On the Eigenvalues of the Manakov System

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(ABSTRACT)

We clear up two issues regarding the eigenvalue problem for the Manakov system; these problems relate directly to the existence of the soliton [*sic*] effect in fiber optic cables. The first issue is a bound on the eigenvalues of the Manakov system: *if* the parameter ξ is an eigenvalue, *then* it must lie in a certain region in the complex plane. The second issue has to do with a chirped Manakov system. We show that if a system is chirped too much, the soliton effect disappears. While this has been known for some time experimentally, there has not yet been a theoretical result along these lines for the Manakov system.

Dedication

I dedicate this thesis to my father, J. C. Keister, who instilled in me a love for mathematics.

Acknowledgments

As usual with Acknowledgements sections, thanks is due to many, many people in no particular order. First I would like to thank my pastor, Chris Hutchinson, along with all of Grace Covenant Presbyterian Church, for helping me stay out of trouble, or at least as much as my outspoken nature allowed. Second, my advisor, Martin Klaus, for pushing me where I needed to go. Third, my family, for supporting me through all the trials, small and great. Thanks to the rest of my committee who so graciously baptized me with fire. I need to thank the Wontrop family for so graciously inviting me into their home, especially Cal Wontrop, without whose distractions this thesis would have been finished much sooner, but perhaps not so pleasantly. Thanks to Jen Nash, Christine Gibson, and Lisa Darlington, who suffered through classes with me and were "foul-weather friends." Thanks to Hannah Swiger, the former Mathematics Department secretary, who was so efficient at cutting through red tape. Thanks to my fiancée, Susan Garrison, who helped spur me on to the finish line. Indeed, both Susan and her mother went through my thesis checking for grammar mistakes, for which I thank them deeply. For any remaining mistakes I, as usual, accept all responsibility.

Last, and certainly greatest by far, thanks be to God the Father, Son and Holy Spirit for everything.

Soli Deo Gloria

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Introduction

As a motivation for studying the Manakov system, allow me to relate the main goal of the people who work in wired data communications and networking: to communicate over a wire. The usual procedure is to send a signal of some kind in an agreed-upon format (i.e., both sender and receiver know what the signals mean) across the wire. However, there is a hitch: all signals degrade over distance and time. The sender may send some signal that he *thinks* is according to the format, but by the time it gets to the receiver, it has lost all intelligibility. The phenomenon is not unlike the game of gossip.

One solution to the problem is to have very reliable *repeaters* at strategic points along the wire. These machines are capable of recognizing an only mildly degraded signal, refreshing it, and sending it farther down the wire. This is the strategy used in some networks. The problem with this strategy is that repeaters are expensive; if the sender has a long wire that needs a lot of repeaters, the finances only get worse.

A different strategy would be to improve the quality of the wire and format; not all wires and formats are created equal. As an example, Unshielded Twisted Pair Category 5 wire (usually called UTP Cat 5) is inferior to UTP Cat 7 wire. The signals usually sent on such wires are electrical in nature. Both, however, are inferior to light waves sent down a fiber optic cable. One reason fiber is superior has a great deal to do with the topic of this thesis: solitons.

Solitons are solitary waves that propagate in some medium without changing shape very much; they can even self-correct their shape. (They also interact with each other in a particle-like way. They "collide," perhaps changing phase; this is the reason they are called soli*tons*.) The reader can see that this would be a boon for the senders and receivers, since repeaters would not be so necessary; at least network designers would not have to use so many. It was John Scott Russell who first discovered the "wave of translation," as he called it, in 1834. He wrote the following description in his *Report on Waves*, Fourteenth meeting of the British Association for the Advancement of Science, 1844:

> "I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great

> velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation."

What I am investigating is the existence of solitons in fiber optic cables. As it turns out, the existence of solitons is closely related to the existence of eigenvalues of the Manakov system, in a way that will hopefully become clear later on.

The significance of this undertaking lies in the improved accuracy of mathematically modeling solitons in optical fibers.

There are several provisos, however. For one thing, it is impossible to send down an optical fiber a solitary wave precisely in a hyperbolic secant shape, the shape of a pure soliton. This is a limitation of current technology at least, if not of physics itself: see Shaw [26], pages 69-70, for an explanation of this limitation. For another, several effects have the unfortunate tendency to mar the self-correction characteristic, including but not limited to "loss, higher order dispersion, or higher order nonlinear effects." - Shaw [26], page 69.

Birefringence is the effect I propose to explore in this study. What is birefringence? It is the effect, in some materials, of differently polarized light waves propagating at different velocities. Not taking birefringence into account leads to the nonlinear Schrödinger equation (NLSE); the interested reader can consult Shaw [26] for a derivation of the NLSE. The Inverse Scattering Transform (IST) maps the NLSE onto the Zakharov-Shabat system. It was Menyuk [22], taking birefringence into account, who derived the coupled nonlinear Schrödinger equations (CNLSE). The IST maps the CNLSE onto the Manakov system. I will explain more about the Inverse Scattering Transform Method in Chapter 1.

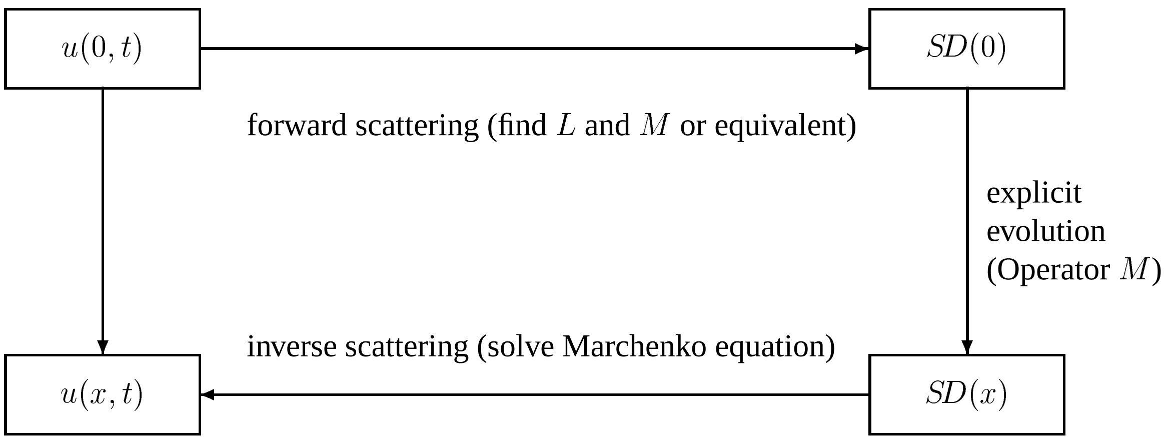
And now, onward!

Chapter 1

Introduction to the Inverse Scattering Transform Method

When a physicist encounters a partial differential equation, his first line of attack, as Griffiths [11] explained in his Chapter 2, is the method of separation of variables. On nonlinear pde's, this method almost never works. One rare example is Dym's equation, which occurs in the study of solitons. One recent (1960's) innovation in the field of nonlinear pde's is the Inverse Scattering Transform Method, invented by a number of people, especially Lax, Ablowitz, Kaup, Newell, and Segur.

The following is a diagram reproduced from Shaw [26] on page 71, illustrating the basic idea of the Inverse Scattering Transform Method.



Here follows the explanation of this diagram. Suppose you have a nonlinear pde governing the behavior of the function $u(x,t)$, and you know the shape of u for all time t, but not for all space

x.[1] You employ the forward scattering technique to obtain the Scattering Data $SD(0)$ at $x = 0$. The Scattering Data consists of the eigenvalues, norming constants, and reflection coefficients, and the forward scattering technique involves inventing the Lax pair. I will explain the Lax pair in more detail below. After you have the Scattering Data at $x = 0$, you employ standard explicit evolution methods to obtain the Scattering Data for all x. When I say standard explicit evolution, I mean that the forward scattering technique has transformed the nonlinear pde into (generally) a system of linear ode's. Therefore, we may employ the standard methods of solving ode's in order to complete the explicit evolution step. Finally, we use the inverse scattering direction to construct the solution $u(x, t)$ from the Scattering Data. This inverse direction involves solving the Marchenko equation, a complicated but linear integral equation. See Ablowitz and Segur [1] for more details.

As the reader might have noticed, this procedure bears a striking similarity in structure to the Laplace Transform, and many other transform techniques. I will now give more details as to each step in the IST method, although the reader will find a yet more detailed treatment in Ablowitz and Segur [1].

1.1 The Forward Direction

The goal of the forward scattering problem is to obtain the Scattering Data at $x = 0$ given the function $u(0, t)$.

We have a pde, typically nonlinear, governing the behavior of some function u. It might be, for example, the Korteweg de-Vries (KdV) equation:

$$u_t - 6uu_x + u_{xxx} = 0. \tag{1.1.1}$$

We will investigate two possible methods of procedure. The first is the Lax Pair method, and secondly we investigate the AKNS (Ablowitz, Kaup, Newell, and Segur) procedure. First the Lax Pair method.

Suppose you could think of two operators, called the Lax pair, which satisfy the following conditions:

$$Lv = \lambda v, \tag{1.1.2}$$

$$v_t = Mv. \tag{1.1.3}$$

[1] A word about temporal versus spatial variables is in order. In Ablowitz and Segur [1], the IST is concerned with evolving the temporal variable. In the context of fiber optics, however, this situation is reversed, and the IST is concerned with evolving the spatial variable. To see this, simply compare the NLSE in Shaw [26] with the same equation in Ablowitz and Segur [1]. Therefore, in this paper I will use the temporal variable t as the variable of differentiation in both the Zakharov-Shabat system and the Manakov system, since the variable of those systems is the opposite of whichever one is used for the IST.

Here λ is any arbitrary spectral parameter, and v is the corresponding eigenfunction, possibly a vector. We assume that λ might depend on t, but definitely not on x. Suppose you were to take Eq. (1.1.2) and differentiate both sides with respect to t. You would get the following:

$$L_t v + L v_t = \lambda_t v + \lambda v_t. \tag{1.1.4}$$

Because of Eq. (1.1.3), we may substitute in for v_t thus:

$$L_t v + LMv = \lambda_t v + \lambda M v = \lambda_t v + M\lambda v = \lambda_t v + MLv. \tag{1.1.5}$$

It follows that

$$L_t v + (LM - ML)v = \lambda_t v. \tag{1.1.6}$$

What we would like is for $\lambda_t = 0$. The reason we would like $\lambda_t = 0$ is because then we can construct $u(x,t)$ much more easily from the scattering data. It is true that $\lambda_t = 0$ will happen if and only if $L_t + (LM - ML) = 0$, since we do not allow the eigenfunction v, by definition, to be identically zero. This equation,

$$L_t + (LM - ML) = 0, \tag{1.1.7}$$

we call Lax's equation. With these integrable PDE's such as the NLSE, the difficulty is determining the Lax pairs L and M. Once we have those, the procedure is much more mechanical. However, as Ablowitz and Segur put it in [1] on page 9, "The difficulties with this method [the Lax Pair method] are that (a) one must 'guess' a suitable L and then find an M in order to satisfy (1.2.4a, b) and (b) it is awkward to work with differential operators (e.g., sine-Gordon (1.2.3))."

Various mathematicians have found Lax pairs for the KdV (1.1.1) equation. One possible choice for the Lax pair for this equation (See Asano and Kato [3], page xiv.) is the following:

$$L = -D^2 + u, \tag{1.1.8}$$
$$M = -4D^3 + 3(Du + uD), \tag{1.1.9}$$

where $D := \frac{\partial}{\partial x}$. The idea is that Eq. (1.1.7) contains within it Eq. (1.1.1) for the above choice of L and M.

To see that this is so, we must assemble the ingredients of Eq. (1.1.7). First, we must have LM. It is best to introduce a test function when working with operators. We shall use y as our test function for all such calculations. Then

$$LMy = (-D^2 + u)(-4D^3 + 3Du + 3uD)y \tag{1.1.10}$$
$$= 4D^5 y - 3D^3 uy - 3D^2 uDy - 4uD^3 y + 3uDuy + 3u^2 Dy. \tag{1.1.11}$$

We need to determine several quantities here. The idea is that we need all the D's to appear on the far right hand side of whatever expression they inhabit, less the test function. The first such expression is $D^3 u$. We append our test function and discover the following:

$$D^3 uy = u_{xxx} y + 3u_{xx} y_x + 3u_x y_{xx} + u y_{xxx} \tag{1.1.12}$$
$$= (u_{xxx} + 3u_{xx} D + 3u_x D^2 + uD^3)y. \tag{1.1.13}$$

Thus,

$$D^3 u = u_{xxx} + 3u_{xx}D + 3u_x D^2 + uD^3. \tag{1.1.14}$$

Performing the same computations on D^2uD and uDu produces

$$D^2 uD = u_{xx}D + 2u_x D^2 + uD^3 \tag{1.1.15}$$

$$uDu = uu_x + u^2 D. \tag{1.1.16}$$

Plugging these into the previous expressions yields

$$LM = 4D^5 - 3u_{xxx} - 9u_{xx}D - 9u_x D^2 - 3uD^3 - 3u_{xx}D - 6u_x D^2 \tag{1.1.17}$$

$$- 3uD^3 - 4uD^3 + 3uu_x + 3u^2 D + 3u^2 D$$

$$= 4D^5 - 3u_{xxx} - 12u_{xx}D - 15u_x D^2 - 10uD^3 + 3uu_x + 6u^2 D. \tag{1.1.18}$$

Next, we calculate ML. We obtain

$$ML = (-4D^3 + 3(Du + uD))(-D^2 + u) \tag{1.1.19}$$

$$= 4D^5 - 4D^3 u - 3DuD^2 + 3Du^2 - 3uD^3 + 3uDu. \tag{1.1.20}$$

The relevant quantities we need are the following:

$$DuD^2 = u_x D^2 + uD^3 \tag{1.1.21}$$

$$Du^2 = 2uu_x + u^2 D. \tag{1.1.22}$$

Substituting these into ML yields

$$ML = 4D^5 - 4u_{xxx} - 12u_{xx}D - 12u_x D^2 - 4uD^3 - 3u_x D^2 - 3uD^3 \tag{1.1.23}$$

$$+ 6uu_x + 3u^2 D - 3uD^3 + 3uu_x + 3u^2 D \tag{1.1.24}$$

$$= 4D^5 - 4u_{xxx} - 12u_{xx}D - 15u_x D^2 - 10uD^3 + 9uu_x + 6u^2 D. \tag{1.1.25}$$

Finally, we need L_t. The non-intuitive fact about L_t is that we only take the time derivative where t explicitly appears in L. The reason is that we would like to have

$$(Ly)_t = L_t y + Ly_t. \tag{1.1.26}$$

On the left hand side, we have

$$\frac{\partial}{\partial t}((-D^2 + u)y) = -\frac{\partial}{\partial t} D^2 y + u_t y + uy_t = -D^2 y_t + u_t y + uy_t = Ly_t + u_t y. \tag{1.1.27}$$

If we compare this calculation with Eq. (1.1.26), we see that $L_t = u_t$, by definition. This is tantamount to setting the $\frac{\partial}{\partial t} D^2$ term in $\frac{\partial}{\partial t} L$ equal to zero.

We have all the pieces for Eq. (1.1.7) now. Putting them together yields the following:

$$\begin{aligned} &L_t + LM - ML \\ &\quad = u_t + 4D^5 - 3u_{xxx} - 12u_{xx}D - 15u_xD^2 - 10uD^3 + 3uu_x + 6u^2D \quad (1.1.28) \\ &\qquad - 4D^5 + 4u_{xxx} + 12u_{xx}D + 15u_xD^2 + 10uD^3 - 9uu_x - 6u^2D \\ &\quad = u_t + u_{xxx} - 6uu_x. \quad (1.1.29) \end{aligned}$$

Therefore, we have recovered the KdV equation; since $L_t + LM - ML = 0$, it must be that $u_t + u_{xxx} - 6uu_x = 0$.

Very well, we have the Lax pair, and we can convince ourselves, given the right pair, that Lax's Equation (1.1.7) is equivalent to the original nonlinear pde.

Secondly, we try the AKNS scheme, which is really a generalization of the Lax Pair method. The AKNS scheme works as follows: suppose we can obtain two operators that govern the time and spatial dependence of the eigenfunctions (See Ablowitz and Segur [1], pages 9ff.):

$$v_x = Xv \quad (1.1.30)$$

$$v_t = Tv. \quad (1.1.31)$$

Note, then, that

$$v_{xt} = X_tv + Xv_t \quad (1.1.32)$$

$$v_{tx} = T_xv + Tv_x. \quad (1.1.33)$$

By Clairaut's Theorem, and given certain smoothness conditions, we may equate the mixed partial derivatives to obtain

$$X_tv + Xv_t = T_xv + Tv_x. \quad (1.1.34)$$

We plug in the expressions for the derivatives of v, and the result is

$$X_tv + XTv = T_xv + TXv. \quad (1.1.35)$$

Therefore,

$$X_tv - T_xv + XTv - TXv = 0, \quad (1.1.36)$$

or since v is not allowed to be zero, we must have

$$X_t - T_x + [X, T] = 0, \quad (1.1.37)$$

the analog of Lax's Equation (1.1.7).

The idea is that this equation produces the original pde. According to Ablowitz and Segur [1], on page 9,

> It turns out that, given X, there is a simple deductive procedure to find a T such that Eq. (1.1.37) contains a nonlinear evolution equation. In order for Eq. (1.1.37) to be effective, the associated operator X should have a parameter which plays the role of an eigenvalue, say ξ, and obeys $\xi_t = 0$.

In this quote, I have changed the letter for the eigenvalue and the equation numbers to be consistent with the rest of this paper.

For details on these calculations, the interested reader can consult Ablowitz and Segur [1], on page 9ff, or also Shaw [26] on pages 74-5.

The next question is, how do we use the pair of operators that we have just produced?

The Lax pair, or equivalently the X and T from the AKNS scheme, plus the corresponding equation (e.g. the AKNS Equation (1.1.37)) give us a linear evolution system of equations governing the eigenfunctions of the operators we just found. We use this system to determine the so-called Scattering Data at $x = 0$ by solving and applying the boundary conditions. The procedure follows.

Firstly, given specific initial conditions for u at one point (possibly infinity, as for the Jost solutions), determine the form of the candidate eigenfunctions. This must happen first because usually the eigenvalues depend on the form of the eigenfunctions and their corresponding boundary conditions (not to be confused with the overall initial conditions for the IST; these boundary conditions at $\pm\infty$ are requirements that the candidate functions must satisfy, by definition, in order to be eigenfunctions). Next, find the eigenvalue(s) for $x = 0$. We do this usually by applying the boundary conditions on the candidate eigenfunctions.

Often, in computations, the form of the eigenfunctions and their boundary conditions is sufficient to determine the eigenvalues. This paper deals mostly with this step of the IST: finding the eigenvalues, and if no closed-form or numerical solution is available, finding theoretical bounds on them. In addition, this paper displays a result concerning how chirp affects the existence of eigenvalues in the Manakov system. Chirp is frequency modulation; for our purposes, we will represent chirp as a complex Gaussian exponential multiplying the complex field envelope function.

Next, find the reflection coefficient for $x = 0$. Currently, all problems with closed-form solutions have rational reflection coefficients, often equal to zero. On page 77 of [26], Shaw states that the "only instance in which $u(\zeta, \tau)$ can be found in explicit form is the *reflectionless* case $R(\xi, 0) = 0$, where one also has $R(\xi, \zeta) = 0$ by (5.22)." (emphasis original) However, on page 4 of [8], Demontis writes that when "the reflection coefficient is a rational matrix function, the inverse scattering problem for eq. (1.7) can be solved in closed form by state space methods." The references Demontis quotes to back his claim, with one exception, all date before Shaw's book [26]. In addition, on pages 105-7 of [20], Lamb derives an explicit closed form solution of the Zakharov-Shabat system with rational reflection coefficients. Therefore, Shaw's book is incorrect on this point.[2]

Next, find the norming constants for $x = 0$. In general, both the reflection coefficient and the norming constants depend on the coefficients of the eigenfunctions expressed as a linear combination of the Jost solutions, which describe the behavior of the solutions at $\pm\infty$. See Ablowitz and Segur [1] on pages 27-8 for the derivation of the norming constants and the reflection coefficients. Together, the eigenvalues, norming constants, and reflection coefficient make up what is called the scattering data.

[2] In direct emails with Dr. Shaw, he has acknowledged the misstatement.

1.2 Direct Linear Evolution

This step involves determining the spatial evolution of the eigenvalues, reflection coefficient, and norming constants. This evolution in space is usually not very complicated since it is often described by a first-order linear differential equation. See Shaw [26] on page 75-6 for a discussion of this evolution.

1.3 Inverse Scattering

The last step of the IST for the NLSE involves solving the Gel'fand-Levitan-Marchenko integral equation for its kernel $K(x, y; t)$, independent of the eigenvalue:

$$K(x,y;t) \mp F^*(x+y;t) \pm \int_x^\infty \int_x^\infty K(x,z;t)\, F(y+s;t)\, F^*(s+z;t)\, ds\, dz = 0, \qquad (1.3.1)$$

where we compute F as the following:

$$F(x;t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{b}{a}\,(\xi,t)\, e^{i\xi x}\, d\xi - i \sum_{j=1}^{N} C_j(t)\, e^{i\xi_j x}. \qquad (1.3.2)$$

Note that the scattering data shows up in the definition of F: b/a is the reflection coefficient, the $C_j(t)$ are the norming constants, and the ξ_j are the eigenvalues. Once you have the kernel, the final solution to the overall problem is $q(x,t) = -2K(x,x;t)$. For a derivation of (1.3.1), see Ablowitz and Segur [1] on page 20ff and also on page 48.

