

A COMPARISON OF SEQUENTIAL PROBABILITY RATIO AND
GENERALIZED ATTRIBUTES ACCEPTANCE SAMPLING PLANS

by

Raymond Allen Shaparenko

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APPROVED:

J. A. Nachlas, Chairman

R. D. Foley

M.  Day

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(ABSTRACT)

A comparison is made between Wald's Sequential Probability Ratio sampling plan, several generalized attributes acceptance sampling plans, and a curtailed single sampling plan.

The evaluation of the plans is based upon a cost function based upon a combination of the Average Sampling Number (ASN) and the variance of an estimator for the proportion of defective items in a lot.

Using a numerical calculation of the defined cost function, the curtailed single sampling plan and also a generalized attributes acceptance sampling plan are shown to be better than Wald's SPR in a number of instances for representative operating characteristics. However, strictly in terms of ASN Wald's SPR is shown to be better.

A computer program is devised which gives a good approximation of the variance of the estimator used for Wald's SPR.

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CHAPTER I. INTRODUCTION

A comparison is made between the Sequential Probability Ratio (SPR) sampling plan as found in Wald [17], several Generalized Attributes Acceptance Sampling Plans [11], and a curtailed single sampling plan [7]. This comparison is based upon the average sampling number (ASN) and the variance of an estimator ($\text{Var}(\hat{p})$) for two specified sets of Operating Characteristic (O.C.) points.

The ASN is the expected number of samples that must be taken in a sampling procedure. Therefore it represents the cost that must be paid in terms of the number of observations required. On the other hand $\text{Var}(\hat{p})$ is the variance of an estimate for p , the proportion defective. Thus it represents the costs connected with rejecting or accepting a lot when the opposite decision should have been made.

Let

$$P[u_i] = \begin{cases} p & \text{if } u_i=1 \\ q=1-p & \text{if } u_i=0 \end{cases},$$

where $\{u_i\}$ is a sequence of independent Bernoulli random variables. As in the works of Girshick, Mosteller, and Savage [7] and of DeGroot [5], consider a path starting at the origin and moving one unit right when $u_i=1$ and one unit up when $u_i=0$. Then various sampling plans may be specified by choosing a boundary and establishing an acceptance region and a rejection region. Also ASN and $\text{Var}(\hat{p})$ may be computed.

The estimator used, as found in Kim and Nachlas [10], is

$$\hat{p} = \frac{x-u}{n-1}$$

where x is the number of defectives observed, n is the accumulated number of observations, $u_n = 1$ if the last item observed is defective, and $u_n = 0$ if that last item is non-defective.

The first of the plans under consideration is Wald's Sequential Probability Ratio (SPR) sampling plan [17]. Let p_1 be the desired quality level and p_2 the lot tolerance fraction defective. Let α be the risk of wrongly rejecting p_1 as the true parameter and β be the risk of wrongly accepting p_1 when p_2 is the true parameter. Observations are taken one by one and the ratio P_{2n}/P_{1n} is examined where P_{in} is the probability of observing the sequence u_1, u_2, u_3, \dots , given that p_i is the true parameter, $i=1,2$.

If $B < P_{2n}/P_{1n} < A$, continue sampling. If $P_{2n}/P_{1n} \geq A$, terminate and reject p_1 as the true parameter. If $P_{2n}/P_{1n} \leq B$, terminate and accept p_1 as the true parameter. One may take $A \approx \frac{1-\beta}{\alpha}$ and $B \approx \frac{\beta}{1-\alpha}$ as approximations for A and B .

In particular, for sequential testing by attributes for the lot proportion defective, if a sequence of n observations has x defectives, then let $P_{1n} = p_1^x (1-p_1)^{n-x}$ and $P_{2n} = p_2^x (1-p_2)^{n-x}$. Therefore

$$\frac{P_{2n}}{P_{1n}} = \frac{p_2^x (1-p_2)^{n-x}}{p_1^x (1-p_1)^{n-x}} .$$

So

$$\log \frac{P_{2n}}{P_{1n}} = x \log \frac{p_2}{p_1} + (n-x) \log \frac{1-p_2}{1-p_1} .$$

The lot is rejected, if

$$x \log \frac{p_2}{p_1} + (n-x) \log \frac{1-p_2}{1-p_1} \geq \log A ,$$

Also it can be shown that the lot would be rejected if

$$x \geq \frac{\log \frac{1-p_1}{1-p_2}}{\log \frac{p_2}{p_1} + \log \frac{1-p_1}{1-p_2}} n + \frac{\log A}{\log \frac{p_2}{p_1} + \log \frac{1-p_1}{1-p_2}}$$

or if $x \geq sn + b_1$, where

$$s = \frac{\log \frac{1-p_1}{1-p_2}}{\log \frac{p_2}{p_1} + \log \frac{1-p_1}{1-p_2}} \quad \text{and} \quad b_1 = \frac{\log A}{\log \frac{p_2}{p_1} + \log \frac{1-p_1}{1-p_2}} .$$

Likewise, it can be shown that the lot would be accepted if

$x \leq sn - b_2$, where

$$b_2 = \frac{-\log B}{\log \frac{p_2}{p_1} + \log \frac{1-p_1}{1-p_2}} .$$

One would continue sampling otherwise.

