partial e_{21}}, \quad (\text{C.11})$$

where the right-hand side derivative is taken at the undeformed state, which is shown to be

$$\frac{\partial \tilde{\alpha}_1}{\partial e_{21}} = O(e_{ij}) = 0, \quad (\text{C.12})$$

which gives the final solution for this component.

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Vita

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