1.4 Summary of the IST

The Inverse Scattering Transform Method as applied to physical pde's is therefore as follows, in outline:

1. Derive the original nonlinear pde from the physics of the situation
2. Forward scattering
    a. Find the Lax pair, or AKNS pair
    b. Find the eigenfunctions for $x = 0$
    c. Find the scattering data for $x = 0$
        i. From the eigenfunctions and their imposed boundary conditions and the potential at $x = 0$, find the eigenvalues for $x = 0$
        ii. Find the norming constants for $x = 0$
        iii. Find the reflection coefficient for $x = 0$ (the zero case is definitely the easiest, and still requires a lot of work)
3. Direct evolution: find the scattering data for all x
4. Inverse Scattering
    a. Using the scattering data at x, solve the Gel'fand-Levitan-Marchenko integral equation
    b. Construct the potential $u(x, t)$ from the solution to the integral equation

Chapter 2

The First Problem: a Zakharov-Shabat System

2.1 Derivation of the Zakharov-Shabat System

2.1.1 The NLSE

The four Maxwell equations govern the electromagnetic field in a fiber optic cable, and they are as follows:

$$\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}, \qquad \text{Faraday's Law} \tag{2.1.1}$$

$$\nabla \times \vec{H} = \frac{\partial}{\partial t}(\varepsilon_0 \vec{E} + \vec{P}), \qquad \text{Ampere's Law} \tag{2.1.2}$$

$$\nabla \cdot \vec{H} = 0, \qquad \text{Gauss's Law for Magnetism} \tag{2.1.3}$$

$$\nabla \cdot \vec{E} = \nabla \cdot \vec{P} = 0, \qquad \text{Gauss's Law} \tag{2.1.4}$$

where $\vec{E}$ is the electric field, μ_0 the magnetic permeability constant, $\vec{H}$ the magnetic field, ε_0 the dielectric permittivity constant, and $\vec{P}$ the polarization vector. They are Maxwell's Equations with no source charges.

The reader can consult Shaw [26] on pages 30-5 for a derivation of the nonlinear Schrödinger equation (NLSE) from the Maxwell equations. The NLSE follows:

$$\frac{\partial A}{\partial z} + i(\beta_2/2)\frac{\partial^2 A}{\partial t^2} = i\gamma|A|^2 A. \tag{2.1.5}$$

Here A is the slowly varying envelope of the electric field, and γ is the Kerr coefficient. β_2 is the second-order Taylor series coefficient in the expansion of $\beta(\omega)$, which is the propagation constant corresponding to the fundamental mode, as described on page 34 of Shaw [26]. It is also known as the group velocity dispersion. The rationale for this terminology is explained on page 36 of Shaw [26], in the last full paragraph. The variable z represents physical length along the fiber, "and t is time, initialized to the pulse center," or local time. (See Shaw [26], page 27.) It is interesting to note that β_1, the first-order Taylor series coefficient in the expansion of $\beta(\omega)$, is usually interpreted as the inverse of the group velocity. Through several changes of variables chosen in order to remove dimensions, this time described on pages 42-3 of Shaw [26], the NLSE appears as

$$i\,u_\zeta + \frac{1}{2}\,u_{\tau\tau} + |u|^2\,u = 0 \tag{2.1.6}$$

for the focusing case, the only case in which solitons appear. The defocusing case has the opposite sign for the $(1/2)\,u_{\tau\tau}$ term, and these terms are conventions in the optical communications literature.

The normalizations used to produce Eq.(2.1.6) from Eq.(2.1.5) in Shaw [26] are as follows: Define the peak power $P_0 = |A(0,0)|^2$. Then let $U(z,t) \equiv A(z,t)/\sqrt{P_0}$. The dimensionless time we define by $\tau = t/T_0$, where T_0 is some reference pulse width such as half-width half-max or root-mean-square width. Similarly, define $L_D = T_0^2/|\beta_2|$ as the dispersion length and let $\zeta = z/L_D$. Then $U(\zeta,\tau)$ satisfies, from Eq.(2.1.5), the following:

$$i\,U_\zeta = \frac{\text{sign}(\beta_2)}{2}\,U_{\tau\tau} - N^2\,|U|^2\,U, \tag{2.1.7}$$

where $N^2 = \gamma\,L_D\,P_0$. Finally, let $u(\zeta,\tau) \equiv N\,U(\zeta,\tau)$. $u(\zeta,\tau)$ satisfies $i\,u_\zeta = \text{sign}(\beta_2/2)\,u_{\tau\tau} - |u|^2\,u$. Shaw [26] writes on p. 43 that in the soliton literature, "...the case sign$(\beta_2) < 0$ is called the *focusing case* and sign$(\beta_2) > 0$ is called the *defocusing case*... Solitons can form in the focusing case but not otherwise."

I should point out that ζ is normalized space (non-dimensional), and τ is normalized local time (non-dimensional). See Hasegawa and Matsumoto [12] on pages 23-4, noting that Shaw's ζ is Hasegawa's Z, and Shaw's τ is Hasegawa's T. For the remainder of this paper, I shall revert back to using x and t as the non-dimensional space and time, respectively.

The next question is that of well-posedness. In other words, given the NLSE, and unique initial conditions, can we be guaranteed unique solutions? The answer is yes. The author who proved this is X. Zhou, in [27]. Zhou showed the well-posedness of Eq.(2.1.6) in a weighted Sobolev space, and that every step of the IST is a one-to-one mapping, meaning reversible. So it makes sense to set about trying to find solutions.

2.1.2 AKNS Operators for the NLSE

Mathematicians have concocted (see Shaw [26] on page 75) the AKNS Pair for the nonlinear Schröd-inger Equation. They are as follows:

$$X = \begin{bmatrix} i(|u|^2 - 2\xi^2) & iu_t + 2\xi u \\ iu_t^* - 2\xi u^* & -i(|u|^2 - 2\xi^2) \end{bmatrix}, \text{ and} \tag{2.1.8}$$

$$T = \begin{bmatrix} -i\xi & u \\ -u^* & i\xi \end{bmatrix}. \tag{2.1.9}$$

Substituting these into (1.1.37) produces the NLSE.

The Zakharov-Shabat (ZS) system is the equation $\vec{v}_t = T\vec{v}$ for the NLSE (Eq. (1.1.31)) in the AKNS scheme, and is therefore

$$\dot{v}_1(t) = -i\,\xi\,v_1(t) + q(t)\,v_2(t) \tag{2.1.10}$$

$$\dot{v}_2(t) = -q^*(t)\,v_1(t) + i\,\xi\,v_2(t). \tag{2.1.11}$$

Here $q(t) := u(0,t)$ and dots denote time derivatives. The v_k are the eigenfunctions for $k = 1, 2$, and ξ is the eigenvalue. The reader may well ask of what operator ξ is an eigenvalue. To show the operator, we will need a helper matrix J defined as follows:

$$J \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.1.12}$$

Note that this is also the third Pauli spin matrix. Also, let $\mathbf{v} \equiv [v_1\ v_2]^T$. Now the ZS we can rewrite as

$$\dot{\mathbf{v}} = \begin{bmatrix} -i\xi & q \\ -q^* & i\xi \end{bmatrix} \mathbf{v}. \tag{2.1.13}$$

Multiply this equation on the left by iJ to obtain

$$iJ\dot{\mathbf{v}} = i \begin{bmatrix} -i\xi & q \\ q^* & -i\xi \end{bmatrix} \mathbf{v} = \begin{bmatrix} \xi & iq \\ iq^* & \xi \end{bmatrix} \mathbf{v}. \tag{2.1.14}$$

From here we can see that

$$iJ\dot{\mathbf{v}} - \begin{bmatrix} 0 & iq \\ iq^* & 0 \end{bmatrix} \mathbf{v} = \xi\mathbf{v}. \tag{2.1.15}$$

Thus ξ is an eigenvalue for the operator

$$iJ\frac{d}{dt} - \begin{bmatrix} 0 & iq \\ iq^* & 0 \end{bmatrix}. \tag{2.1.16}$$

2.2 Statement of the Problem

For the first problem, we assign a particular function for q in the Zakharov-Shabat system. We could choose most any function, but we will go with

$$q(t) = h(\theta(t+d) - \theta(t-d)), \tag{2.2.1}$$

where $\theta(t)$ is the Heaviside unit step function defined by

$$\theta(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}, \tag{2.2.2}$$

h the height, and $2d$ the width of this square pulse.

Now in order to solve this system, we have to make some assumptions. The first assumption is that our solution, $v(t)$, is continuous and a member of L^2, i.e., $\int_{-\infty}^{\infty} \|v(t)\|_2 \, dt < \infty$. The second assumption is that the eigenvalues ξ have positive imaginary part: Im$(\xi) > 0$. The reason for this assumption is that the Zakharov-Shabat system has a great deal of symmetry. If ξ is an eigenvalue, then so are ξ^*, $-\xi$, and $-\xi^*$. See Klaus [15], on page 28. Without loss of generality, therefore, it suffices to take Im$(\xi) > 0$.

Our goal for the first problem is to find the eigenvalue, ξ, that solves the linear homogeneous ODE above subject to the appropriate boundary conditions, and to investigate how it changes with respect to a particular parameter. We will take the height of the potential, h as the parameter. This is a technique we will use extensively in calculations.

2.3 Solving the System

There are three regions for time t, which we call R1, R2, and R3. R1 corresponds to $(-\infty, -d)$, R2 to $(-d, d)$, and R3 to (d, ∞).

In R1, $q(t) = 0$. Hence the Zakharov-Shabat system appears as follows:

$$\dot{v}_1 = -i\,\xi\, v_1, \tag{2.3.1}$$
$$\dot{v}_2 = i\,\xi\, v_2. \tag{2.3.2}$$

The solution is $v_1(t) = Ae^{-i\,\xi\,t}$ and $v_2(t) = Be^{i\,\xi\,t}$. Since the v_j are members of L^1, it must be that $B = 0$, since otherwise v_2 would blow up as $t \to -\infty$. We normalize A such that $v_1(-d) = 1$. This implies that $v_1(t) = e^{-i\,\xi\,(t+d)}$ and $v_2(t) \equiv 0$ in R1.

In R2, the solution is a bit more complicated. The system written in matrix form is

$$\dot{V} = \begin{bmatrix} -i\xi & h \\ -h & i\xi \end{bmatrix} V = \mathbf{A}\, V. \tag{2.3.3}$$

The solution here is $V(t) = e^{(t+d)\mathbf{A}} V_0$, where $V_0 = (1, 0)^T$. Let $\psi = \sqrt{h^2 + \xi^2}$. We compute the matrix exponential using Mathematica; the result, after simplifying, factoring i out to form the ψ terms, and changing exponential to trigonometric functions, is

$$V(t) = \begin{bmatrix} \cos((t+d)\,\psi) - \frac{i\,\xi\,\sin((t+d)\,\psi)}{\psi} & \frac{h\,\sin((t+d)\,\psi)}{\psi} \\ -\left(\frac{h\,\sin((t+d)\,\psi)}{\psi}\right) & \cos((t+d)\,\psi) + \frac{i\,\xi\,\sin((t+d)\,\psi)}{\psi} \end{bmatrix} V_0. \tag{2.3.4}$$

It follows that

$$V(t) = \begin{bmatrix} \cos((t+d)\,\psi) - \frac{i\,\xi\,\sin((t+d)\,\psi)}{\psi} \\ -\left(\frac{h\,\sin((t+d)\,\psi)}{\psi}\right) \end{bmatrix}. \tag{2.3.5}$$

This is the solution in R2.

The solution in R3 is $v_1(t) = C\,e^{-i\,\xi\,t}$ and $v_2(t) = D\,e^{i\,\xi\,t}$. But again, since the solution has to be exponentially decaying (or L^2), we need $C = 0$, else the solution will blow up as $t \to \infty$. Therefore, in order to match up the solutions in R2 and R3 at $+d$, we force

$$\psi\,\cos(2d\psi) = i\xi\sin(2d\psi)\,, \tag{2.3.6}$$

or

$$\cot(2d\psi) = \frac{i\xi}{\psi}\,. \tag{2.3.7}$$

Substituting back in what ψ is gives us

$$\cot(2d\sqrt{h^2+\xi^2}) = \frac{i\xi}{\sqrt{h^2+\xi^2}}\,. \tag{2.3.8}$$

Finally, we note that in Klaus and Shaw [18], the authors proved that if q is a single-lobe potential with $\left|\int_{\mathbb{R}} q(t)\,dt\right| > \pi/2$, then all eigenvalues are pure imaginary. Our particular q will satisfy the conditions of the theorem provided $h > \pi/4$. Therefore, we may take $\xi = is$, $s \in \mathbb{R}^+$. Recall that the imaginary part was positive by assumption. Plugging this expression into our equation gives

$$\cot(2d\sqrt{h^2-s^2}) = -\frac{s}{\sqrt{h^2-s^2}}\,. \tag{2.3.9}$$

The goal now is to see how s changes as a function of h. Let $d = 1$. Solving this equation is impossible analytically, and thus we are forced to try a numerical scheme. Simply using ImplicitPlot in Mathematica produces a figure with good and “bad” curves (the “bad” ones being very jagged). While it is possible to rule out the bad curves as not being eigenvalue curves, using Theorem 4.6 in Klaus [15], there is a better way to plot them.

The idea is to take the h derivative of Eqn. 2.3.9, and then use a numerical scheme to find $s(h)$. Theorem 4.6 in Klaus [15] gives us the initial conditions for each eigenvalue curve. Note that we end up using this scheme to great advantage in the second problem. Mathematica reveals that

$$s'(h) = \frac{h\left(2h^2 - 2s^2(h) + s(h)\,\sin^2\left(2\sqrt{h^2 - s^2(h)}\right)\right)}{2h^2 s(h) - 2s^3(h) + h^2\,\sin^2\left(2\sqrt{h^2 - s^2(h)}\right)}. \tag{2.3.10}$$

Using the NDSolve command and utilizing odd-integer multiples of $\pi/4$ as initial conditions $(s((2N-1)\pi/4) = 0)$ produces Figure 2.1.

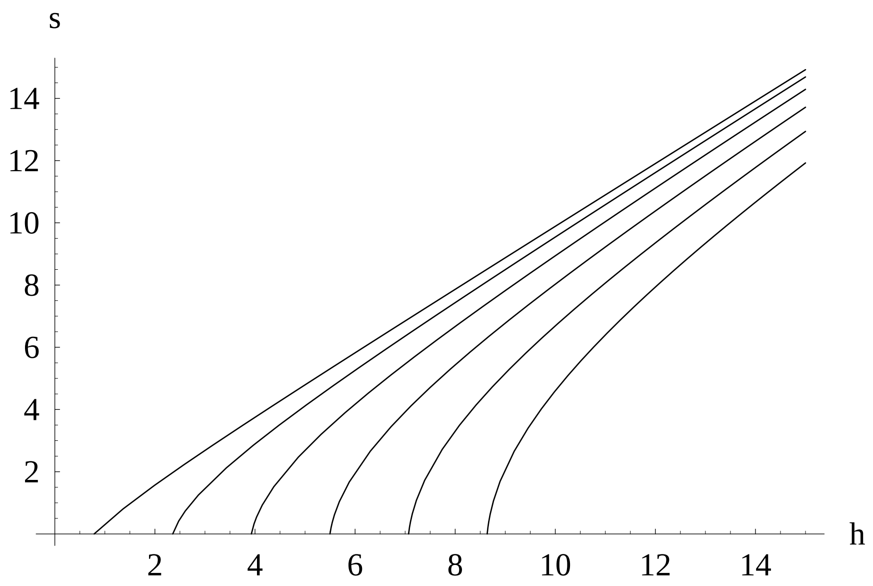


Figure 2.1: Only true eigenvalue curves

This was the goal of the first problem: plot the eigenvalue as a function of the initial parameter h. The numerical techniques used will show up again in later chapters, hence the main usefulness of these computations lies in the learning process.

Chapter 3

The Second Problem: Relating the Manakov System to its Two Associated Zakharov-Shabat Systems

3.1 Introduction to the Manakov System

The reader may see for himself in Shaw [26] how to derive the Zakharov-Shabat system from the Maxwell equations and the appropriate boundary conditions. The first step is the nonlinear Schrödinger Equation, followed by the forward scattering direction. The result is the Zakharov-Shabat system.

However, the Zakharov-Shabat system does not account for birefringence in the fiber optic cable. According to the McGraw-Hill Dictionary of Physics [9], birefringence is the

> **1.** Splitting of a light beam into two components, which travel at different velocities, by a material. **2.** For a light beam that has been split into two components by a material, [it is] the difference in the indices of refraction of the components within the material. It is also known as double refraction.

Fiber optic cables produce a birefringent effect partially because of the presence of anisotropies and asymmetries. See Menyuk [22], page 118.

It was Menyuk, in 1999, who published [22] the first complete derivation of the coupled nonlinear Schrödinger equations (cNLSE's), although scientists and mathematicians were aware of their existence before. The cNLSE's, as he derived them, follow:

$$i\,\frac{\partial \mathbf{U}}{\partial z} - ig\mathbf{U} + (\cos\,\theta\sigma_3 + \sin\,\theta\sigma_1)\left(\Delta\beta\mathbf{U} + i\Delta\beta'\frac{\partial \mathbf{U}}{\partial t}\right) - \tfrac{1}{2}\,\beta''\frac{\partial^2 \mathbf{U}}{\partial t^2} - \tfrac{1}{6}\,i\beta'''\frac{\partial^3 \mathbf{U}}{\partial t^3} + \gamma[|\mathbf{U}|^2\mathbf{U} - \tfrac{1}{3}(\mathbf{U}^\dagger\sigma_2\mathbf{U})\sigma_2\mathbf{U}] = 0. \tag{3.1.1}$$

To explain what $\mathbf{U}$ is, first consider the vector $\mathbf{F}$, the slowly varying envelope of the electric field (page 117)[1]. On page 120, Menyuk expands $\mathbf{F}$ into a linear combination of two vectors $\mathbf{R}_1$ and $\mathbf{R}_2$. The coefficients are u_1 and u_2, respectively, and the $\mathbf{R}_j$ "...are the two orthogonal eigenmodes of a single mode fiber..." (page 120) z is the distance along the fiber. t is a somewhat complicated clock. According to Menyuk [23], on page 2679, the

> ...time variable s is not proportional to t, absolute time in the laboratory frame. Its variation at any point ξ is proportional to time measured in the laboratory frame, but its origin is ξ dependent. The origin is chosen so that if a signal moved at the group velocity intermediate between that of the two modes, the evolution in ξ of its s profile would appear to be frozen. Hence, terms proportional to $\partial u/\partial s$ and $\partial v/\partial s$ appear with opposite sign in (49) to account for the group velocity difference between the two modes.

Note that Menyuk's variable names in this different paper are different from mine. g is defined as $g(\omega_0) \equiv -\text{Im}[\beta(\omega_0)]$, where $\beta(\omega)$ is the central wavenumber, chosen "so that $\mathbf{F}$ is slowly varying relative to the wavelength of light." (page 117) The variable θ is an orientation angle, the matrices $\sigma_1, \sigma_2,$ and σ_3 are the standard Pauli spin matrices. Finally, γ is a constant, equal to $\omega_0 n_2/cA_{\text{eff}}$, where ω_0 is the carrier frequency of the signal (or central frequency), n_2 is the Kerr coefficient, c is the speed of light, and A_{eff} is the effective area defined by

$$\frac{1}{A_{\text{eff}}} \equiv \int_0^{2\pi}\int_0^{\infty}(\mathbf{R}_j^* \cdot \mathbf{R}_j)^2\,\rho\,d\rho\,d\theta. \tag{3.1.2}$$

In this definition, $j = 1$ or 2.

The fact that scientists and mathematicians were aware of the cNLSE's before Menyuk explains why Manakov [21], in 1974, was able to show how the Inverse Scattering Transform method could work on the cNLSE's. The version of the cNLSE's that Manakov used follows:

$$i\,\frac{\partial E_1}{\partial t} + \frac{\partial^2 E_1}{\partial x^2} + \kappa(|E_1|^2 + |E_2|^2)E_1 = 0, \tag{3.1.3}$$

$$i\,\frac{\partial E_2}{\partial t} + \frac{\partial^2 E_2}{\partial x^2} + \kappa(|E_1|^2 + |E_2|^2)E_2 = 0. \tag{3.1.4}$$

[1] In this paragraph, all page numbers refer to Menyuk's article [22].

Suppose $\mathbf{E}$ is the slowly varying complex envelope of the electric field. Represent $\mathbf{E}$ "as a sum of right- and left-hand polarized waves: $\mathbf{E} = E_1 c_R + E_2 c_L$, where c_R and c_L are the complex unit vectors corresponding to right-hand and left-hand polarizations." (See Manakov [21], page 249.) The constant κ is dimensionless.

The Lax pair L and A that he used to represent these two equations in the form $\partial L/\partial t = i[L, A]$ was the following:

$$L = i\begin{bmatrix} 1-p & 0 & 0 \\ 0 & 1+p & 0 \\ 0 & 0 & 1+p \end{bmatrix}\frac{\partial}{\partial x} + \begin{bmatrix} 0 & E_1 & E_2 \\ E_1^* & 0 & 0 \\ E_2^* & 0 & 0 \end{bmatrix}, \text{ and} \tag{3.1.5}$$

$$A = -p\frac{\partial^2}{\partial x^2} + \begin{bmatrix} -(|E_1|^2+|E_2|^2)/(1-p) & -iE_{1x} & -iE_{2x} \\ iE_{1x}^* & |E_1|^2/(1+p) & E_2E_1^*/(1+p) \\ iE_{2x}^* & E_2^*E_1/(1+p) & |E_2|^2/(1+p) \end{bmatrix}, \tag{3.1.6}$$

where $\kappa = 2/(1-p^2)$. Thus the eigenvalue problem associated with the cNLSE's is the following:

$$\begin{aligned} v_{1t} + i\xi v_1 &= q_1 v_2 + q_2 v_3, \\ v_{2t} - i\xi v_2 &= -q_1^* v_1 \\ v_{3t} - i\xi v_3 &= -q_2^* v_1. \end{aligned} \tag{3.1.7}$$

Another version, which I will call the modified Manakov system and is due to Kaup and Malomed [14], follows:

$$\begin{aligned} \dot{v}_1 &= -i\,\xi\, v_1 + q_1\, e^{ikt}\, v_2 + q_2\, e^{-ikt}\, v_3 \\ \dot{v}_2 &= -q_1\, e^{-ikt}\, v_1 + i\,\xi\, v_2 \\ \dot{v}_3 &= -q_2\, e^{ikt}\, v_1 + i\,\xi\, v_3. \end{aligned} \tag{3.1.8}$$

The complex exponentials come from Eqs. (1a) and (1b) in Kaup and Malomed [14] and the phase transformation Eqs. (2a) and (2b). The δ in the phase transformation is relabeled k in Eqs. (21a) and (21b). Note that in Eqs. (2a) and (2b) of Kaup and Malomed [14], we set $x = 0$ as per the procedure of the Inverse Scattering Transform.