The first question to be answered is how should one describe Wald's SPR sampling plan in terms of a generalized sampling diagram. This is an easy question to answer. If x is the number of defectives, y the number non-defectives, and n the total number sampled, then the

question is answered algebraically by setting $n=x+y$. Thus the boundaries $x=sn+b_1$ and $x=sn-b_2$ in Wald's SPR sampling plan would be transformed into

$$y = \frac{1-s}{s} x - \frac{b_1}{s} \quad \text{and} \quad y = \frac{1-s}{s} x + \frac{b_2}{s} ,$$

respectively, $0 < s \leq 1$. This is represented graphically in Figure 1.

Note that the upper line would be the acceptance boundary and the lower line would be the rejection boundary.

Secondly generalized attributes plans are considered. In particular plans called S_6 and S_8 in Kim and Nachlas [10], are considered. S_6 may be viewed as having a minimum sample size n_1 , a maximum sample size n_2 , and a minimum number of defectives m_1 . This plan is depicted in Figure 2. S_8 may be viewed as containing a maximum sample size n_2 , a minimum number defective m_1 , and a minimum number non-defective m_2 , as shown in Figure 3.

S_6 and S_8 may be considered as generalized attribute acceptance sampling plans by letting the lot be accepted if a sample path reaches point A or above and are rejected if a sample path reaches the part of the boundary below or to the right of point A as shown in Figures 2 and 3.

Thirdly a curtailed single sampling plan is constructed so that observation would be terminated upon reaching either a fixed number K_1 of defective items or a fixed number K_2 of non-defective items. This plan, denoted $S^*(K_1, K_2)$, was considered by Girshick, Mosteller, and Savage [7]. It is depicted in Figure 4. Note that once again the