We will have occasion to use both of these models at various points.

3.2 Existence of the Jost Solutions to the Manakov System and Analyticity of the First Component in ξ

We need to prove that the Jost solutions to the Manakov system exist and that the first component v_1 is analytic in ξ. We require this latter fact in Chapter 3.3.2.

Since it is much easier to work with the integral equation version of the Manakov system, Eq. (3.1.7), we derive the integral equations governing the Jost solutions as follows.

First, multiply through by the integrating factor:

$$\begin{aligned} e^{i\xi t}v_{1t} + i\xi e^{i\xi t}v_1 &= e^{i\xi t}q_1 v_2 + e^{i\xi t}q_2 v_3, \\ e^{-i\xi t}v_{2t} - i\xi e^{-i\xi t}v_2 &= -e^{-i\xi t}q_1^* v_1 \\ e^{-i\xi t}v_{3t} - i\xi e^{-i\xi t}v_3 &= -e^{-i\xi t}q_2^* v_1. \end{aligned} \tag{3.2.1}$$

Each left-hand-side is now a perfect derivative, thus:

$$\begin{aligned} \frac{d}{dt}(e^{i\xi t}v_1) &= e^{i\xi t}q_1 v_2 + e^{i\xi t}q_2 v_3, \\ \frac{d}{dt}(e^{-i\xi t}v_2) &= -e^{-i\xi t}q_1^* v_1 \\ \frac{d}{dt}(e^{-i\xi t}v_3) &= -e^{-i\xi t}q_2^* v_1. \end{aligned} \tag{3.2.2}$$

Integrate both sides to obtain the following most general version of the integral equations:

$$\begin{aligned} \lim_{\tau\to b}[e^{i\xi\tau}v_1(\tau)] - \lim_{\tau\to a}[e^{i\xi\tau}v_1(\tau)] &= \int_a^b e^{i\xi\tau}\,[q_1(\tau)\,v_2(\tau) + q_2(\tau)\,v_3(\tau)]\,d\tau \\ \lim_{\tau\to b}[e^{-i\xi\tau}v_2(\tau)] - \lim_{\tau\to a}[e^{-i\xi\tau}v_2(\tau)] &= -\int_a^b e^{-\xi\tau}q_1^*(\tau)\,v_1(\tau)\,d\tau \\ \lim_{\tau\to b}[e^{-i\xi\tau}v_3(\tau)] - \lim_{\tau\to a}[e^{-i\xi\tau}v_3(\tau)] &= -\int_a^b e^{-\xi\tau}q_2^*(\tau)\,v_1(\tau)\,d\tau. \end{aligned} \tag{3.2.3}$$

Note here that we have suppressed the dependence of the eigenfunctions v_j on the eigenvalues. In addition, a and b could be any real number, or $\pm\infty$, or a variable. Eq. (3.2.3) is the integral equation version of the Manakov system, Eq.(3.1.7).

The Jost solutions are the equivalent of solutions satisfying some specified initial conditions: they obey asymptotic conditions. If we wish to investigate the Jost solutions, denoted $f_j^\pm$ depending on whether we are interested in the conditions at $\pm\infty$, we will need to plug in the corresponding asymptotic conditions. For the Jost functions as $t \to -\infty$, the conditions are as follows:

$$\begin{aligned} \lim_{\tau\to-\infty}[e^{i\xi\tau}f_1^-(\tau)] &= 1 \\ \lim_{\tau\to-\infty}[e^{-i\xi\tau}f_2^-(\tau)] &= 0 \\ \lim_{\tau\to-\infty}[e^{-i\xi\tau}f_3^-(\tau)] &= 0. \end{aligned} \tag{3.2.4}$$

The integral equations governing the Jost solutions as $t \to -\infty$ are as follows:

$$\begin{aligned} f_1^-(t) &= e^{-i\xi t}\left(1 + \int_{-\infty}^t e^{i\xi\tau}\,(q_1\,f_2^- + q_2\,f_3^-)\,d\tau\right) \\ f_2^-(t) &= -e^{i\xi t}\int_{-\infty}^t e^{-i\xi\tau}\,q_1^*\,f_1^-\,d\tau \\ f_3^-(t) &= -e^{i\xi t}\int_{-\infty}^t e^{-i\xi\tau}\,q_2^*\,f_1^-\,d\tau. \end{aligned} \tag{3.2.5}$$

To obtain these equations, set $b = t$, $a = -\infty$, use the conditions for the Jost solution as $t \to -\infty$, and rearrange slightly. Here again we have suppressed the dependence of the Jost solutions on ξ.

To prove existence and analyticity, we will need to set up the kernel iteration scheme for the integral equations governing the Jost solutions. We set up the following iteration scheme using $g_j^{[n]}$, where j is the component number, and $[n]$ is the iteration number starting at $n = 0$.

$$\begin{aligned} g_1^{[n+1]}(t) &= e^{-i\xi t}\left(1 + \int_{-\infty}^{t} e^{i\xi\tau}\,(q_1\, g_2^{[n]}(t) + q_2\, g_3^{[n]}(t))\, d\tau\right) \\ g_2^{[n+1]}(t) &= -e^{i\xi t}\int_{-\infty}^{t} e^{-i\xi\tau}\, q_1^*\, g_1^{[n]}(t)\, d\tau \\ g_3^{[n+1]}(t) &= -e^{i\xi t}\int_{-\infty}^{t} e^{-i\xi\tau}\, q_2^*\, g_1^{[n]}(t)\, d\tau. \end{aligned} \tag{3.2.6}$$

We simply set the first iterates equal to zero: $g_j^{[0]}(t) = 0$ for $j = 1, 2, 3$.

3.2.1 Existence of the Jost Solutions to the Manakov System

Assume that q_1 and q_2 are in $L^1(\mathbb{R})$. We need a bound on $|g_j^{[n+1]} - g_j^{[n]}|$. So let $\Delta_j^{[n]} \equiv g_j^{[n+1]} - g_j^{[n]}$. We seek a system of integral equations that governs the $\Delta_j^{[n]}$'s. We have from Eq.(3.2.6) that

$$\begin{aligned} \Delta_1^{[n]} = g_1^{[n+1]} - g_1^{[n]} = e^{-i\,\xi\,t}&\left[1 + \int_{-\infty}^{t} e^{i\xi\tau}(q_1\, g_2^{[n]} + q_2\, g_3^{[n]})\, d\tau\right] \\ &- e^{-i\,\xi\,t}\left[1 + \int_{-\infty}^{t} e^{i\xi\tau}(q_1\, g_2^{[n-1]} + q_2\, g_3^{[n-1]})\, d\tau\right] \end{aligned} \tag{3.2.7}$$

$$= e^{-i\,\xi\,t}\int_{-\infty}^{t} e^{i\xi\tau}\left[q_1\, \Delta_2^{[n-1]} + q_2\, \Delta_3^{[n-1]}\right]\, d\tau. \tag{3.2.8}$$

We rewrite this as

$$\Delta_1^{[n+1]} = e^{-i\,\xi\,t}\int_{-\infty}^{t} e^{i\xi\tau}\left[q_1\, \Delta_2^{[n]} + q_2\, \Delta_3^{[n]}\right]\, d\tau. \tag{3.2.9}$$

Similarly, we have

$$\Delta_2^{[n+1]} = -e^{i\,\xi\,t}\int_{-\infty}^{t} e^{-i\xi\tau}\, q_1^*\, \Delta_1^{[n]}\, d\tau, \tag{3.2.10}$$

and

$$\Delta_3^{[n+1]} = -e^{i\,\xi\,t}\int_{-\infty}^{t} e^{-i\xi\tau}\, q_2^*\, \Delta_1^{[n]}\, d\tau. \tag{3.2.11}$$

Examining our iteration scheme, we decide to define $\tilde{q}(t) = \max\{|q_1(t)|, |q_2(t)|\}$ for all t. Also, suppose $\xi = \alpha + i\beta$, where $\alpha, \beta \in \mathbb{R}$, and $\beta > 0$. We work with the inequalities thus:

$$\left|\Delta_1^{[n+1]}\right| \leq e^{\beta t} \int_{-\infty}^{t} e^{-\beta t} \left(\tilde{q} \left|\Delta_2^{[n]}\right| + \tilde{q} \left|\Delta_3^{[n]}\right| \right) \, d\tau \tag{3.2.12}$$

$$\left|\Delta_2^{[n+1]}\right| \leq \int_{-\infty}^{t} \tilde{q} \left|\Delta_1^{[n]}\right| \, d\tau \tag{3.2.13}$$

$$\left|\Delta_3^{[n+1]}\right| \leq \int_{-\infty}^{t} \tilde{q} \left|\Delta_1^{[n]}\right| \, d\tau. \tag{3.2.14}$$

Plugging both the second and third inequalities back into the first gives us the following:

$$\left|\Delta_1^{[n+2]}\right| \leq 2\, e^{\beta t} \int_{-\infty}^{t} e^{-\beta \tau} \tilde{q}(\tau) \int_{-\infty}^{\tau} \tilde{q}(s) \left|\Delta_1^{[n]}(s)\right| \, ds \, d\tau. \tag{3.2.15}$$

Note that this is an inequality relating a particular iteration to the iteration before the last. In other words, we skip one iteration in this inequality. By examining the $g_j^{[n]}$'s, we see that $\Delta_1^{[0]} = e^{-i\,\xi\, t}$ and $\Delta_1^{[1]} = 0$. Consequently, $\Delta_1^{[n]} = 0$ for all n odd. The above inequality yields the following:

$$\left|\Delta_1^{[2]}\right| \leq 2\, e^{\beta t} \int_{-\infty}^{t} e^{-\beta \tau} \tilde{q}(\tau) \int_{-\infty}^{\tau} \tilde{q}(s)\, e^{\beta s} \, ds \, d\tau \tag{3.2.16}$$

$$\leq 2\, e^{\beta t} \int_{-\infty}^{t} \tilde{q}(\tau) \int_{-\infty}^{\tau} \tilde{q}(s) \, ds \, d\tau \tag{3.2.17}$$

$$\leq 2\, e^{\beta t} \, \frac{\left(\int_{-\infty}^{t} \tilde{q} \right)^2}{2!} \tag{3.2.18}$$

$$\leq e^{\beta t} \, \frac{\left(\sqrt{2} \int_{-\infty}^{t} \tilde{q} \right)^2}{2!}. \tag{3.2.19}$$

Therefore, we claim that

$$\left|\Delta_1^{[n]}\right| \leq e^{\beta t} \frac{\left(\sqrt{2} \int_{-\infty}^{t} \tilde{q} \right)^n}{n!}, \quad \text{for } n \text{ even}, \tag{3.2.20}$$

and $\left|\Delta_1^{[n]}\right| = 0$, for n odd. The latter is immediate from Eq.(3.2.15). Certainly the claim holds for

$n = 0$. Assume the result holds for n even; we want to prove it holds for $n + 2$. We have

$$\left|\Delta_1^{[n+2]}\right| \leq 2\, e^{\beta t} \int_{-\infty}^{t} e^{-\beta\tau} \tilde{q}(\tau) \int_{-\infty}^{\tau} \tilde{q}(s) \left|\Delta_1^{[n]}(s)\right| \, ds \, d\tau \tag{3.2.21}$$

$$\leq 2\, e^{\beta t} \int_{-\infty}^{t} e^{-\beta\tau} \tilde{q}(\tau) \int_{-\infty}^{\tau} \tilde{q}(s)\, e^{\beta s} \frac{\left(\sqrt{2}\int_{-\infty}^{s} \tilde{q}(u)\, du\right)^n}{n!} \, ds \, d\tau \tag{3.2.22}$$

$$\leq 2\, (\sqrt{2})^n\, e^{\beta t} \int_{-\infty}^{t} \tilde{q}(\tau) \int_{-\infty}^{\tau} \tilde{q}(s)\, \frac{\left(\int_{-\infty}^{s} \tilde{q}(u)\, du\right)^n}{n!} \, ds \, d\tau \tag{3.2.23}$$

$$\leq 2\, (\sqrt{2})^n\, e^{\beta t} \int_{-\infty}^{t} \tilde{q}(\tau)\, \frac{\left(\int_{-\infty}^{\tau} \tilde{q}(s)\, ds\right)^{n+1}}{(n+1)!} \, d\tau \tag{3.2.24}$$

$$\leq (\sqrt{2})^{n+2}\, e^{\beta t}\, \frac{\left(\int_{-\infty}^{t} \tilde{q}(\tau)\, d\tau\right)^{n+2}}{(n+2)!} \tag{3.2.25}$$

$$\leq e^{\beta t}\, \frac{\left(\sqrt{2}\int_{-\infty}^{t} \tilde{q}(\tau)\, d\tau\right)^{n+2}}{(n+2)!}. \tag{3.2.26}$$

Thus, we have a bound on $|g_1^{[n+1]} - g_1^{[n]}|$. From Eq. (3.2.10) and Eq. (3.2.11), we can see that $\Delta_2^n = 0 = \Delta_3^n$ for n even, and that, by Eq. (3.2.20),

$$\left|\Delta_j^{[n]}\right| \leq e^{\beta t}\, \frac{\left(\sqrt{2}\int_{-\infty}^{t} \tilde{q}(\tau)\, d\tau\right)^{n}}{n!} \tag{3.2.27}$$

for $j = 2, 3$ and n odd. The next step is to show that the sequence $\{g_1^{[n]}\}$ is uniformly Cauchy on any interval of the form $(-\infty, b)$. Suppose $n > m$. Then we have

$$|g_1^{[n]} - g_1^{[m]}| \leq |g_1^{[n]} - g_1^{[n-1]} + g_1^{[n-1]} - g_1^{[n-2]} + \cdots + g_1^{[m+2]} - g_1^{[m+1]} + g_1^{[m+1]} - g_1^{[m]}| \tag{3.2.28}$$

$$\leq |g_1^{[n]} - g_1^{[n-1]}| + |g_1^{[n-1]} - g_1^{[n-2]}| + \cdots + |g_1^{[m+2]} - g_1^{[m+1]}| + |g_1^{[m+1]} - g_1^{[m]}| \tag{3.2.29}$$

$$\leq |\Delta_1^{[n-1]}| + |\Delta_1^{[n-2]}| + \cdots + |\Delta_1^{[m+1]}| + |\Delta_1^{[m]}| \tag{3.2.30}$$

$$\leq e^{\beta t} \sum_{j=m}^{n-1} \frac{\left(\sqrt{2}\int_{-\infty}^{t} \tilde{q}\right)^{j}}{j!}. \tag{3.2.31}$$

Since

$$\sum_{j=0}^{\infty} \frac{\left(\sqrt{2}\int_{-\infty}^{t} \tilde{q}\right)^{j}}{j!} = e^{\sqrt{2}\int_{-\infty}^{t} \tilde{q}}, \tag{3.2.32}$$

the tails as described by the Cauchy difference above must vanish as $m \to \infty$.

In view of the factor $e^{\beta t}$, the sequence $\{g_1^{[n]}\}$ is uniformly Cauchy on any semi-infinite interval: for any right-hand endpoint a, we have

$$|g_1^{[n]}(t) - g_1^{[m]}(t)| \leq e^{\beta a} \sum_{j=m}^{n-1} \frac{\left(\sqrt{2}\,\|\tilde{q}\|_1\right)^j}{j!} \leq e^{\beta a + \sqrt{2}\|\tilde{q}\|_1} \tag{3.2.33}$$

for all $t < a$.

Suppose we are working in the space of continuous functions on the whole line with norm $\|f\| = \sup_{t\in(-\infty,a)} |e^{-\beta t} f(t)|$. This space is complete, and hence the sequence $\{g_1^{[n]}\}$ converges uniformly to the limit function g_1 inside the space.

Similarly, we can show that $\{g_2^{[n]}\}$ converges uniformly to the limit function g_2, and $\{g_3^{[n]}\}$ converges uniformly to the limit function g_3.

Recall that the $g_j^{[n]}$ satisfied the iteration scheme 3.2.6, which in turn we used to approximate the Jost solutions. In that iteration scheme, we may take the limit on both sides of all three equations as $n \to \infty$. The result is the following:

$$\begin{aligned} g_1(t) &= e^{-i\xi t}\left(1 + \int_{-\infty}^{t} e^{i\xi\tau}\,(q_1\, g_2(\tau) + q_2\, g_3(\tau))\, d\tau\right) \\ g_2(t) &= -e^{i\xi t} \int_{-\infty}^{t} e^{-i\xi\tau}\, q_1^*\, g_1(\tau)\, d\tau \\ g_3(t) &= -e^{i\xi t} \int_{-\infty}^{t} e^{-i\xi\tau}\, q_2^*\, g_1(\tau)\, d\tau. \end{aligned} \tag{3.2.34}$$

Note that the convergence is uniform, so all interchanging of limits and integration is allowed. The right-hand sides are all differentiable with respect to t, hence the left-hand sides are (minimally) absolutely continuous with L^1 derivative. If you differentiate 3.2.34 you get the Manakov system. Here we use the assumption that $q_1, q_2 \in L^1(\mathbb{R})$. Hence, we have proven the existence of the Jost solutions from the left, because $g_1 = f_1^-$ and similarly for g_2 and g_3. In an analogous manner, we may prove the existence of the Jost solutions from the right, which are defined as follows:

$$\lim_{\tau\to\infty} [\, e^{i\xi\tau}\, f_{21}^+(\tau)] = 0 \tag{3.2.35}$$

$$\lim_{\tau\to\infty} [\, e^{-i\xi\tau}\, f_{22}^+(\tau)] = 1 \tag{3.2.36}$$

$$\lim_{\tau\to\infty} [\, e^{-i\xi\tau}\, f_{23}^+(\tau)] = 0, \tag{3.2.37}$$

and

$$\lim_{\tau\to\infty} [\, e^{i\xi\tau}\, f_{31}^+(\tau)] = 0 \tag{3.2.38}$$

$$\lim_{\tau\to\infty} [\, e^{-i\xi\tau}\, f_{32}^+(\tau)] = 0 \tag{3.2.39}$$

$$\lim_{\tau\to\infty} [\, e^{-i\xi\tau}\, f_{33}^+(\tau)] = 1. \tag{3.2.40}$$

3.2.2 Analyticity of the First Component in ξ

Analyticity will be a relatively immediate consequence of existence, because of a theorem in Berenstein and Gay [4], in which our iteration scheme for proving existence allows a short digression to prove analyticity. We are more interested in bounds than explicit formulas for each iteration.

By Eq.(3.2.31), we have that

$$|g_1^{[n]}| \le e^{\beta t} \sum_{j=0}^{n-1} \frac{\left(\sqrt{2}\int_{-\infty}^{t} \tilde{q}\right)^j}{j!}. \tag{3.2.41}$$

Just set $m = 0$ in Eq.(3.2.33). Let

$$h^{[n]} \equiv e^{\beta\, t} \sum_{j=0}^{n-1} \frac{\left(\sqrt{2}\, \|\tilde{q}\|_1\right)^j}{j!}. \tag{3.2.42}$$

Also, let

$$h \equiv \lim_{n\to\infty} h^{[n]} = \lim_{n\to\infty} e^{\beta\, t} \sum_{j=0}^{n-1} \frac{\left(\sqrt{2}\|\tilde{q}\|_1\right)^j}{j!} = e^{\beta\, t} e^{\sqrt{2}\|\tilde{q}\|_1}, \tag{3.2.43}$$

and this is finite, since $\tilde{q}$ is L^1. Now

$$\int_{-\infty}^{t} h(\tau)\, d\tau = \int_{-\infty}^{t} e^{\beta\, t} e^{\sqrt{2}\|\tilde{q}\|_1}\, d\tau = e^{\sqrt{2}\|\tilde{q}\|_1} \left.\frac{e^{\beta\,\tau}}{\beta}\right|_{-\infty}^{t} = e^{\sqrt{2}\|\tilde{q}\|_1} \frac{e^{\beta\, t}}{\beta} < \infty. \tag{3.2.44}$$

Also, $h^{[n]} \le h\ \forall\, n$, since we are adding positive terms. It follows that the $h^{[n]}$ are integrable on $(-\infty, t]$ for all n. This in turn implies that the $|g_1^{[n]}|$ are integrable on $(-\infty, t]$ for all n, since $|g_1^{[n]}| \le |h^{[n]}|$.

We want to show that the $g_j^{[n]}$ are analytic in ξ for all n. The proof is by induction. Now $g_j^{[0]} = 0$ are certainly analytic for $j = 1, 2, 3$. Assume the $g_j^{[n]}$ are analytic in ξ. It is known that $(\mathbb{R}, m)$ is a σ-finite measure space, where m is the Lebesgue measure, and $m > 0$. Note that $e^{-i\,\xi\, t}$ is analytic in ξ. Also, $e^{i\,\xi\, t}(q_1\, g_2^{[n]} + q_2\, g_3^{[n]})$ is, $\forall\, \xi \in \mathbb{C}^+$, measurable in t and defined a.e. in $\mathbb{R}$. Let $z \in \mathbb{C}^+$. Let $\varepsilon > 0$ such that $\overline{B}(z, \varepsilon) \subseteq \mathbb{C}^+$. Write $z = a + ib$. Let

$$\gamma^{[n]}(t) \equiv e^{(b+\varepsilon)\, t} \sum_{j=0}^{n-1} \frac{\left(\sqrt{2}\int_{-\infty}^{t} \tilde{q}(\tau)\, d\tau\right)^j}{j!}. \tag{3.2.45}$$

We can see that $\gamma^{[n]}$ is integrable with respect to m on $(-\infty, t]$, and

$$|e^{i\,\xi\, t}(q_1\, g_2^{[n]} + q_2\, g_3^{[n]})| \le \gamma^{[n]}(t) \quad \text{a.e. for } \xi \in \overline{B}(z, \varepsilon). \tag{3.2.46}$$

Also, for a.e. t, we have that $e^{i\,\xi\,t}(q_1\,g_2^{[n]} + q_2\,g_3^{[n]})$ is analytic in $\mathbb{C}^+$. By Remark 2.1.9 on page 101 of Berenstein and Gay [4], it follows that $\int_{-\infty}^{t} e^{i\,\xi\,\tau}(q_1\,g_2^{[n]}+q_2\,g_3^{[n]})\,d\tau$ is analytic in $\mathbb{C}^+$. Therefore, $g_1^{[n]}$ is analytic in $\mathbb{C}^+$ with respect to ξ. Similarly, $g_2^{[n]}$ and $g_3^{[n]}$ are analytic in $\mathbb{C}^+$ with respect to ξ.

Since $\{g_j^{[n]}\}$ converges uniformly, then by the first theorem on p. 136 of Gamelin [10], we have that the limit function, our candidate for the solution, is also analytic in ξ.

3.3 Statement of the Relating Problem

The Relating Problem is to relate the Modified Manakov system (with Kaup's potentials [14]) to its two associated Zakharov-Shabat systems. The basic idea is that as the birefringence factor k gets large, the eigenvalue curves of the Manakov system approach those of the associated Zakharov-Shabat systems.

The Relating Problem Part One is to say something substantive about how the eigenvalues ξ of the Modified Manakov System, given an explicit pair of potential functions q_1 and q_2, relate to those of the two associated Zakharov-Shabat systems; we will accomplish this by some extensive computer calculations. The Relating Problem Part Two is to prove a theorem illustrating this relationship.

3.3.1 The Relating Problem Part One

There being no known exact solution to the Modified Manakov system, I decided to use some numerical schemes in order to try to see a pattern. I experimented in Mathematica for a while, and did some theoretical work. For the potentials

$$d_1 = h_1 = 1, \tag{3.3.1}$$

$$d_2 = h_2 = 2, \tag{3.3.2}$$

$$q_1(t) = \begin{cases} h_1 & |t| \le d_1 \\ 0 & |t| > d_1 \end{cases}, \tag{3.3.3}$$

$$q_2(t) = \begin{cases} h_2 & |t| \le d_2 \\ 0 & |t| > d_2 \end{cases}, \tag{3.3.4}$$

we see that in order for ξ to be an eigenvalue for the Manakov system, it is necessary and sufficient that $v_1(\xi, k, d_2) \equiv 0$. The reason this component has to vanish is that it has to go to zero in the far right region, where $t > d_2$. Otherwise this component would blow up at infinity and thus not belong to an L^1 solution.

If we consider the Manakov system without any particular boundary conditions, then, like with the Zakharov-Shabat system, there are symmetries which enable us to focus solely on either $\mathrm{Im}(\xi) > 0$

or Im$(\xi) < 0$ (see Klaus [16] on page 357). Arbitrarily, therefore, we restrict ourselves to the investigation of eigenvalues ξ with Im$(\xi) > 0$.

In order to plot the curve of ξ on the complex plane numerically with k as the parameter, we decide to dream up some differential equation for $\frac{d\xi}{dk}$. If we can produce this equation, we will have a way to plot $\xi(k)$. So we start with the eigenvalue condition $v_1(\xi, k, d_2) = 0$, and take the derivative of the entire equation with respect to k. The result is

$$v_{1k}(\xi, k, d_2) + v_{1\xi}(\xi, k, d_2)\,\frac{d\xi}{dk} = 0. \tag{3.3.5}$$

Solving for $\frac{d\xi}{dk}$ gives

$$\frac{d\xi}{dk} = -\frac{v_{1k}(\xi, k, d_2)}{v_{1\xi}(\xi, k, d_2)}. \tag{3.3.6}$$

Numerically speaking, now, we have to find a way to calculate v_{1k} and $v_{1\xi}$. Well, it is back to the Manakov system. We differentiate the entire Manakov system once with respect to k, and once with respect to ξ. Here is what we get for the k partial differentiation:

$$\dot{v}_{1k} = -i\,\xi\,v_{1k} + q_1(i\,t\,e^{ikt}\,v_2 + e^{ikt}v_{2k}) + q_2(-i\,t\,e^{-ikt}\,v_3 + e^{-ikt}\,v_{3k}) \tag{3.3.7}$$

$$\dot{v}_{2k} = -q_1(-i\,t\,e^{-ikt}\,v_1 + e^{-ikt}\,v_{1k}) + i\,\xi\,v_{2k} \tag{3.3.8}$$

$$\dot{v}_{3k} = -q_2(i\,t\,e^{ikt}\,v_1 + e^{ikt}\,v_{1k}) + i\,\xi\,v_{3k}, \tag{3.3.9}$$

and the corresponding initial conditions are that $v_{jk}(-d_2) = 0$ for $j = 1, 2, 3$. The ξ differentiation results in the following:

$$\dot{v}_{1\xi} = -i\,v_1 - i\,\xi\,v_{1\xi} + q_1\,e^{ikt}\,v_{2\xi} + q_2\,e^{-ikt}\,v_{3\xi} \tag{3.3.10}$$

$$\dot{v}_{2\xi} = -q_1\,e^{-ikt}\,v_{1\xi} + i(v_2 + \xi\,v_{2\xi}) \tag{3.3.11}$$

$$\dot{v}_{3\xi} = -q_2\,e^{ikt}v_{1\xi} + i(v_3 + \xi\,v_{3\xi}), \tag{3.3.12}$$

and the corresponding initial conditions are $v_{j\xi}(-d_2) = 0$ for $j = 1, 2, 3$.

The idea behind all this is to obtain v_{1k} and $v_{1\xi}$, solve Eq. (3.3.6), and thus be able to plot the solutions. Quite important to the success of this scheme was the ability of Mathematica to define delayed functions.

After working with my advisor Dr. Klaus, I produced the Mathematica code in Appendix A. One assumption I made was that if $k = 0$, then the corresponding eigenvalues were pure imaginary. In order to solve Eq. (3.3.6) numerically, I knew I would need an initial condition; in other words, I would need to know $\xi(0)$ for the function $\xi(k)$. The code in Appendix A.1 finds this value. Notice the appearance of the Manakov system with $k = 0$.

I needed the 'epsi' variable in order for the NDSolve command to work properly. Perhaps, if the reader tries to duplicate this work, he may not need it; Dr. Klaus does not have the variable in his code. The NDSolve command returns an interpolating function, and going a little outside the required limits of $[-d_2, d_2]$ makes the procedure work. It does not work (for me) otherwise.

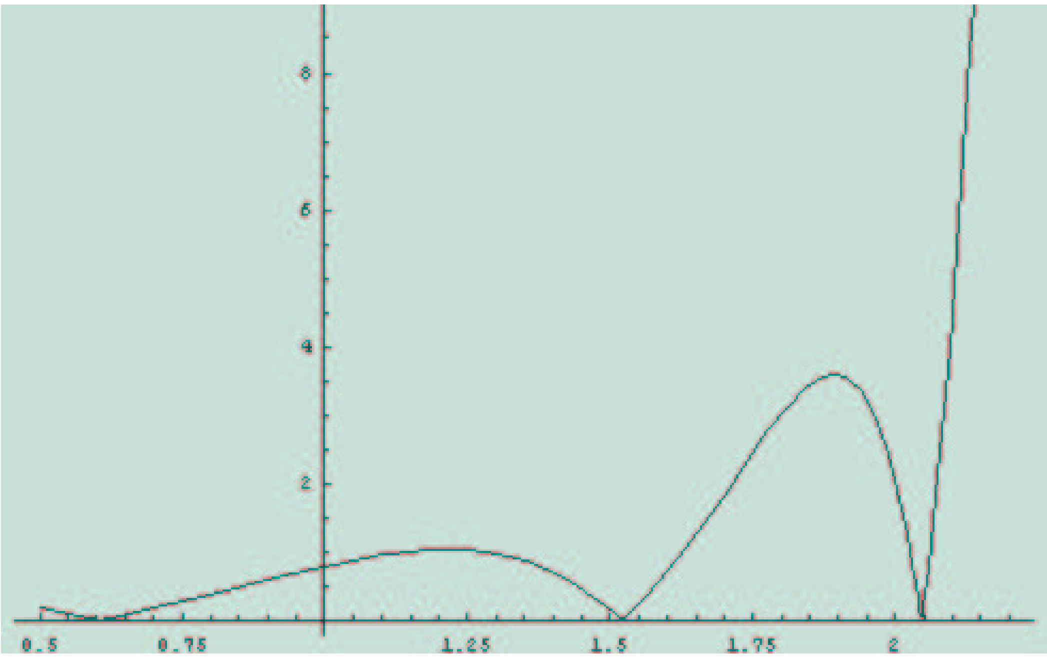


Figure 3.1: First Component $|v_1(is, 0, d_2)|$ as a function of s

The result of this computation is Figure 3.1. The next step is to discover the roots of this function to an acceptable degree of accuracy. The reader will recall that $v_1(\xi, k, d_2) = 0$ in order for ξ to be an eigenvalue. Each root corresponds to an eigenvalue curve. The code in Appendix A.2 finds these roots. I found the numbers that appear in the FindRoot commands by simply looking at Figure 3.1 and noting approximately where the roots were.

Each of these solutions is significant. I decided to use the biggest root first. For the reader's benefit, I should explain that the basic idea here is to find the v_j first using a delayed definition. Next, we find the partial derivatives v_{jk} and $v_{j\xi}$ again using the Mathematica delayed definition of :=.

Finally, I solved Eq.(3.3.6) numerically by coding it myself, since Mathematica's NDSolve routine was not working. Runga-Kutta 4 was the method. The last few lines extract the solution as a list of the imaginary parts of the eigenvalues plotted versus the real parts. The code is in Appendix A.3.

The result of this computation is Figure 3.2. Notice how the curve flattens out as k gets large. The theorem in the Main Problem will explain the theoretic basis for how exactly this curve behaves for large k, also known as the asymptotic behavior. In addition to this potential, I fiddled around with different potentials. The basic behavior of this eigenvalue curve does not change for any potential with compact support, although there is some oscillation for potentials with more total energy (area under the curve).

The other roots of the function $v_1(is, 0, d_2)$ correspond to other eigenvalue curves that are lower. I had, in fact, earlier used a different algorithm to solve this same problem. It was a brute force shooting algorithm in Mathematica, and it appears as Appendix A.4. The result of this computa-

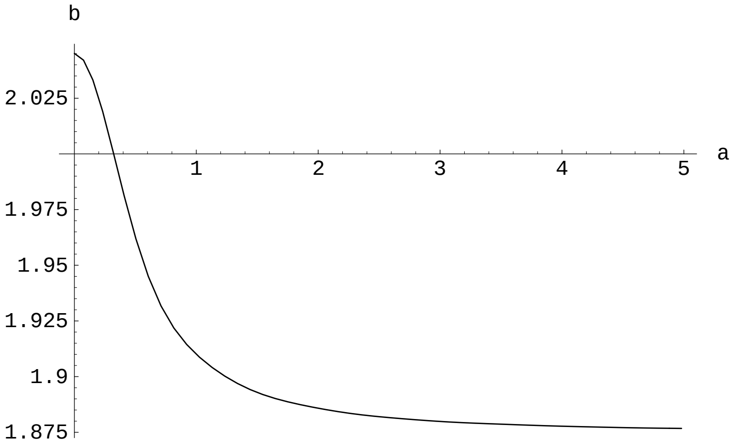


Figure 3.2: Eigenvalue curve: Im(ξ) versus Re(ξ) with k as parameter

tionally intensive algorithm is Figure 3.3.

As the reader can see, there were more curves than we might have expected from the more elegant calculation, that is, the calculations that depend on differentiating the eigenvalue condition and solving the resulting differential equation. In particular, there was one curve coming up from the lower half plane that the elegant solution did not discover on its own. Thus, the usefulness of the rough brute force algorithm was that it revealed another eigenvalue curve. I decided to use the rough algorithm to obtain a good initial condition for the more elegant solution. So I modified the code to “zero in” on the more interesting initial condition on the real axis.

Then I fed those numbers into the elegant solution, thus obtaining a new curve. Plotting all the curves resulting from the zeros of the v_1 component plus the odd curve all on the same axes gives Figure 3.4.

All of this is very nice, but the real problem is to relate the Modified Manakov system to its two associated Zakharov-Shabat systems. That is, relate Eq. (3.3.13) to Eq. (3.3.14) and Eq. (3.3.15). So the strategy is to use the elegant code on the two associated Zakharov-Shabat systems, then use it on the Manakov system, plot all curves on the same axes, and then see if we can make anything of it.

This was quite simple to do, because using the Manakov code on the Zakharov-Shabat system only required deleting a few lines. So the overall strategy here was to find the eigenvalue curves for each associated Zakharov-Shabat system, which ended up being straight lines. Then, I used the elegant code on the original Manakov system and showed how the Manakov curves approach the associated Zakharov-Shabat curves. The result of solving the above problem is Figure 3.5.

The “brute force” algorithm was so labor-intensive with regard to my computer, that I decided not to use it on any other systems. However, the “elegant” solution only took about two hours for each problem and was generally satisfactory. Here is another result from running the code above,

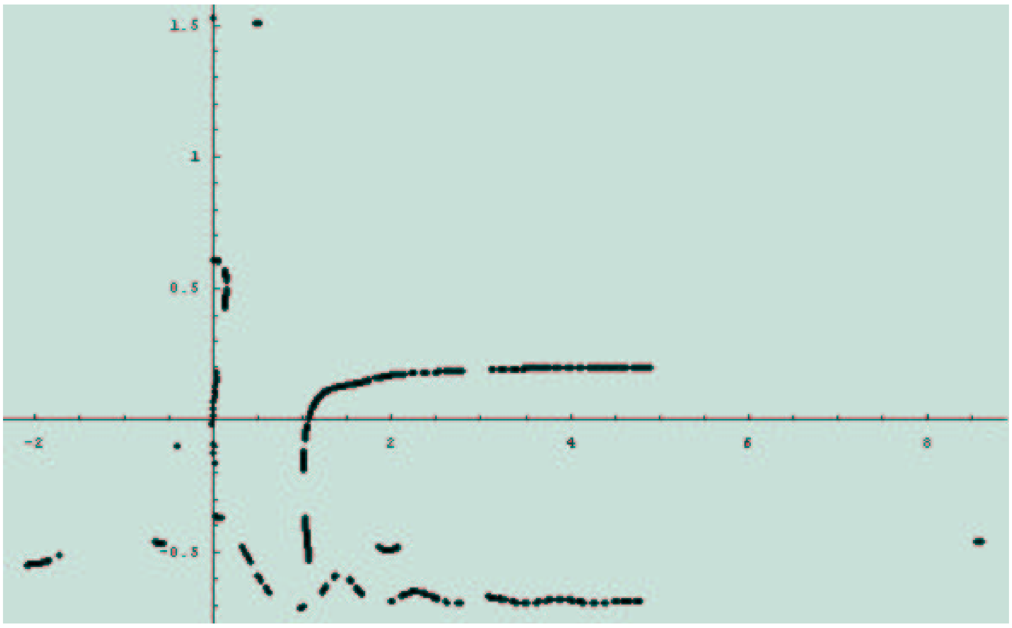


Figure 3.3: Eigenvalue curves from the Brute Force Algorithm: Im(ξ) versus Re(ξ) with k as parameter

including the Zakharov-Shabat systems.

For $q_2(t) = 2\,u(t+4) - 2\,u(t-4)$, and $q_1(t) = 2\,u(t+3.5) - 2\,u(t-3.5)$, we obtain Figure 3.6.

3.3.2 The Relating Problem Part Two

Consider the modified Manakov system due to Malomed [14], which follows:

$$\begin{aligned}\dot{v}_1 &= -i\xi v_1 + q_1 e^{ikt} v_2 + q_2 e^{-ikt} v_3 \\ \dot{v}_2 &= -q_1 e^{-ikt} v_1 + i\xi v_2 \\ \dot{v}_3 &= -q_2 e^{ikt} v_1 + i\xi v_3\end{aligned} \tag{3.3.13}$$

where Im$(\xi) > 0$. The real parameter k is a measure of the birefringence in the fiber optic cable. In addition to Eq.(3.3.13) we introduce the following two Zakharov-Shabat systems:

$$\begin{aligned}\dot{v}_1 &= -i\xi v_1 + q_1 e^{ikt} v_2 \\ \dot{v}_2 &= -q_1 e^{-ikt} v_1 + i\xi v_2\end{aligned} \tag{3.3.14}$$

and

$$\begin{aligned}\dot{v}_1 &= -i\xi v_1 + q_2 e^{-ikt} v_3 \\ \dot{v}_3 &= -q_2 e^{ikt} v_1 + i\xi v_3.\end{aligned} \tag{3.3.15}$$

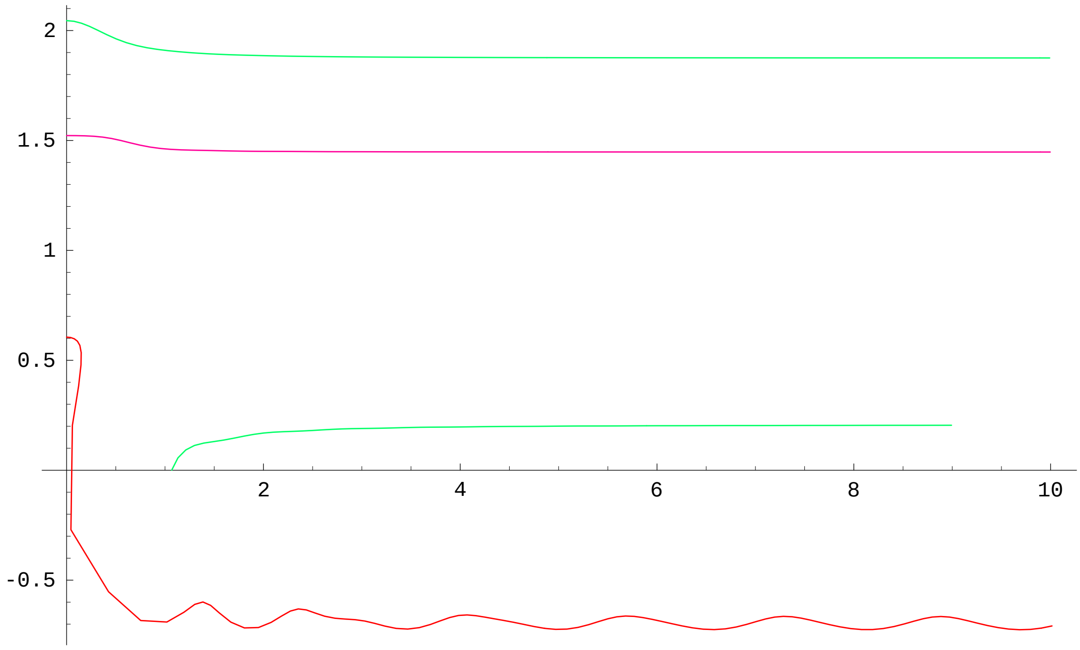


Figure 3.4: Eigenvalue curves: Im(ξ) versus Re(ξ) with k as parameter

Note that we obtain Eq. (3.3.14) from Eq. (3.3.13) by setting $q_2 = 0$ and ignoring the third equation in Eq. (3.3.13). Also, we obtain Eq. (3.3.15) from Eq. (3.3.13) by setting $q_1 = 0$ and ignoring the middle equation.

We obtain the spectrum of Eq. (3.3.14) from that of the case $k = 0$ by a translation; this follows from the following lemma.

Lemma 3.3.1 *For the Zakharov-Shabat system*

$$\dot{v}_1(t) = -i\,\xi\,v_1(t) + q_1(t)\,v_2(t) \tag{3.3.16}$$
$$\dot{v}_2(t) = -q_1^*(t)\,v_1(t) + i\,\xi\,v_2(t), \tag{3.3.17}$$

if ξ is an eigenvalue corresponding to the potential q_1, then $\xi - \frac{c}{2}$ is an eigenvalue for the potential $q_1\,e^{ict}$.

Proof: We perform the substitution $v_1(t) = z_1(t)\,e^{-ict/2}$ and $v_2(t) = z_2(t)\,e^{ict/2}$. By the Zakharov-Shabat system, we have

$$\dot{z}_1(t) - \frac{i\,c}{2}\,z_1(t) = -i\,\xi\,z_1(t) + q_1(t)\,z_2(t)\,e^{ict} \tag{3.3.18}$$
$$\dot{z}_2(t) + \frac{i\,c}{2}\,z_2(t) = -q_1^*(t)\,z_1(t)\,e^{-ict} + i\,\xi\,z_2(t). \tag{3.3.19}$$

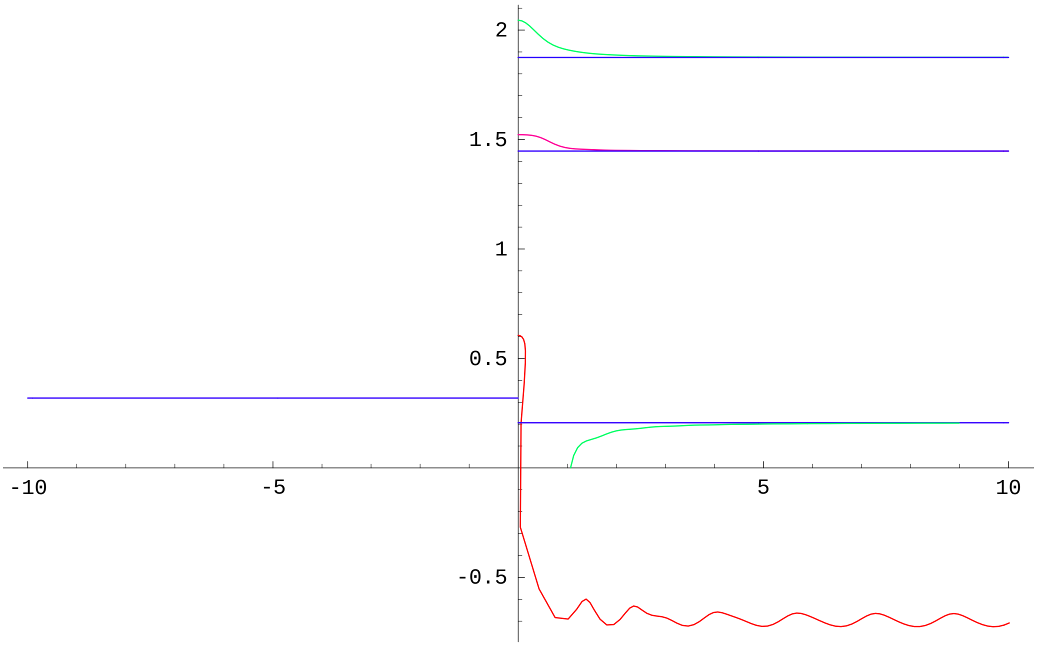


Figure 3.5: Eigenvalue curves: Im(ξ) versus Re(ξ) with k as parameter

Rearranging by throwing the second terms on the left hand sides over to the right gives

$$\dot{z}_1(t) = -i\left(\xi - \frac{c}{2}\right) z_1(t) + q_1(t)\, e^{ict}\, z_2(t) \tag{3.3.20}$$

$$\dot{z}_2(t) = -q_1^*(t)\, e^{-ict}\, z_1(t) + i\left(\xi - \frac{c}{2}\right) z_2(t). \tag{3.3.21}$$

After staring at this result for a while, the reader will realize this is just another Zakharov-Shabat system with potential $q_1(t)\, e^{ict}$ and eigenvalues $\xi - \frac{c}{2}$.

Q. E. D..

As $k \to \infty$ the eigenvalues of Eq. (3.3.14) moves to the left. A similar conclusion holds for Eq. (3.3.15). As $k \to +\infty$, the eigenvalues of Eq. (3.3.15) move to the right.

Our goal is to show that in the limit of large k the eigenvalues of Eq. (3.3.13) are located near the points $\xi_0 \pm \frac{k}{2}$ where ξ_0 stands for any eigenvalue of Eq. (3.3.14) for $k = 0$. Note that ξ_0 is an eigenvalue of Eq. (3.3.14) with $k = 0$ and not of Eq. (3.3.13) with $k = 0$.

Theorem 3.3.1 *Let ξ_0 be an eigenvalue of the associated Zakharov-Shabat system Eq. (3.3.14) for $k = 0$. Then for any $\delta > 0$ there is a number $K_\delta > 0$ such that for $k > K_\delta$ the disk $|\xi - \xi_0 - \frac{k}{2}| < \delta$ contains exactly one eigenvalue of the Manakov system Eq. (3.3.13). Similarly, there is a number $\tilde{K}_\delta$ such that for $k > \tilde{K}_\delta$ the disk $|\xi - \xi_0 + \frac{k}{2}| < \delta$ contains exactly one eigenvalue of Eq. (3.3.13).*[2]

[2]This theorem and its proof are altered from some notes of M. Klaus and are used by permission.

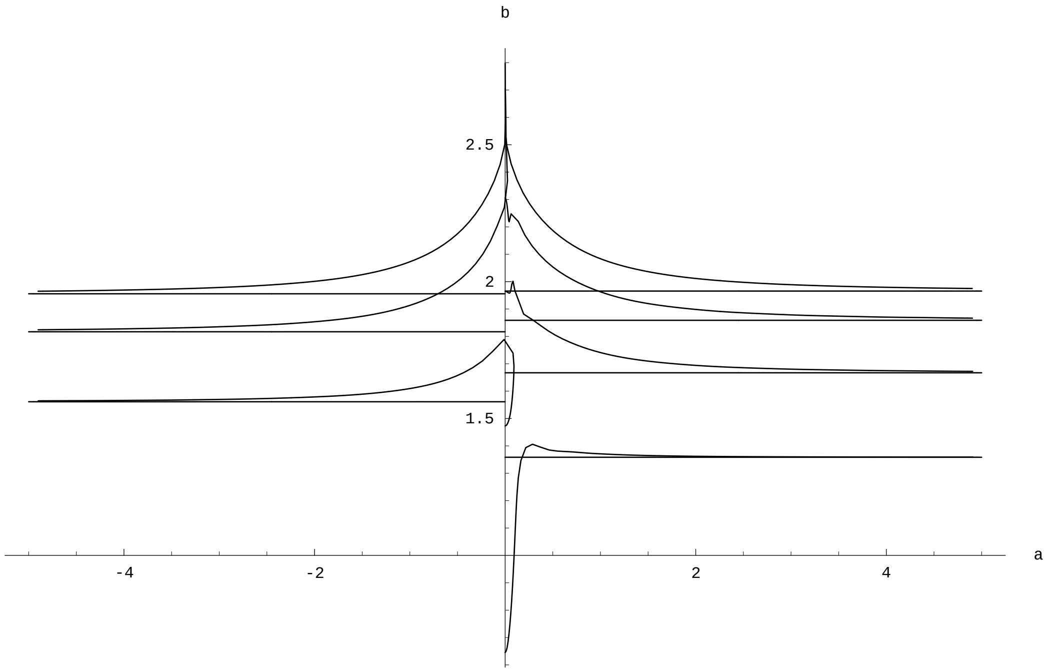


Figure 3.6: Eigenvalue curves: Im(ξ) versus Re(ξ) with k as parameter

Proof: Let us write the second associated Zakharov-Shabat system Eq. (3.3.15) as

$$\begin{aligned} \dot{u}_1 &= -i\xi u_1 + q_2 e^{-ikt} u_3 \\ \dot{u}_3 &= -q_2 e^{ikt} u_1 + i\xi u_3, \end{aligned} \tag{3.3.22}$$

i.e. we have only replaced v_1, v_3 with u_1, u_3. We recall that eigenvalues are given as the roots $\xi(k)$ of the equation $u_1(d; \xi, k) = 0$, where $[-d, d]$ is an interval that contains the support of both q_1 and q_2, and where $\vec{u} = (u_1, u_3)$ is normalized such that $u_1(-d; \xi, k) = e^{i\xi d}$, and $u_3(-d; \xi, k) = 0$. The requirement that $u_3(-d; \xi, k) = 0$ is necessary, else u_3 will blow up as $t \to -\infty$. The normalization $u_1(-d; \xi, k) = e^{i\xi d}$ is for convenience. Eigenvalues are given as the roots $\xi(k)$ of the equation $u_1(d; \xi, k) = 0$ because if it is not the case that $u_1(d; \xi, k) = 0$, then u_1 will blow up as $t \to +\infty$, since $u_1 = Ae^{-i\xi t}$ in the region $t > d$. We have assumed that Im$(\xi) > 0$, hence the magnitude $|u_1| = |A|e^{\text{Im}(\xi)\, t}$, which blows up as $t \to +\infty$ unless $|A| = 0$.

Then we claim

$$\begin{aligned} v_1(t; \xi, k) &:= e^{-\frac{ikt}{2}}\, u_1(t; \xi - \frac{k}{2}, 0) \\ v_3(t; \xi, k) &:= e^{\frac{ikt}{2}}\, u_3(t; \xi - \frac{k}{2}, 0) \end{aligned}$$

is a solution of the second associated Zakharov-Shabat system Eq. (3.3.15) and thus Eq. (3.3.22). This follows from the inspection of Eq. (3.3.15) and straight-forward calculations.

It follows that the eigenvalues of Eq.(3.3.15) are given as the roots of the equation

$$u_1(d; \xi - \frac{k}{2}, 0) = 0. \tag{3.3.23}$$

Let ξ_0 with Im$(\xi_0) > 0$ satisfy $u_1(d; \xi_0, 0) = 0$, i.e., $\xi = \xi_0 + \frac{k}{2}$ satisfies Eq.(3.3.23). Now u_1 is an analytic function of ξ, a fact we proved in Chapter 3.2. Alternatively, one could see that Ablowitz and Segur mention this on page 17 of Ablowitz and Segur [1] It is a consequence of the theory of Volterra integral equations on a finite interval. Furthermore, we know from Gamelin [10], page 156, that if D is a domain, and $f(z)$ is an analytic function on D that is not identically zero, then the zeros of $f(z)$ are isolated. As it turns out, u_1 is an *entire* function of ξ, even though ξ is not an eigenvalue if Im$(\xi) \leq 0$. This fact, in addition to the theorem just mentioned, shows that the roots of $u_1(d; \xi, 0)$ are isolated points in the domain of the upper half plane. We will show later on that if ξ is a root of $u_1(d; \xi, 0)$, then $|\xi|$ can be no larger than a certain amount. Therefore, if there were infinitely many roots, they would have to cluster around some finite point in the complex plane. This would imply that the zeros were not isolated, and therefore, since we definitely have $u_1 \not\equiv 0$, it must be that there are finitely many zeros. Let $\delta > 0$ be so small that ξ_0 is the only root of $u_1(d; \xi, 0)$ in the disk $|\xi - \xi_0| \leq \delta$ and this disk lies in the upper half plane. Then there is an $\alpha > 0$ such that $|u_1(d; \xi, 0)| \geq \alpha$ on $|\xi - \xi_0| = \delta$. That we may do this is a consequence of continuity since we constructed the disk so there were no roots in it other than ξ_0. Thus $|u_1(d; \xi - \frac{k}{2}, 0)| \geq \alpha$ on $|\xi - \xi_0 - \frac{k}{2}| = \delta$. Now let $\vec{v}_1 = (v_1, v_2, v_3)$ be a solution of Eq.(3.3.13) with $v_1(-d; \xi, k) = e^{i\xi d}$ and $v_2(-d; \xi, k) = v_3(-d; \xi, k) = 0$. In order to prove the theorem we will show that

$$|e^{\frac{ikd}{2}} v_1(d; \xi, k) - u_1(d; \xi - \frac{k}{2}, 0)| < \alpha \tag{3.3.24}$$

on $|\xi - \xi_0 - \frac{k}{2}| = \delta$, provided that $k > K_\delta$ where K_δ will be determined below.

Rouche's Theorem as stated on page 229 of Gamelin [10] reads as follows: Let D be a bounded domain with piecewise smooth boundary ∂D. Let $f(z)$ and $h(z)$ be analytic on $D \cup \partial D$. If $|h(z)| < |f(z)| \ \forall z \in \partial D$, then $f(z)$ and $f(z) + h(z)$ have the same number of zeros in D, counting multiplicities.

We seek to apply Rouche's Theorem to our circumstances here; let

$$f(\xi) = u_1(d; \xi - \frac{k}{2}, 0), \quad \text{and} \tag{3.3.25}$$

$$h(\xi) = e^{ikd/2} v_1(d; \xi, k) - u_1(d; \xi - \frac{k}{2}, 0). \tag{3.3.26}$$

Note that if we prove Eq.(3.3.24), then we have

$$|h(\xi)| = |e^{ikd/2} v_1(d; \xi, k) - u_1(d; \xi - \frac{k}{2}, 0)| < \alpha \leq |u_1(d; \xi - \frac{k}{2})| = |f(\xi)| \tag{3.3.27}$$

on $|\xi - \xi_0 - \frac{k}{2}| = \delta$. Since $u_1(d; \xi - \frac{k}{2}, 0)$ has only one root in the disk, then Rouche's theorem implies that $f(\xi) + h(\xi) = e^{ikd/2} v_1(d; \xi, k)$ has exactly one root in $|\xi - \xi_0 - \frac{k}{2}| < \delta$, and hence $v_1(d; \xi, k)$ has exactly one root in $|\xi - \xi_0 - \frac{k}{2}| < \delta$. This would prove Theorem 3.3.1.

We perform the same trick with the Manakov system that we did with the Zakharov-Shabat systems in multiplying through by the integrating factor. We obtain

$$\begin{aligned} \frac{d}{dt}(v_1 e^{ikt/2}) &= \dot{v}_1\, e^{ikt/2} + v_1\, i\, \frac{k}{2}\, e^{ikt/2} \\ &= [-i\,\xi\, v_1 + q_1\, e^{ikt}\, v_2 + q_2\, e^{-ikt}\, v_3]\, e^{ikt/2} + v_1\, i\, \frac{k}{2}\, e^{ikt/2} \\ &= -i(\xi - \frac{k}{2})(e^{ikt/2} v_1) + q_1 e^{2ikt}(e^{-ikt/2} v_2) + q_2 e^{-ikt/2} v_3. \end{aligned} \tag{3.3.28}$$

Repeating these calculations for the other two components and restating gives

$$\begin{aligned} \frac{d}{dt}(v_1 e^{ikt/2}) &= -i(\xi - \frac{k}{2})(e^{ikt/2} v_1) + q_1 e^{2ikt}(e^{-ikt/2} v_2) + q_2 e^{-ikt/2} v_3, \\ \frac{d}{dt}(v_2 e^{-ikt/2}) &= -q_1 e^{-2ikt}(e^{ikt/2} v_1) + i(\xi - \frac{k}{2}) e^{-ikt/2} v_2, \text{ and} \\ \frac{d}{dt}(v_3 e^{-ikt/2}) &= -q_2 (e^{ikt/2} v_1) + i(\xi - \frac{k}{2}) e^{-ikt/2} v_3. \end{aligned} \tag{3.3.29}$$

Note that $\vec{u}(t; \xi - \frac{k}{2}, 0)$ satisfies

$$\begin{aligned} \dot{u}_1 &= -i(\xi - \frac{k}{2}) u_1 + q_2 u_3 \\ \dot{u}_3 &= -q_2 u_1 + i(\xi - \frac{k}{2}) u_3. \end{aligned} \tag{3.3.30}$$

Let $z_1 = v_1 e^{ikt/2}, z_2 = v_2 e^{-ikt/2}, z_3 = v_3 e^{-ikt/2}$. Assume

$$\vec{u}(t = -d) = (e^{id(\xi - k/2)}, 0)^T \text{ and } \vec{z}(t = -d) = (e^{id(\xi - k/2)}, 0, 0)^T.$$

We convert Eq. (3.3.29) and Eq. (3.3.30) to integral equations. Begin with

$$\begin{aligned} \dot{z}_1 &= -i\,(\xi - \frac{k}{2})\, z_1 + q_1\, e^{2ikt}\, z_2 + q_2\, z_3 \\ \dot{z}_2 &= -q_1\, e^{-2ikt}\, z_1 + i\,(\xi - \frac{k}{2})\, z_2 \\ \dot{z}_3 &= -q_2\, z_1 + i\,(\xi - \frac{k}{2})\, z_3. \end{aligned} \tag{3.3.31}$$

This we obtained simply by substituting our definition of the z_j's into Eq. (3.3.29). If we focus on the first component z_1 for now, please follow these computations:

$$\dot{z}_1 + i\,(\xi - \frac{k}{2})\, z_1 = q_1\, e^{2ikt}\, z_2 + q_2\, z_3 \tag{3.3.32}$$

$$\dot{z}_1\, e^{i(\xi - k/2)t} + i\,(\xi - \frac{k}{2})\, z_1\, e^{i(\xi - k/2)t} = q_1\, e^{2ikt}\, z_2\, e^{i(\xi - k/2)t} + q_2\, z_3\, e^{i(\xi - k/2)t} \tag{3.3.33}$$

$$\frac{d}{dt}(z_1\, e^{i(\xi - k/2)t}) = q_1\, e^{2ikt}\, z_2\, e^{i(\xi - k/2)t} + q_2\, z_3\, e^{i(\xi - k/2)t} \tag{3.3.34}$$

$$\int_{-d}^{t} \frac{\partial}{\partial \tau}(z_1\, e^{i(\xi - k/2)\tau})\, d\tau = \int_{-d}^{t} e^{i(\xi - k/2)\tau} \left(q_1\, e^{2ik\tau}\, z_2 + q_2\, z_3\right)\, d\tau. \tag{3.3.35}$$

And we continue

$$z_1\, e^{i(\xi-k/2)t} - e^{id(\xi-k/2)}\, e^{i(\xi-k/2)(-d)} = \int_{-d}^{t} e^{i(\xi-k/2)\tau}\left(q_1\, e^{2ik\tau}\, z_2 + q_2\, z_3\right)\, d\tau \tag{3.3.36}$$

$$z_1\, e^{i(\xi-k/2)t} = 1 + \int_{-d}^{t} e^{i(\xi-k/2)\tau}\left(q_1\, e^{2ik\tau}\, z_2 + q_2\, z_3\right)\, d\tau. \tag{3.3.37}$$

Thus, we end up with

$$z_1 = e^{-i(\xi-k/2)t} + e^{-i(\xi-k/2)t}\int_{-d}^{t} e^{i(\xi-k/2)\tau}\left(q_1\, e^{2ik\tau}\, z_2 + q_2\, z_3\right)\, d\tau. \tag{3.3.38}$$

Similar calculations for the other components reveal that, as a whole,

$$\begin{aligned} z_1 &= e^{-i(\xi-\frac{k}{2})t} + e^{-i(\xi-\frac{k}{2})t}\left[\int_{-d}^{t}\left(q_1 e^{i(\xi+\frac{3}{2}k)\tau} z_2 + q_2 e^{i(\xi-\frac{k}{2})\tau} z_3\right) d\tau\right] \\ z_2 &= -e^{i(\xi-\frac{k}{2})t}\int_{-d}^{t} q_1 e^{-i(\xi+\frac{3k}{2})\tau} z_1\, d\tau \\ z_3 &= -e^{i(\xi-\frac{k}{2})t}\int_{-d}^{t} q_2 e^{-i(\xi-\frac{k}{2})\tau} z_1\, d\tau; \end{aligned} \tag{3.3.39}$$

and for the u_j's, we have

$$\begin{aligned} u_1(t;\xi-\frac{k}{2},0) &= e^{-i(\xi-\frac{k}{2})t} + e^{-i(\xi-\frac{k}{2})t}\int_{-d}^{t} q_2 e^{i(\xi-\frac{k}{2})\tau} u_3\, d\tau \\ u_3(t;\xi-\frac{k}{2},0) &= -e^{i(\xi-\frac{k}{2})t}\int_{-d}^{t} q_2 e^{-i(\xi-\frac{k}{2})\tau} u_1\, d\tau. \end{aligned} \tag{3.3.40}$$

We can solve the systems Eq.(3.3.39) and Eq.(3.3.40) by iteration. We substitute z_1 into the second and third equation (after changing variables appropriately):

$$\begin{aligned} z_2 = &-e^{i(\xi-\frac{k}{2})t}\int_{-d}^{t} q_1 e^{-i(2\xi+k)\tau} d\tau \\ &- e^{i(\xi-\frac{k}{2})t}\int_{-d}^{t} q_1 e^{-i(2\xi+k)\tau}\left[\int_{-d}^{\tau} q_1 e^{i(\xi+\frac{3k}{2})s} z_2\, ds + \int_{-d}^{\tau} q_2 e^{i(\xi-\frac{k}{2})s} z_3\, ds\right] d\tau \end{aligned} \tag{3.3.41}$$

$$\begin{aligned} z_3 = &-e^{i(\xi-\frac{k}{2})t}\int_{-d}^{t} q_2 e^{-i(2\xi-k)\tau}\, d\tau \\ &- e^{i(\xi-\frac{k}{2})t}\int_{-d}^{t} q_2 e^{-2i(\xi-\frac{k}{2})\tau}\left[\int_{-d}^{\tau} q_1 e^{i(\xi+\frac{3k}{2})s} z_2\, ds + \int_{-d}^{\tau} q_2 e^{i(\xi-\frac{k}{2})s} z_3\, ds\right]\, d\tau. \end{aligned} \tag{3.3.42}$$

Now let's look at the integral $\int_{-d}^{t} q_1 e^{-i(2\xi+k)\tau}\, d\tau$. Let $\xi = \xi_0 + \frac{k}{2} + \rho$ where $|\rho| \leq \delta$ (we need only the more mild condition $|\rho| = \delta$). Then

$$\left|\int_{-d}^{t} q_1 e^{-i(2\xi+k)\tau}\, d\tau\right| = \left|\int_{-d}^{t} q_1 e^{-i(2(\xi_0+\frac{k}{2}+\rho)+k)\tau}\, d\tau\right| = \left|\int_{-d}^{t} q_1 e^{-2i(\xi_0+k+\rho)\tau}\, d\tau\right| \to 0$$

as $k \to \infty$ and $t \in [-d, d]$, uniformly in ρ for $|\rho| \le \delta$. This follows from the Riemann-Lebesgue lemma, which is stated as follows in Rudin [25] on page 103:

> 5.14 As in Sec. 4.26, we associate to every $f \in L^1(T)$ a function $\hat{f}$ on $\mathbb{Z}$ defined by
>
> $$(1) \qquad \hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt \qquad (n \in \mathbb{Z}).$$
>
> It is easy to prove that $\hat{f}(n) \to 0$ as $|n| \to \infty$, for every $f \in L^1$. For we know that $C(T)$ is dense in $L^1(T)$ (Theorem 3.14) and that the trigonometric polynomials are dense in $C(T)$ (Theorem 4.25). If $\varepsilon > 0$ and $f \in L^1(T)$, this says that there is a $g \in C(T)$ and a trigonometric polynomial P such that $\|f-g\|_1 < \varepsilon$ and $\|g-P\|_\infty < \varepsilon$. Since
>
> $$\|g - P\|_1 \le \|g - P\|_\infty$$
>
> it follows that $\|f - P\|_1 < 2\varepsilon$; and if $|n|$ is large enough (depending on P), then
>
> $$(2) \qquad |\hat{f}(n)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\{f(t) - P(t)\}\,e^{-int}\,dt\right| \le \|f - P\|_1 < 2\varepsilon.$$
>
> Thus $\hat{f}(n) \to 0$ as $n \to \pm\infty$. This is known as the Riemann-Lebesgue lemma.

This proof carries over for our case; the existence of the 2 in the exponent makes no difference, and we may absorb the rest of the exponential into the q_1 function and still retain the L^1 properties. As for the fact that this integral goes to zero uniformly in ρ, that follows from first approximating q_1 by a C^∞ function, and then integrating by parts. The integral goes to zero like $\frac{1}{k}$.

Let $\gamma > 0$ be arbitrary and assume that k is so large that

$$\left|\int_{-d}^{t} q_1 e^{-2i(\xi_0+k+\rho)\tau}\,d\tau\right| \le \gamma \qquad t \in [-d, d]. \tag{3.3.43}$$

Before we estimate z_2 we need to look at the last term in Eq.(3.3.41), namely

$$\int_{-d}^{t} q_1 e^{-i(2\xi+k)\tau}\int_{-d}^{\tau} q_2 e^{i(\xi-\frac{k}{2})s} z_3\,ds\,d\tau. \tag{3.3.44}$$

We omit the factor $e^{i(\xi-k/2)t}$. We integrate this term by parts and get

$$\left(\int_{-d}^{\tau} q_1 e^{-i(2\xi+k)s}\,ds\right)\left(\int_{-d}^{\tau} q_2 e^{i(\xi-\frac{k}{2})s} z_3\,ds\right)\Bigg|_{-d}^{t} - \int_{-d}^{t}\left(\int_{-d}^{\tau} q_1 e^{-i(2\xi+k)s}\,ds\right) q_2(\tau)\,e^{i(\xi-\frac{k}{2})s} z_3\,d\tau. \tag{3.3.45}$$

We have the integral in Eq. (3.3.43) two times. From Eq. (3.3.41) and Eq. (3.3.42) it follows that $|z_1| \le C, |z_2| \le C, |z_3| \le C$ uniformly in all parameters, assuming ξ is near ξ_0. The proof of these estimates follows.

Note that for $t \in [-d, d]$, we have $|e^{i(\xi - \frac{k}{2})t}| < e^{\mathrm{Im}(\xi)d}$, since $\mathrm{Im}(\xi) > 0$. Similarly, we see that for similar terms, all that matters for magnitude is whatever pure imaginary number multiplies the i in the rest of the exponent.

We have the following for $|z_2|$:

$$\begin{aligned} |z_2| &\le e^{\mathrm{Im}(\xi)d} \int_{-d}^{d} |q_1| e^{2\mathrm{Im}(\xi)\tau}\, d\tau \\ &\quad + e^{\mathrm{Im}(\xi)d} \int_{-d}^{t} |q_1|\, e^{2\mathrm{Im}(\xi)\tau} \left[\int_{-d}^{\tau} (|q_1|\, e^{\mathrm{Im}(\xi)s}\, |z_2| + |q_2|\, e^{\mathrm{Im}(\xi)s}\, |z_3|)\, ds \right] d\tau && (3.3.46) \\ &\le C_1 + e^{4\mathrm{Im}(\xi)d} \int_{-d}^{t} |q_1| \left[\int_{-d}^{\tau} (|q_1|\, |z_2| + |q_2|\, |z_3|)\, ds \right] d\tau && (3.3.47) \\ &\le C_1 + e^{4\mathrm{Im}(\xi)d} \int_{-d}^{d} |q_1|\, d\tau \int_{-d}^{t} (|q_1|\, |z_2| + |q_2|\, |z_3|)\, d\tau && (3.3.48) \\ &\le C_1 + C_2 \int_{-d}^{t} (|q_1| + |q_2|)\, (|z_2| + |z_3|)\, d\tau. && (3.3.49) \end{aligned}$$

Here $C_1 = e^{3\,\mathrm{Im}(\xi)d} \int_{-d}^{d} |q_1|\, d\tau$ and $C_2 = e^{4\,\mathrm{Im}(\xi)d} \int_{-d}^{d} |q_1|\, d\tau$. We can see the necessity of bounds on $\mathrm{Im}(\xi)$. I shall prove something even stronger, namely bounds on $|\xi|$, in Chapter 4. Note that in going from the second step to the third, we have used the fact that integrating a positive integrand over a superset of the previous interval produces a larger number, hence the t appearing as the upper limit of the inner integral of the second term. Furthermore, in going from the third step to the fourth, we have used the fact that if you multiply out the integrand of the final integral, you get the terms of the previous integral plus two more. The step follows because the integrand becomes larger.

If we use similar crude estimates on $|z_3|$, we will obtain the following:

$$|z_3| \le C_3 + C_4 \int_{-d}^{t} (|q_1| + |q_2|)\, (|z_2| + |z_3|)\, d\tau, \qquad (3.3.50)$$

where $C_3 = e^{3\,\mathrm{Im}(\xi)d} \int_{-d}^{d} |q_2|\, d\tau$, and $C_4 = C_3 = e^{4\,\mathrm{Im}(\xi)d} \int_{-d}^{d} |q_2|\, d\tau$. We add these two estimates together to obtain the following:

$$|z_2| + |z_3| \le C_1 + C_2 + (C_3 + C_4) \int_{-d}^{t} (|q_1| + |q_2|)\, (|z_2| + |z_3|)\, d\tau. \qquad (3.3.51)$$

Let $C_5 = C_1 + C_2$, and let $C_6 = C_3 + C_4$. Then we have

$$|z_2| + |z_3| \le C_5 + C_6 \int_{-d}^{t} (|q_1| + |q_2|)\, (|z_2| + |z_3|)\, d\tau. \qquad (3.3.52)$$

The Gronwall inequality now comes into play, implying that

$$|z_2| + |z_3| \le C_5\, e^{C_6 \int_{-d}^{t} (|q_1|+|q_2|)\, d\tau} \le C_5\, e^{C_6 \int_{-d}^{d} (|q_1|+|q_2|)\, d\tau} =: C. \tag{3.3.53}$$

It follows immediately that $|z_2| \le C$ and $|z_3| \le C$ as claimed. To show that $|z_1| \le C$, note that, from Eq. (3.3.39), we have

$$|z_1| \le e^{\mathrm{Im}(\xi) d} + e^{\mathrm{Im}(\xi) d} \int_{-d}^{t} (|q_1|\, e^{\mathrm{Im}(\xi)\tau}\, |z_2| + |q_2|\, e^{\mathrm{Im}(\xi)\tau}\, |z_3|)\, d\tau \tag{3.3.54}$$

$$\le e^{\mathrm{Im}(\xi) d} + C\, e^{2\,\mathrm{Im}(\xi) d} \int_{-d}^{t} (|q_1| + |q_2|)\, d\tau \tag{3.3.55}$$

$$\le e^{\mathrm{Im}(\xi) d} + C\, e^{2\,\mathrm{Im}(\xi) d} \int_{-d}^{d} (|q_1| + |q_2|)\, d\tau =: \tilde{C}. \tag{3.3.56}$$

If you like, let $\hat{C} = \max(C, \tilde{C})$, and then certainly $|z_1|, |z_2|$, and $|z_3|$ are all less than $\hat{C}$.

It follows that Eq. (3.3.44) is $< \gamma$ for large enough k, since we have bounds on every term in it. Thus we can estimate z_2 as

$$|z_2| \le 2\gamma + e^{|\mathrm{Im}(\xi)| t} \int_{-d}^{t} |q_1| \int_{-d}^{\tau} |q_1|\, |z_2| \tag{3.3.57}$$

$$\le 2\gamma + e^{|\mathrm{Im}(\xi)| d} \left(\int_{-d}^{d} |q_1| \right) \int_{-d}^{t} |q_1|\, |z_2|. \tag{3.3.58}$$

By Gronwall, $|z_2| \le 2\gamma C$ where C is a suitable constant that we can choose independently of ξ. This shows that $|z_2| \to 0$ as $k \to \infty$ for $|\xi - \xi_0 - \frac{k}{2}| \le \delta$.

Going back to Eq. (3.3.39) and Eq. (3.3.40) we see that

$$z_1(t; \xi, k) - u_1(t; \xi - \frac{k}{2}, 0) = e^{-i(\xi - \frac{k}{2})t} \int_{-d}^{t} q_1 e^{i(\xi + \frac{3k}{2})\tau} z_2\, d\tau + e^{-i(\xi - \frac{k}{2})t} \int_{-d}^{t} q_2 e^{i(\xi - \frac{k}{2})\tau} (z_3 - u_3)\, d\tau \tag{3.3.59}$$

$$z_3(t; \xi, k) - u_3(t; \xi - \frac{k}{2}, 0) = -e^{i(\xi - \frac{k}{2})t} \int_{-d}^{t} q_2 e^{-i(\xi - \frac{k}{2})\tau} (z_1 - u_1)\, d\tau. \tag{3.3.60}$$

We can use Gronwall (on $|z_1 - u_1| + |z_3 - u_3|$, for example) with the result that $|z_1 - u_1| \le 2\gamma\tilde{C}$, where we have redefined $\tilde{C}$ suitably. This proves Eq. (3.3.24), just make $2\gamma\tilde{C} < \alpha$. So the K_δ we need for the theorem is a number large enough to make $|z_1 - u_1| \le 2\gamma\tilde{C}$, and similarly for the $\tilde{K}_\delta$. To recapitulate the logic from here, Rouche's Theorem implies v_1 has the same number of zeros in the disk $|\xi - \xi_0 - \frac{k}{2}| \le \delta$, namely, one.

Q. E. D.

Chapter 4

The Third Problem: Bounds on the Eigenvalues

4.1 Statement of the Problem

We start with the Manakov system:

$$v_{1t} + i\xi v_1 = q_1 v_2 + q_2 v_3, \tag{4.1.1}$$

$$v_{2t} - i\xi v_2 = -q_1^* v_1 \tag{4.1.2}$$

$$v_{3t} - i\xi v_3 = -q_2^* v_1. \tag{4.1.3}$$

4.2 The Problem

The main goal that we need to accomplish is to extend, for the Manakov system, as much as possible, Klaus's recent work on the Zakharov-Shabat system in Klaus [15]. In particular, the most desirable result, and the goal of the Problem, is to establish bounds on the eigenvalue ξ. In [5], Bronski mentions the desirability of such bounds, and how they work to reduce the complexity of computations of eigenvalues.

4.2.1 Improved Bounds on the Jost Solutions

What follows is a generalization of Klaus [15], Lemma 2.1, for the Jost solutions from the left for the Manakov system. We will need these bounds in Chapter 4.2.2.

Lemma 4.2.1 *Assume $\beta = Im(\xi) > 0$, and both q_1 and q_2 are in $L^1(\mathbb{R})$. Then*

$$|f_k^-(t,\xi)| \le e^{\beta t}, \quad k = 1,2,3. \tag{4.2.1}$$

Proof: We have already proved existence of solutions in Section 3.2. We can rewrite the Manakov system as

$$\dot{\mathbf{v}} = A(q_1,q_2)\,\mathbf{v}, \text{ where } A(q_1,q_2) = \begin{bmatrix} -i\xi & q_1 & q_2 \\ -q_1^* & i\xi & 0 \\ -q_2^* & 0 & i\xi \end{bmatrix}. \tag{4.2.2}$$

Let

$$J \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{4.2.3}$$

Now

$$\frac{d}{dt}(\langle \mathbf{v},\mathbf{v}\rangle) = \langle \dot{\mathbf{v}},\mathbf{v}\rangle + \langle \mathbf{v},\dot{\mathbf{v}}\rangle = \langle A\mathbf{v},\mathbf{v}\rangle + \langle \mathbf{v},A\mathbf{v}\rangle = \langle \mathbf{v},(A+A^\dagger)\mathbf{v}\rangle. \tag{4.2.4}$$

I went ahead and calculated what $A + A^\dagger$ was, and found out that $A + A^\dagger = 2\beta J$. Thus we have

$$\frac{d}{dt}(\langle \mathbf{v},\mathbf{v}\rangle) = 2\beta\langle \mathbf{v},J\mathbf{v}\rangle. \tag{4.2.5}$$

If we take magnitudes on the right hand side, we end up with

$$\frac{d}{dt}(\|\mathbf{v}\|_{\mathbb{C}^2}^2) \le 2\beta\|\mathbf{v}\|_{\mathbb{C}^2}^2. \tag{4.2.6}$$

This is a differential inequality, which, when coupled with the particular boundary conditions f^- must satisfy, namely Eq.(3.2.4), produces the inequality

$$\|f^-(t,\xi)\|_{\mathbb{C}^2} \le e^{\beta t}. \tag{4.2.7}$$

The Lemma follows, because one component of f^- must be smaller in magnitude than $\|f^-\|_{\mathbb{C}^2}$.

Q. E. D.

4.2.2 Existence of the Wronskian Integral

Lemma 4.2.2 *Suppose $q_1, q_2 \in L^1(\mathbb{R})$. Then the integral*

$$\int_{\mathbb{R}} \left[e^{i\,\xi\,\tau}\, q_1\, f_2^- + e^{i\,\xi\,\tau}\, q_2\, f_3^-\right]\, d\tau \tag{4.2.8}$$

exists.

Proof: All we need to show is that the integrand is in $L^1(\mathbb{R})$. By Lemma 4.2.1, we have that $|e^{i\xi\tau} f_k^-(\tau)| \le 1$, for $k = 1,2,3$. Since $q_1, q_2 \in L^1(\mathbb{R})$, the assertion follows.

Q. E. D.

4.2.3 Lemma of the Wronskian

This Lemma introduces Eq. (4.2.9), which we will use in the next section to prove the eigenvalue bound.

Lemma 4.2.3 *Assume Im*$(\xi) > 0$*. Then* ξ *is an eigenvalue of the Manakov system if and only if*

$$1 + \int_{\mathbb{R}} e^{i\,\xi\,\tau}\left[q_1\,f_2^- + e^{i\,\xi\,\tau}\,q_2\,f_3^-\right]\,d\tau = 0. \tag{4.2.9}$$

Proof: $(\Rightarrow)$ Assume Im$(\xi) > 0$. Further assume that ξ is an eigenvalue of the Manakov system. We seek to show that Eq. (4.2.9) holds. If the reader will recall, Eq. (3.2.5) is the equation for the Jost solution for $t \to -\infty$. In order for ξ to be an eigenvalue, it must be the case that the eigenfunctions obey the restrictions placed on them. One such restriction is that f_1^- must vanish at $+\infty$. Otherwise, the solution would blow up there. This implies

$$e^{-i\xi t}\left(1 + \int_{-\infty}^{t} e^{i\xi\tau}\,(q_1\,f_2^- + q_2\,f_3^-)\,d\tau\right) \to 0 \tag{4.2.10}$$

as $t \to \infty$. However, note that the exponential $e^{-i\xi t}$ behaves like $e^{\beta t}$ in magnitude, where $\xi = \alpha + i\beta$, and recall that $\beta > 0$ since Im$(\xi) > 0$. This exponential blows up as $t \to \infty$. Moreover, note that the integral in Eq.Eq.(4.2.10) remains finite as $t \to \infty$. Therefore, it has to be that

$$1 + \int_{-\infty}^{\infty} e^{i\xi\tau}\,(q_1\,f_2^- + q_2\,f_3^-)\,d\tau = 0. \tag{4.2.11}$$

$(\Leftarrow)$ Assume that Eq.(4.2.9) holds. Now recall that

$$f_1^-(\xi,t) = e^{-i\,\xi\,t}\left(1 + \int_{-\infty}^{t} e^{i\,\xi\,\tau}\,(q_1\,f_2^- + q_2\,f_3^-)\,d\tau\right). \tag{4.2.12}$$

Since Eq.(4.2.9) holds, it follows that

$$e^{-i\,\xi\,t}\left(1 + \int_{\mathbb{R}} e^{i\,\xi\,\tau}\left[q_1\,f_2^- + q_2\,f_3^-\right]\,d\tau\right) = 0\;\forall\,t \in \mathbb{R}. \tag{4.2.13}$$

We subtract this result from f_1^- to see that

$$f_1^-(\xi,t) = -e^{-i\,\xi\,t}\int_{t}^{\infty} e^{i\,\xi\,\tau}\left[q_1\,f_2^- + q_2\,f_3^-\right]\,d\tau. \tag{4.2.14}$$

By Theorem 1 on page 88 of Coppel [6], we may write f_1^- as a linear combination of the columns $\varphi_1, \varphi_2, \varphi_3$ of a fundamental matrix consisting of solutions to the Manakov system. These solutions behave thus as $t \to \infty$:

$$\varphi_1 \sim e^{-i\,\xi\,t}\begin{bmatrix}\alpha_1 + \mathrm{o}(1)\\ \alpha_2 + \mathrm{o}(1)\\ \alpha_3 + \mathrm{o}(1)\end{bmatrix}, \quad \varphi_2 \sim e^{i\,\xi\,t}\begin{bmatrix}\beta_1\\ \beta_2\\ \beta_3\end{bmatrix}, \quad \varphi_3 \sim e^{i\,\xi\,t}\begin{bmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{bmatrix}, \tag{4.2.15}$$

where the vectors $(\alpha_1 \;\; \alpha_2 \;\; \alpha_3)^T$, $(\beta_1 \;\; \beta_2 \;\; \beta_3)^T$, and $(\gamma_1 \;\; \gamma_2 \;\; \gamma_3)^T$ are all linearly independent and nonzero. Since by virtue of Eq. (4.2.14) we may write

$$e^{-i\,\xi\,t} \int_t^{\infty} e^{i\,\xi\,\tau} \left[\, q_1\, f_2^- + q_2\, f_3^- \right] \, d\tau = C_1\, \varphi_1 + C_2\, \varphi_2 + C_3\, \varphi_3, \tag{4.2.16}$$

it follows that

$$\int_t^{\infty} e^{i\,\xi\,\tau} \left[\, q_1\, f_2^- + q_2\, f_3^- \right] \, d\tau = (C_1\, \varphi_1 + C_2\, \varphi_2 + C_3\, \varphi_3)\, e^{i\,\xi\,t}. \tag{4.2.17}$$

Note that the LHS goes to zero as $t \to \infty$. On the RHS, the terms corresponding to φ_2 and φ_3 go to zero strongly, but $\varphi_1\, e^{i\,\xi\,t} \to \vec{\alpha} \neq 0$. Therefore, it must be that $C_1 = 0$. This implies that f_1^- decays exponentially as it should and is thus an eigenfunction with ξ as the corresponding eigenvalue.

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4.2.4 Eigenvalue Bounds

We seek a bound, $C > 0$, such that if ξ is an eigenvalue of the Manakov System Eq. (3.1.7), then $|\xi| \leq C$. Let

$$X(\xi) \equiv 1 + \int_{-\infty}^{\infty} e^{i\xi\tau}\, (q_1\, f_2^- + q_2\, f_3^-)\, d\tau. \tag{4.2.18}$$

Theorem 4.2.1 *If ξ is an eigenvalue of the Manakov system Eq. (3.1.7), and q_1, q_2 and their first derivatives are all L^1, then*

$$|\xi| \leq \frac{1}{2} \left(\|q_1'\|_1\, \|q_1\|_1 + \|q_2'\|_1\, \|q_2\|_1 \right). \tag{4.2.19}$$

Proof: We begin with the assumption that ξ is an eigenvalue, and that q_1, q_2 are both L^1 and differentiable. As proven earlier, in Section 4.2.3, the former implies that

$$1 + \int_{-\infty}^{\infty} e^{i\xi\tau}\, (q_1\, f_2^- + q_2\, f_3^-)\, d\tau = 0. \tag{4.2.20}$$

The argument goes like this: the integral in Eq.(4.2.20) is, for large $|\xi|$, smaller than 1 in magnitude. If that happens, then ξ cannot be an eigenvalue, since Eq. (4.2.20) will not hold. As it turns out, the bigger $|\xi|$ is, the smaller the integral, and thus the less likely it is for ξ to be an eigenvalue. Let

$$M_1(t, \xi) \equiv \sup_{\tau \geq t} e^{2\beta\tau} \left| \int_{\tau}^{\infty} q_1(s)\, e^{2i\xi s}\, ds \right|, \text{ and} \tag{4.2.21}$$

$$M_2(t, \xi) \equiv \sup_{\tau \geq t} e^{2\beta\tau} \left| \int_{\tau}^{\infty} q_2(s)\, e^{2i\xi s}\, ds \right|. \tag{4.2.22}$$

I claim that

$$|X(\xi) - 1| \leq M_1(-\infty, \xi)\, \|q_1\|_1 + M_2(-\infty, \xi)\, \|q_2\|_1. \tag{4.2.23}$$

To show this, we follow the logic in Klaus [15], on pages 26-7, where Klaus proves the analogous claim for the Zakharov-Shabat system. Note that

$$X(\xi) - 1 = \int_{-\infty}^{\infty} e^{i\xi\tau}\,(q_1\, f_2^- + q_2\, f_3^-)\, d\tau. \tag{4.2.24}$$

Let us analyze the integral $\int_{-\infty}^{t} e^{i\xi\tau} q_1\, f_2^-\, d\tau$. We have

$$\int_{-\infty}^{t} e^{i\xi\tau} q_1(\tau)\, f_2^-(\tau)\, d\tau = -\int_{-\infty}^{t} e^{2i\xi\tau} q_1(\tau)\, \left[\int_{-\infty}^{\tau} e^{-i\xi s}\, q_1^*(s)\, f_1^-(s)\, ds\right]\, d\tau \tag{4.2.25}$$

$$= \Big[\int_{-\infty}^{\tau} e^{-2i\xi s}\, q_1^*(s)[e^{i\xi s}\, f_1^-(s)]\, ds \cdot \int_{\tau}^{\infty} e^{2i\xi s}\, q_1(s)\, ds\Big|_{-\infty}^{t} \tag{4.2.26}$$

$$- \int_{-\infty}^{t} \left(\int_{\tau}^{\infty} e^{2i\xi s}\, q_1(s)\, ds\right) q_1^*(\tau)\, e^{-2i\xi\tau}[e^{i\xi\tau}\, f_1^-(t)]\, d\tau\Big].$$

I claim that

$$\lim_{\tau\to-\infty} \int_{-\infty}^{\tau} e^{-2i\xi s}\, q_1^*(s)[e^{i\xi s}\, f_1^-(s)]\, ds \cdot \int_{\tau}^{\infty} e^{2i\xi\tau}\, q_1(s) = 0. \tag{4.2.27}$$

Proof of claim. By Lemma 4.2.1, we have

$$\left|\int_{-\infty}^{\tau} e^{-2i\xi s}\, q_1^*(s)[e^{i\xi s}\, f_1^-(s)]\, ds \cdot \int_{\tau}^{\infty} e^{2i\xi s}\, q_1(s)\, ds\right| \tag{4.2.28}$$

$$\leq \left[\int_{-\infty}^{\tau} e^{2\beta s}\, |q_1(s)|\, ds \cdot \int_{\tau}^{\infty} e^{-2\beta s}\, |q_1(s)|\, ds\right] \tag{4.2.29}$$

$$\leq \left[\int_{-\infty}^{\tau} |q_1(s)|\, ds \cdot \int_{\tau}^{\infty} |q_1(s)|\, ds\right] \tag{4.2.30}$$

$$\leq \|q_1\|_1 \int_{-\infty}^{\tau} |q_1(s)|\, ds. \tag{4.2.31}$$

This last expression vanishes because of the Lebesgue Dominated Convergence Theorem (LDCT). Letting $\tau \to -\infty$ proves the claim Eq.(4.2.27). The claim, in turn, implies that

$$\int_{-\infty}^{t} e^{i\xi\tau} q_1(\tau)\, f_2^-(\tau)\, d\tau = \int_{-\infty}^{t} e^{-2i\xi\tau}\, q_1^*(\tau)[e^{i\xi\tau}\, f_1^-(\tau)]\, d\tau \cdot \int_{t}^{\infty} e^{2i\xi\tau}\, q_1(\tau)\, d\tau \tag{4.2.32}$$

$$- \int_{-\infty}^{t} \left(\int_{\tau}^{\infty} e^{2i\xi s}\, q_1(s)\, ds\right) q_1^*(\tau)\, e^{-2i\xi\tau}[e^{i\xi\tau}\, f_1^-(t)]\, d\tau.$$

Again, by Lemma 4.2.1, we have that

$$\left|\int_{-\infty}^{t} e^{i\xi\tau} q_1(\tau)\, f_2^-(\tau)\, d\tau\right| \leq [M_1(t, \xi) + M_1(-\infty, \xi)] \int_{-\infty}^{t} |q_1(\tau)|\, d\tau. \tag{4.2.33}$$

Letting $t \to \infty$ gives us

$$\left| \int_{-\infty}^{\infty} e^{i\xi\tau} q_1(\tau)\, f_2^-(\tau)\, d\tau \right| \leq M_1(-\infty, \xi)\, \|q_1\|_1, \tag{4.2.34}$$

since $M_1(\infty, \xi) = 0$. Similarly,

$$\left| \int_{-\infty}^{\infty} e^{i\xi\tau} q_2(\tau)\, f_3^-(\tau)\, d\tau \right| \leq M_2(-\infty, \xi)\, \|q_2\|_1. \tag{4.2.35}$$

This proves Eq. (4.2.23).

Next we examine the following integral:

$$\int_{\tau}^{\infty} q_1(s)\, e^{2i\xi s}\, ds = q_1(s)\, \frac{e^{2i\xi s}}{2i\xi}\Big|_{\tau}^{\infty} - \int_{\tau}^{\infty} \frac{e^{2i\xi s}}{2i\xi}\, q_1'(s)\, ds, \tag{4.2.36}$$

thus

$$\left| \int_{\tau}^{\infty} q_1(s)\, e^{2i\xi s}\, ds \right| \leq \frac{e^{-2\beta\tau}}{2|\xi|} \left(|q_1(\tau)| + \int_{\tau}^{\infty} |q_1'(s)|\, ds \right), \tag{4.2.37}$$

and therefore

$$e^{2\beta\tau} \left| \int_{\tau}^{\infty} q_1(s)\, e^{2i\xi s}\, ds \right| \leq \frac{1}{2|\xi|} \left(|q_1(\tau)| + \int_{\tau}^{\infty} |q_1'(s)|\, ds \right). \tag{4.2.38}$$

Examining the RHS of the last expression reveals that as $\tau \to -\infty$, the RHS goes to $\frac{1}{2|\xi|}\, \|q_1'\|_1$. Therefore, referencing the definition of M_1 above, we get that

$$M_1(-\infty, \xi) \leq \frac{1}{2|\xi|}\, \|q_1'\|_1. \tag{4.2.39}$$

Given the assumptions that q_1 and q_1' are both L^1, then q_1 must vanish at negative infinity. We may multiply both sides by $\|q_1\|_1$ to obtain

$$M_1(-\infty, \xi)\, \|q_1\|_1 \leq \frac{1}{2|\xi|}\, \|q_1'\|_1\, \|q_1\|_1. \tag{4.2.40}$$

Similarly, we have

$$M_2(-\infty, \xi)\, \|q_2\|_1 \leq \frac{1}{2|\xi|}\, \|q_2'\|_1\, \|q_2\|_1. \tag{4.2.41}$$

The theorem follows from Eq. (4.2.23), and the fact that if its right-hand side is less than 1, there is no way ξ can be an eigenvalue. Since a necessary but not sufficient condition for ξ to be an eigenvalue is that $|X(\xi) - 1| = 1$, we have

$$1 \leq M_1(-\infty, \xi)\, \|q_1\|_1 + M_2(-\infty, \xi)\, \|q_2\|_1 \leq \frac{1}{2|\xi|} \left(\|q_1'\|_1\, \|q_1\|_1 + \|q_2'\|_1\, \|q_2\|_1 \right). \tag{4.2.42}$$

Multiplying both sides by $|\xi|$ proves the theorem.

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Chapter 5

The Fourth Problem: Bounds on the Chirp for a Chirped Manakov System

5.1 Introduction to Chirp

Chirp corresponds to a variation in frequency. It is possible to chirp a pulse of light before sending it down an optical fiber. The purpose of this chapter is to illustrate that too much chirping destroys the soliton effect.

In the context of the Inverse Scattering Transform method as applied to the cNLSE's, linear chirping corresponds to multiplying the potentials by a complex Gaussian thus:

$$q_1(t) = e^{iC_1 t^2}\, \varphi_1(t) \tag{5.1.1}$$

$$q_2(t) = e^{iC_2 t^2}\, \varphi_2(t), \tag{5.1.2}$$

where $\varphi_{1,2}$ are real and symmetric. See Agrawal [2] on pages 56ff for a discussion of why *linear* chirping corresponds to a *Gaussian* multiplying the potential. The results for this chapter will be some computations concerning chirp for a concrete example Manakov system, and a bound on C_1 and C_2 such that if they exceed the bound, there are no eigenvalues, and hence no solitons. First, the computations.

5.2 Computations on the Chirped Manakov System

The general methodology used was very similar to that used in Chapter 3.3.1. This time, we are calculating the dependence of the eigenvalue on the two chirp parameters C_1 and C_2. For this

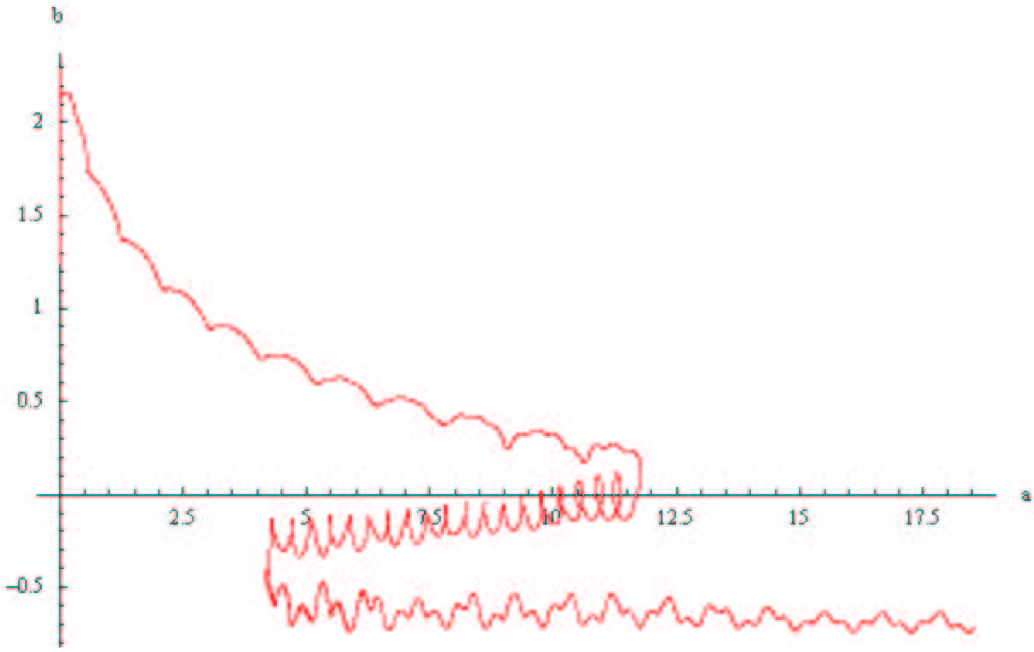

Figure 5.1: Eigenvalue curves: Im(ξ) versus Re(ξ) with s as parameter

problem, I used the two square barriers:

$$q_1(t) = 2 \cdot \chi_{(-3.5,3.5)}(t), \tag{5.2.1}$$
$$q_2(t) = 3 \cdot \chi_{(-4,4)}(t), \tag{5.2.2}$$

where χ is the characteristic function used before. Furthermore, I parameterized the two chirp parameters as $C_1(s) = 2s$ and $C_2(s) = s$. The differential equation for $d\xi/ds$ is as follows:

$$\frac{d\xi}{ds} = -\frac{v_{1C_1}C_1' + v_{1C_2}C_2'}{v_{1\xi}}\bigg|_{\xi,s,d_2}, \tag{5.2.3}$$

which derive from the eigenvalue condition that $v_1(\xi, s, d_2) = 0$.

Note that the calculations began with ξ on the imaginary axis, and traveling on the imaginary axis until colliding with another eigenvalue. Theorem 2.10 from Klaus [16] is relevant to this assumption, since for chirp values prior to the collisions I used only the imaginary part of the slope Eq.(5.2.3), as well as using only the real part of the first component to check accuracy.

The computations were sensitive at the point of collision, and therefore I used smaller increment sizes, and continually checked the accuracy by seeing how close to zero the first component was at the points calculated. This eigenvalue, as shown in Figure 5.1, permanently left the upper half-plane when $C_2 \approx 7.1605$ and $C_1 \approx 14.33$.

5.3 A Preliminary Result

Theorem 5.3.1 *Suppose an operator A is self-adjoint, and consider the eigenvalue problem*

$$(A+B)\,x = \lambda\,x. \tag{5.3.1}$$

Then $|Im(\lambda)| \leq \|B\|$.

Proof: From the assumption, it follows that $(A-\lambda)\,x = -B\,x$. If $\lambda \notin \sigma(A)$, then we have $x = -(A-\lambda)^{-1}Bx$. Consequently, $\|x\| \leq \|(A-\lambda)^{-1}\|\,\|B\|\,\|x\|$. Note that the resolvent operator $R(\xi, A) = (A-\xi)^{-1}\ \forall\,\xi \notin \sigma(A)$. By Theorem 3.16 on page 271 of Kato [13], we have that $\|R(\xi, A)\| \leq \frac{1}{|\text{Im}(\xi)|}$. Therefore, it follows that $\|x\| \leq \frac{\|B\|\,\|x\|}{|\text{Im}(\lambda)|}$. Note that $x \neq 0$, or by definition it would not be an eigenvector. The result follows by dividing out $\|x\|$.

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The reason for proving this result is that the Manakov system may be decomposed into just such a system. Therefore, we may deduce a bound on the imaginary part of the eigenvalues. We derive the system similar to that in Theorem 5.3.1. Let the matrix J be defined as follows:

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{5.3.2}$$

Note that $J^2 = I$, the identity. We may write the Manakov system as

$$\dot{v}_1 - q_1\,v_2 - q_2\,v_3 = -i\,\xi\,v_1 \tag{5.3.3}$$

$$\dot{v}_2 + \overline{q}_1\,v_1 = i\,\xi\,v_2 \tag{5.3.4}$$

$$\dot{v}_3 + \overline{q}_2\,v_1 = i\,\xi\,v_3. \tag{5.3.5}$$

Multiplying both sides by i and writing in matrix form gives

$$i\,\frac{d}{dt}\,\vec{v} - i\begin{bmatrix} 0 & q_1 & q_2 \\ -\overline{q}_1 & 0 & 0 \\ -\overline{q}_2 & 0 & 0 \end{bmatrix}\vec{v} = \xi\,J\,\vec{v}. \tag{5.3.6}$$

Multiplying through by J gives

$$i\,J\,\frac{d}{dt}\,\vec{v} - i\begin{bmatrix} 0 & q_1 & q_2 \\ \overline{q}_1 & 0 & 0 \\ \overline{q}_2 & 0 & 0 \end{bmatrix}\vec{v} = \xi\,\vec{v}. \tag{5.3.7}$$

Here is the eigenvalue problem akin to that in Theorem 5.3.1. Note that the operator $i\,J\,\partial_t$ is self-adjoint; hence, if we define B as

$$B \equiv -i\begin{bmatrix} 0 & q_1 & q_2 \\ \overline{q}_1 & 0 & 0 \\ \overline{q}_2 & 0 & 0 \end{bmatrix}, \tag{5.3.8}$$

then the theorem tells us that $|\text{Im}(\xi)| \leq \|B\|$, or since $\text{Im}(\xi) = \beta > 0$, we know that $\beta \leq \|B\|$.

Theorem 5.3.2 *Define* $\hat{q}(t) \equiv \max(\,|q_1(t)|, |q_2(t)|\,)\ \forall\, t$. *Then* $\beta \leq 2\,\|\hat{q}\|_\infty$.[1]

Proof: Define $C \equiv i\,B$. Note that

$$\|B\| = \|C\| = \left\| \begin{bmatrix} 0 & q_1 & q_2 \\ \overline{q}_1 & 0 & 0 \\ \overline{q}_2 & 0 & 0 \end{bmatrix} \right\|. \tag{5.3.9}$$

By inspection, we see that C is self-adjoint. We would like to find a K such that $\|Cx\|^2 \leq K\,\|x\|^2\ \forall\, x$. Now

$$\begin{aligned} \|Cx\|^2 &= \langle x|\, C^\dagger\, C\, |x\rangle & (5.3.10)\\ &= \langle x|\, C^2\, |x\rangle & (5.3.11)\\ &= \int_{\mathbb{R}} \left[(|q_1|^2 + |q_2|^2)\, |x_1|^2 + |q_1|^2\, |x_2|^2 + |q_2|^2\, |x_3|^2\right]\, dt & (5.3.12)\\ &\quad + \int_{\mathbb{R}} (\overline{q}_1\, q_2\, \overline{x}_2\, x_3 + q_1\, \overline{q}_2\, x_2\, \overline{x}_3)\, dt. & (5.3.13) \end{aligned}$$

Then

$$\int_{\mathbb{R}} \left[(|q_1|^2 + |q_2|^2)\, |x_1|^2 + |q_1|^2\, |x_2|^2 + |q_2|^2\, |x_3|^2\right]\, dt \leq 2\,\|\hat{q}\|_\infty^2\, \|x\|^2. \tag{5.3.14}$$

Note that $\|x\| = \|x\|_2$. Also, by examining the remaining terms we see that

$$\begin{aligned} \left|\int_{\mathbb{R}} (\overline{q}_1\, q_2\, \overline{x}_2\, x_3 + q_1\, \overline{q}_2\, x_2\, \overline{x}_3)\, dt\right| &= \left|2\,\mathrm{Re}\left[\int_{\mathbb{R}} \overline{q}_1\, q_2\, \overline{x}_2\, x_3\, dt\right]\right| & (5.3.15)\\ &\leq 2 \int_{\mathbb{R}} |q_1\, q_2\, x_2\, x_3|\, dt & (5.3.16)\\ &\leq 2\,\|\hat{q}\|_\infty^2 \int_{\mathbb{R}} |x_2\, x_3|\, dt & (5.3.17)\\ &\leq 2\,\|\hat{q}\|_\infty^2 \sqrt{\int_{\mathbb{R}} |x_2|^2\, dt}\, \sqrt{\int_{\mathbb{R}} |x_3|^2\, dt} & (5.3.18)\\ &\leq 2\,\|\hat{q}\|_\infty^2 \left(\sqrt{\int_{\mathbb{R}} (|x_1|^2 + |x_2|^2 + |x_3|^2)\, dt}\right)^2 & (5.3.19)\\ &\leq 2\,\|\hat{q}\|_\infty^2\, \|x\|^2. & (5.3.20) \end{aligned}$$

Hence, we have that

$$\|Cx\|^2 \leq 4\,\|\hat{q}\|_\infty^2\, \|x\|^2, \tag{5.3.21}$$

[1]My advisor Dr. Klaus has shown me that the best possible bound is $\|C\| = \text{ess}\ \sup\left(\sqrt{|q_1|^2 + |q_2|^2}\,\right)$. However, I proved this bound, so I will use it.

and therefore

$$\|C\| \le 2\,\|\hat{q}\|_\infty. \tag{5.3.22}$$

From this we may deduce that

$$\mathrm{Im}(\xi) = \beta \le \|B\| = \|C\| \le 2\,\|\hat{q}\|_\infty, \tag{5.3.23}$$

and the theorem is proved.

Q. E. D.

We now move on to prove our main result about the chirped Manakov system.

5.4 Bounds on the Chirps of the Manakov System

Theorem 5.4.1 *Let K_0 be the smallest number such that $|\int_m^n e^{-i\omega^2}\,d\omega| \le K_0\ \forall\, m,n$. Assume C_1 and C_2 are both nonnegative*[2]*, and let $C_0 \equiv \min(C_1, C_2)$ be the minimum of the two chirp parameters in the following chirped Manakov system:*

$$\begin{aligned}\dot{v}_1(t) &= -i\,\xi\,v_1(t) + e^{i\,C_1\,t^2}\,\varphi_1(t)\,v_2(t) + e^{i\,C_2\,t^2}\,\varphi_2(t)\,v_3(t)\\ \dot{v}_2(t) &= i\,\xi\,v_2(t) - e^{-i\,C_1\,t^2}\,\varphi_1(t)\,v_1(t)\\ \dot{v}_3(t) &= i\,\xi\,v_3 - e^{-i\,C_2\,t^2}\,\varphi_2(t)\,v_1(t),\end{aligned} \tag{5.4.1}$$

where the φ_j are real, differentiable, and $\|\varphi_j\|_\infty < \infty$ for $j = 1, 2$. Let

$$\hat{q}(t) \equiv \max(\,|\varphi_1(t)|, |\varphi_2(t)|\,)\ \forall\, t. \tag{5.4.2}$$

Suppose there exists a d such that the φ_j both vanish outside $(-d, d)$. Let

$$D_1 \equiv K_0[\,\|\varphi_1\|_\infty + 8\,d\,\|\hat{q}\|_\infty\,e^{8\,d\,\|\hat{q}\|_\infty}\,\|\varphi_1\|_\infty + 2\,d\,e^{8\,d\,\|\hat{q}\|_\infty}\,\|\varphi_1'\|_\infty] \tag{5.4.3}$$

$$D_2 \equiv K_0[\,\|\varphi_2\|_\infty + 8\,d\,\|\hat{q}\|_\infty\,e^{8\,d\,\|\hat{q}\|_\infty}\,\|\varphi_2\|_\infty + 2\,d\,e^{8\,d\,\|\hat{q}\|_\infty}\,\|\varphi_2'\|_\infty]. \tag{5.4.4}$$

Suppose

$$\begin{aligned}\tilde{F} \equiv \max(&D_1[\|\varphi_1\|_\infty \cdot 2\,d\,e^{8\,d\,\|\hat{q}\|_\infty} + 2\,d\,\|\varphi_1\|_\infty],\\ &D_2[\|\varphi_2\|_\infty \cdot 2\,d\,e^{8\,d\,\|\hat{q}\|_\infty} + 2\,d\,\|\varphi_2\|_\infty]).\end{aligned} \tag{5.4.5}$$

Then if $C_0 > 4\,\tilde{F}^2$, it follows that ξ is not an eigenvalue of Eq.(5.4.1).

[2] We could also use the magnitude of the chirps; then we would not need to assume nonnegativity. The logic will work the same way in either case. The negative case is analogous.

Proof: Let

$$\vec{w} \equiv \begin{bmatrix} e^{i\,\xi\,t} & 0 & 0 \\ 0 & e^{-i\,\xi\,t} & 0 \\ 0 & 0 & e^{-i\,\xi\,t} \end{bmatrix} \vec{v}. \tag{5.4.6}$$

Then $\vec{w}$ satisfies the system

$$\begin{aligned} \dot{w}_1 &= e^{2\,i\,\xi\,t+i\,C_1\,t^2}\,\varphi_1\,w_2 + e^{2\,i\,\xi\,t+i\,C_2\,t^2}\,\varphi_2\,w_3 \\ \dot{w}_2 &= -e^{-2\,i\,\xi\,t-i\,C_1\,t^2}\,\varphi_1\,w_1 \\ \dot{w}_3 &= -e^{-2\,i\,\xi\,t-i\,C_2\,t^2}\,\varphi_2\,w_1, \end{aligned} \tag{5.4.7}$$

with the normalization

$$\vec{w}(-d) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{5.4.8}$$

Note that the vector $\vec{v}$ is the same as the vector f^-, the Jost solutions. It follows from Lemma 4.2.1 that $|w_1| \le 1$. Also, though we will not use this bound, $|w_j(t)| \le e^{2\,\beta\,t}$ for $j = 2, 3$. We will examine the case where the φ_j have compact support.

We convert the system Eq. (5.4.7) to integral equations, using the initial condition just mentioned. The result is the following:

$$\begin{aligned} w_1(t) &= 1 + \int_{-d}^{t} e^{i\,C_1\,x^2+2\,i\,\xi\,x}\,\varphi_1(x)\,w_2(x)\,dx + \int_{-d}^{t} e^{i\,C_2\,x^2+2\,i\,\xi\,x}\,\varphi_2(x)\,w_3(x)\,dx \\ w_2(t) &= -\int_{-d}^{t} e^{-i\,C_1\,x^2-2\,i\,\xi\,x}\,\varphi_1(x)\,w_1(x)\,dx \\ w_3(t) &= -\int_{-d}^{t} e^{-i\,C_2\,x^2-2\,i\,\xi\,x}\,\varphi_2(x)\,w_1(x)\,dx. \end{aligned} \tag{5.4.9}$$

We plug the second and third of Eq. (5.4.9) into the first to obtain

$$\begin{aligned} w_1(t) = 1 &- \int_{-d}^{t} e^{i\,C_1\,x^2+2\,i\,\xi\,x}\,\varphi_1(x) \int_{-d}^{x} e^{-i\,C_1\,s^2-2\,i\,\xi\,s}\,\varphi_1(s)\,w_1(s)\,ds\,dx \\ &- \int_{-d}^{t} e^{i\,C_2\,x^2+2\,i\,\xi\,x}\,\varphi_2(x) \int_{-d}^{x} e^{-i\,C_2\,s^2-2\,i\,\xi\,s}\,\varphi_2(s)\,w_1(s)\,ds\,dx. \end{aligned} \tag{5.4.10}$$

Let

$$B_1(x) = \int_{-d}^{x} e^{i\,C_1\,s^2+2\,i\,\xi\,s}\,\varphi_1(s)\,ds, \tag{5.4.11}$$

$$B_2(x) = \int_{-d}^{x} e^{i\,C_2\,s^2+2\,i\,\xi\,s}\,\varphi_2(s)\,ds. \tag{5.4.12}$$

Suppose $\xi = a + ib$, with $a, b \in \mathbb{R}$, and $b > 0$. Then

$$B_1(x) = \int_{-d}^{x} e^{i\,C_1\,s^2+2\,i\,s\,(a+ib)}\,\varphi_1(s)\,ds \tag{5.4.13}$$

$$= \int_{-d}^{x} e^{i\,C_1\,s^2+2\,a\,i\,s}\,e^{-2\,b\,s}\varphi_1(s)\,ds \tag{5.4.14}$$

$$= \left[\int_{-d}^{s} e^{i\,C_1\,u^2+2\,i\,a\,u}\,du\cdot e^{-2\,b\,s}\,\varphi_1(s)\right]\Big|_{-d}^{x} - \int_{-d}^{x}\int_{-d}^{t} e^{i\,C_1\,s^2+2\,i\,a\,s}\,ds\,\left[e^{-2\,b\,t}\,\varphi_1(t)\right]'\,dt \tag{5.4.15}$$

$$= \int_{-d}^{x} e^{i\,C_1\,s^2+2\,i\,a\,s}\,ds\cdot e^{-2\,b\,x}\,\varphi_1(x) - \int_{-d}^{x}\int_{-d}^{t} e^{i\,C_1\,s^2+2\,i\,a\,s}\,ds\,\left[e^{-2\,b\,t}\,\varphi_1(t)\right]'\,dt. \tag{5.4.16}$$

We examine the integral $\int_{-d}^{x} e^{i\,C_1\,s^2+2\,i\,a\,s}\,ds$. We may complete the square in the exponent to obtain

$$i\,C_1\,s^2 + 2\,i\,a\,s = -\frac{i\,a^2}{C_1} + i\left(\sqrt{C_1}\left(s+\frac{a}{C_1}\right)\right)^2. \tag{5.4.17}$$

Let $\omega \equiv \sqrt{C_1}\left(s+\frac{a}{C_1}\right)$. Then $d\omega = \sqrt{C_1}\,ds$. We have

$$\int_{-d}^{x} e^{i\,C_1\,s^2+2\,i\,a\,s}\,ds = \int_{-d}^{x} e^{-\frac{i\,a^2}{C_1}}\,e^{i\left(\sqrt{C_1}\left(s+\frac{a}{C_1}\right)\right)^2}\,ds \tag{5.4.18}$$

$$= \frac{e^{-\frac{i\,a^2}{C_1}}}{\sqrt{C_1}}\int_{\sqrt{C_1}\left(-d+\frac{a}{C_1}\right)}^{\sqrt{C_1}\left(x+\frac{a}{C_1}\right)} e^{i\,\omega^2}\,d\omega. \tag{5.4.19}$$

We would like a K_0 such that $\left|\int_m^n e^{i\,\omega^2}\,d\omega\right| \le K_0\ \forall\, m, n \in \mathbb{R}$. As it turns out,

$$\left|\int_m^n e^{i\,\omega^2}\,d\omega\right| = \left|\overline{\left(\int_m^n e^{i\,\omega^2}\,d\omega\right)}\right| = \left|\int_m^n e^{-i\,\omega^2}\,d\omega\right| \le K_0, \tag{5.4.20}$$

where the K_0 is the same as mentioned in Klaus [17] on pages 496-7. This implies that

$$\left|\int_{-d}^{x} e^{i\,C_1\,s^2+2\,i\,a\,s}\,ds\right| \le \frac{K_0}{\sqrt{C_1}}. \tag{5.4.21}$$

From Eq. (5.4.16), we can see that

$$|B_1(x)| \leq \frac{K_0}{\sqrt{C_1}}\, e^{-2\,b\,x}\, \|\varphi_1\|_\infty + \frac{K_0}{\sqrt{C_1}} \int_{-d}^{x} \left[\, 2\,b\,e^{-2\,b\,t}\,|\varphi_1(t)| + e^{-2\,b\,t}\,|\varphi_1'(t)|\,\right]\, dt \tag{5.4.22}$$

$$\leq \frac{K_0}{\sqrt{C_1}}\, e^{-2\,b\,x} \left[\|\varphi_1\|_\infty + 2\,b \int_{-d}^{x} e^{2\,b\,(x-t)}\,|\varphi_1(t)|\,dt + \int_{-d}^{x} e^{2\,b\,(x-t)}\,|\varphi_1'(t)|\,dt\right] \tag{5.4.23}$$

$$\leq \frac{K_0}{\sqrt{C_1}}\, e^{-2\,b\,x} \left[\|\varphi_1\|_\infty + 4\,\|\hat{q}\|_\infty\, e^{8\,d\,\|\hat{q}\|_\infty}\, \|\varphi_1\|_\infty \cdot 2\,d + e^{8\,d\,\|\hat{q}\|_\infty} \|\varphi_1'(t)\|_\infty \cdot 2\,d\right] \tag{5.4.24}$$

$$= \frac{K_0}{\sqrt{C_1}}\, e^{-2\,b\,x} \left[\|\varphi_1\|_\infty + 8\,d\,\|\hat{q}\|_\infty\, e^{8\,d\,\|\hat{q}\|_\infty}\, \|\varphi_1\|_\infty + 2\,d\,e^{8\,d\,\|\hat{q}\|_\infty} \|\varphi_1'(t)\|_\infty\right]. \tag{5.4.25}$$

Let D_1 and D_2 be defined as stated in the theorem. Then we have

$$|B_1(x)| \leq \frac{D_1}{\sqrt{C_1}}\, e^{-2\,b\,x}, \tag{5.4.26}$$

and similarly

$$|B_2(x)| \leq \frac{D_2}{\sqrt{C_2}}\, e^{-2\,b\,x}. \tag{5.4.27}$$

Recall that $w_1 = 1 - \int\int_{1,2} - \int\int_{2,3}$. We examine the $-\int\int$. Let

$$F_1(t) \equiv -\int_{-d}^{t} e^{i\,C_1\,x^2 + 2\,i\,\xi\,x}\, \varphi_1(x) \int_{-d}^{x} e^{-i\,C_1\,s^2 - 2\,i\,\xi\,s}\, \varphi_1(s)\, w_1(s)\, ds\, dx. \tag{5.4.28}$$

Similarly, let

$$F_2(t) \equiv -\int_{-d}^{t} e^{i\,C_2\,x^2 + 2\,i\,\xi\,x}\, \varphi_2(x) \int_{-d}^{x} e^{-i\,C_2\,s^2 - 2\,i\,\xi\,s}\, \varphi_2(s)\, w_1(s)\, ds\, dx. \tag{5.4.29}$$

We analyze F_1 by integrating it by parts, obtaining

$$\begin{aligned} F_1(t) = -\Bigg[&\left(\int_{-d}^{x} e^{-i\,C_1\,s^2 - 2\,i\,\xi\,s}\, \varphi_1(s)\, w_1(s)\, ds \cdot \int_{-d}^{x} e^{i\,C_1\,s^2 + 2\,i\,\xi\,s}\, \varphi_1(s)\, ds \right) \Bigg|_{-d}^{t} \\ &- \int_{-d}^{t} \int_{-d}^{x} e^{i\,C_1\,s^2 + 2\,i\,\xi\,s}\, \varphi_1(s)\, ds \cdot e^{-i\,C_1\,x^2 - 2\,i\,\xi\,x}\, \varphi_1(x)\, w_1(x)\, dx \Bigg] \end{aligned} \tag{5.4.30}$$

$$\begin{aligned} = -\Bigg[&\int_{-d}^{t} e^{-i\,C_1\,s^2 - 2\,i\,\xi\,s}\, \varphi_1(s)\, w_1(s)\, ds \cdot \int_{-d}^{t} e^{i\,C_1\,s^2 + 2\,i\,\xi\,s}\, \varphi_1(s)\, ds \\ &- \int_{-d}^{t} \int_{-d}^{x} e^{i\,C_1\,s^2 + 2\,i\,\xi\,s}\, \varphi_1(s)\, ds \cdot e^{-i\,C_1\,x^2 - 2\,i\,\xi\,x}\, \varphi_1(x)\, w_1(x)\, dx \Bigg]. \end{aligned} \tag{5.4.31}$$

Therefore, since $|w_1(x)| \leq 1$, we have

$$|F_1(t)| \leq \int_{-d}^{t} e^{2\,b\,s}\,|\varphi_1(s)|\,ds \cdot \frac{D_1}{\sqrt{C_1}}\,e^{-2\,b\,t} + \int_{-d}^{t} \frac{D_1}{\sqrt{C_1}}\,e^{-2\,b\,x} \cdot e^{2\,b\,x}\,|\varphi_1(x)|\,dx \tag{5.4.32}$$

$$\leq \frac{D_1}{\sqrt{C_1}} \left[\|\varphi_1\|_\infty\,e^{2\,b\,d} \cdot 2\,d \cdot e^{-2\,b\,t} + \int_{-d}^{t} \|\varphi_1\|_\infty\,dx \right] \tag{5.4.33}$$

$$\leq \frac{D_1}{\sqrt{C_1}} \left[\,2\,d\,\|\varphi_1\|_\infty\,e^{8\,d\,\|\hat{q}\|_\infty} + 2\,d\,\|\varphi_1\|_\infty\,\right]. \tag{5.4.34}$$

Similarly, we have

$$|F_2(t)| \leq \frac{D_2}{\sqrt{C_2}} \left[\,2\,d\,\|\varphi_2\|_\infty\,e^{8\,d\,\|\hat{q}\|_\infty} + 2\,d\,\|\varphi_2\|_\infty\,\right]. \tag{5.4.35}$$

Let $\tilde{F}$ and C_0 both be defined as in the theorem. It follows that

$$|F_1(t)| \leq \frac{\tilde{F}}{\sqrt{C_0}},\ \text{and}\ |F_2(t)| \leq \frac{\tilde{F}}{\sqrt{C_0}}\,. \tag{5.4.36}$$

Since $w_1(t) = 1 + F_1(t) + F_2(t)$, it must be that

$$|w_1(t)| \geq 1 - |F_1 + F_2| \tag{5.4.37}$$

$$\geq 1 - \frac{2\tilde{F}}{\sqrt{C_0}}. \tag{5.4.38}$$

If $w_1 > 0$, then ξ is not an eigenvalue. This happens when $1 - \frac{2\tilde{F}}{\sqrt{C_0}} > 0$, or equivalently $\sqrt{C_0} \geq 2\tilde{F}$, or $C_0 > 4\tilde{F}^2$. The theorem is proved.

Q. E. D.

Conclusion

In the context of the Inverse Scattering Transform, we have worked on an important eigenvalue problem. Since the completion of the IST requires information about the eigenvalues in question, it follows that our work has significant value, especially in computations when it is nice to know bounds on the eigenvalues in order to make calculations more efficient. In addition, we have extended Klaus's work on chirped solitons, showing that soliton solutions of the Manakov system disappear given enough initial chirping. This result is one step towards showing the connections among experiment, computations, and theory. In the process we have proven several minor theorems, existence of the Jost solutions for the Manakov system, and the new bound on the imaginary part of the eigenvalues of the Manakov system.

Future areas of research include an investigation of exact solutions for particular potentials, especially a complete IST construction; investigation of the norming constants and reflection coefficients; and work on solving the Gel'fand-Levitan-Marchenko equation.

Appendix A

Mathematica Code

A.1 First Segment

$d_1 = 1; d_2 = 2;$

$h_2 = 2; h_2 = 2;$

$q_1[t\_] := \text{If}[\text{Abs}[t] < d_1, h_1, 0];$

$q_2[t\_] := \text{If}[\text{Abs}[t] < d_2, h_2, 0];$

$\text{epsi} = 0.1;$

$\text{sol1}[\xi\_] := \text{NDSolve}[\{$

$\quad v_1'[t] == -i\,\xi\, v_1[t] + q_1[t]\, v_2[t] + q_2[t]\, v_3[t],$

$\quad v_2'[t] == -q_1[t]\, v_1[t] + i\,\xi\, v_2[t],$

$\quad v_3'[t] == -q_2[t]\, v_1[t] + i\,\xi\, v_3[t],$

$v_1[-d_2] == 1,\ v_2[-d_2] == 0,\ v_3[-d_2] == 0\},\ \{v_1, v_2, v_3\}, \{t, -d_2 - \text{epsi}, d_2 + \text{epsi}\},$

$\text{MaxStepSize} \to 0.01, \text{MaxSteps} \to 5000];$

$g1[\xi\_, t\_] := v_1[t]/.\text{sol1}[\xi][[1]][[1]];$

$\text{Plot}[\text{Abs}[g1[i\, s, d_2]], \{s, .5, 2.2\}]$

A.2 Second Segment

[begin]

FindRoot$[$Abs$[g1[is, d_2]] == 0, \{s, \{2, 2.1\}\}]$

FindRoot$[$Abs$[g1[is, d_2]] == 0, \{s, \{1.5, 1.6\}\}]$

FindRoot$[$Abs$[g1[is, d_2]] == 0, \{s, \{.5, .6\}\}]$

[end]

Executing these commands in Mathematica gave

[begin]

$\{s \to 2.04512\}$

$\{s \to 1.52222\}$

$\{s \to 0.605503\}$

[end]

Many times in the process of finding these roots, Mathematica would confront me with the message that the method failed to converge after 15 iterations. Whenever that happened, I would refine my initial guesses to be closer to the root as shown by the graph produced in the code above.

A.3 Third Segment

sol1$[\xi_, k_] :=$ NDSolve$[\{$

$v_1'[t] == -i\,\xi\,v_1[t] + q_1[t]\,e^{ikt}\,v_2[t] + q_2[t]\,e^{-ikt}\,v_3[t],$

$v_2'[t] == -q_1[t]\,e^{-ikt}\,v_1[t] + i\,\xi\,v_2[t],$

$v_3'[t] == -q_2[t]\,e^{ikt}\,v_1[t] + i\,\xi\,v_3[t],$

$v_1[-d_2] == 1,\ v_2[-d_2] == 0,\ v_3[-d_2] == 0\},\ \{v_1, v_2, v_3\}, \{t, -d_2 - \text{epsi}, d_2 + \text{epsi}\},$

MaxStepSize $\to 0.01,$ MaxSteps $\to 5000];$

$g1[\xi_, k_, t_] := v_1[t]/.$sol1$[\xi, k][[1]][[1]];$

$g2[\xi_, k_, t_] := v_2[t]/.$sol1$[\xi, k][[1]][[2]];$

$g3[\xi_, k_, t_] := v_3[t]/.$sol1$[\xi, k][[1]][[3]];$

solxi$[\xi_, k_] :=$ NDSolve$[\{$

$\text{vxi}_1'[t] == -i\,g1[\xi, k, t] - i\,\xi\,\text{vxi}_1[t] + q_1[t]\,e^{ikt}\,\text{vxi}_2[t] + q_2[t]\,e^{-ikt}\,\text{vxi}_3[t],$

$\text{vxi}_2'[t] == -q_1[t]\,e^{-ikt}\text{vxi}_1[t] + i(g2[\xi, k, t] + \xi\,\text{vxi}_2[t]),$

$\text{vxi}_3'[t] == -q_2[t]\,e^{ikt}\text{vxi}_1[t] + i(g3[\xi, k, t] + \xi\,\text{vxi}_3[t]),$

```
    vxi₁[-d₂] == 0, vxi₂[-d₂] == 0, vxi₃[-d₂] == 0}, {vxi₁, vxi₂, vxi₃}, {t, -d₂, d₂},
 MaxStepSize -> 0.1, MaxSteps -> 5000];
gxi1[ξ_, k_, t_] := vxi₁[t]/. solxi[ξ, k][[1]][[1]];
solk[ξ_, k_] := NDSolve[{
    vk₁'[t] == -i ξ vk₁[t] + q₁[t](i t e^(ikt) g2[ξ, k, t]
        +e^(ikt) vk₂[t]) + q₂[t](-i t e^(-ikt) g3[ξ, k, t] + e^(-ikt) vk₃[t]),
    vk₂'[t] == -q₁ e^(-ikt)(-i t g1[ξ, k, t] + vk₁[t]) + i ξ vk₂[t],
    vk₃'[t] == -q₂ e^(ikt)(i t g1[ξ, k, t] + vk₁[t]) + i ξ vk₃[t],
    vk₁[-d₂] == 0, vk₂[-d₂] == 0, vk₃[-d₂] == 0}, {vk₁, vk₂, vk₃}, {t, -d₂, d₂},
 MaxStepSize -> 0.1, MaxSteps -> 5000];
gk1[ξ_, k_, t_] := vk₁[t]/. solk[ξ, k][[1]][[1]];
slope[ξ_, k_] := -gk1[ξ, k, d₂]/gxi1[ξ, k, d₂];
minc := 0.2;
personal[t_, y_] := {t, y};
rkstep[{ξ_, m_}] := Block[{k1, k2, k3, k4},
    k1 = slope[ξ, m];
    k2 = slope[ξ + minc * k1/2, m + minc/2];
    k3 = slope[ξ + minc * k2/2, m + minc/2];
    k4 = slope[ξ + minc * k3, m + minc];
    {ξ + minc * (k1 + 2 * k2 + 2 * k3 + k4)/6, m + minc}
];
l = NestList[rkstep, {i * 2.045119271994138, 0}, 50];
temp = Flatten[Map[Take[#, 1]&, %]];
z1 = Re[temp]; z2 = Im[temp];
Thread[personal[z1, z2]];
Biggest = ListPlot[%, PlotStyle -> Hue[.7], PlotJoined -> True]
```

A.4 Brute Force Algorithm

$d_1 = 1;$
$d_2 = 2;$
$h_1 = 1;$
$h_2 = 2;$
$k = 0.1;$
$q_1[t\_] = \text{If}[\text{Abs}[t] < d_1, h_1, 0];$
$q_2[t\_] = \text{If}[\text{Abs}[t] < d_2, h_2, 0];$
kDependence =
  $\text{Table}[\{k,$
    RealCenter = 0; ImagCenter = 0;
    RealRadius = 10; ImagRadius = 10;
    RealStep = 2; ImagStep = 2;
    $\text{For}\,[m = 1,\ m < 10,\ m{+}{+},$
      ListOfSols =
        $\text{Table}[\{\text{Imagine}, \text{Really},$
          $\xi = \text{Really} + \text{imagine} * i;$
          $\text{Sols}[\xi\_] =$
            $\text{NDSolve}[\{v_1'[t] == -i\,\xi\, v_1[t] + q_1[t]\, e^{ikt}\, v_2[t] + q_2[t]\, e^{-ikt}\, v_3[t],$
              $v_2'[t] == -q_1[t]\, e^{-ikt}\, v_1[t] + i\,\xi\, v_2[t],\ v_3'[t] == -q_2[t]\, e^{ikt}\, v_1[t] + i\,\xi\, v_3[t],$
              $v_1[-d_2] == 1,\ v_2[-d_2] == 0,\ v_3[-d_2] == 0\}, \{v_1, v_2, v_3\}, \{t, -d_2, d_2\}$
            $];$
          $g_1[\xi\_, t\_] = v_1[t] /. \text{Sols}[\xi][[1]][[1]];$
          $\text{Abs}[g_1[\xi, d_2]]\},$
        $\{\text{Imagine}, \text{ImagCenter} - \text{ImagRadius}, \text{ImagCenter} + \text{ImagRadius}, \text{ImagStep}\},$
        $\{\text{Really}, \text{RealCenter} - \text{RealRadius}, \text{RealCenter} + \text{RealRadius}, \text{RealStep}\}$
      $];$

```
        RealNums = (RealRadius * 2 + RealStep)/RealStep;
        ImagNums = (ImagRadius * 2 + ImagStep)/ImagStep;
        Answers = Table[ListOfSols[[i]][[j]][[3]], {i, 1, RealNums}, {j, 1, ImagNums}];
        Minimum = Min[Answers];
        Pos = Position[ListOfSols, Minimum];
        RealZeroInPos = Pos[[1]][[1]]; ImagZeroInPos = Pos[[1]][[2]];
        RealCenter = ListOfSols[[RealZeroInPos]][[ImagZeroInPos]][[2]];
        ImagCenter = ListOfSols[[RealZeroInPos]][[ImagZeroInPos]][[1]];
        RealRadius = RealRadius/4.0; ImagRadius = ImagRadius/4.0;
        RealStep = RealStep/4.0; ImagStep = ImagStep/4.0
    ];
    RealCenter + ImagCenter * i
}, {k, 0.0, 10.00, .01}];
kDependence
ComplexNumber = {1, 2}
ShortAnswer = Array[ComplexNumber, 1001];
For [j = 1, j < 1002, j + +,
    ShortAnswer[[j]] = {Re[kDependence[[j]][[2]], Im[kDependence[[j]][[2]]]}];
ShortAnswer
ShortAnswerPerm = *stuff*
ListPlot[ShortAnswer]
```

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