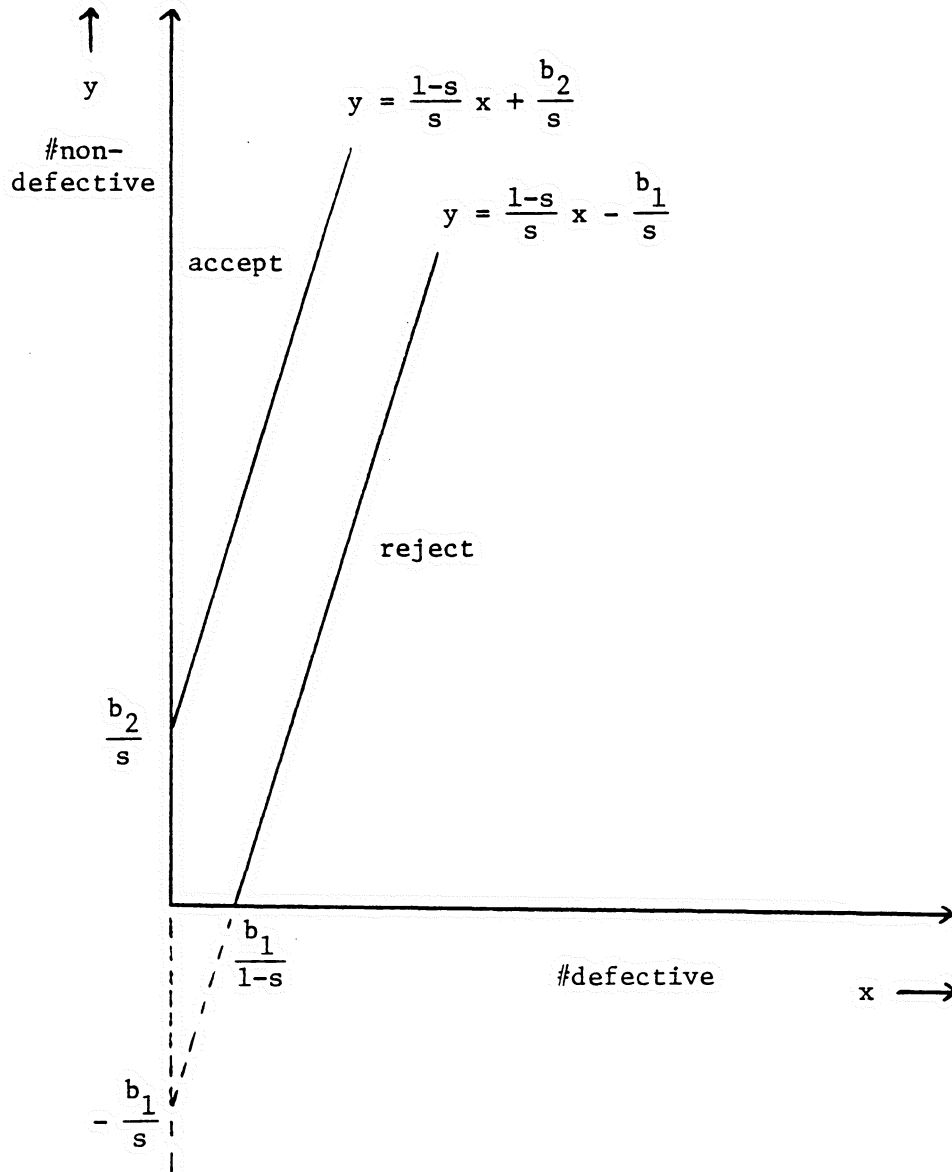
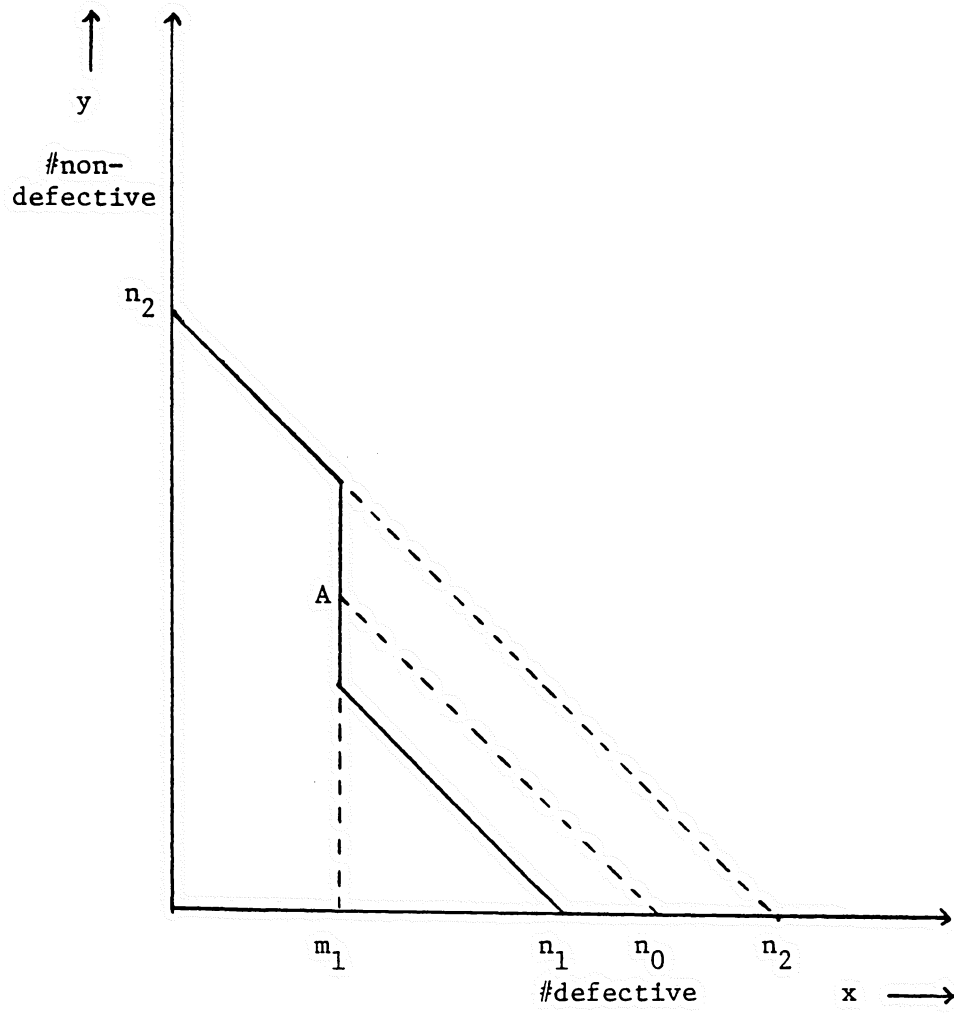
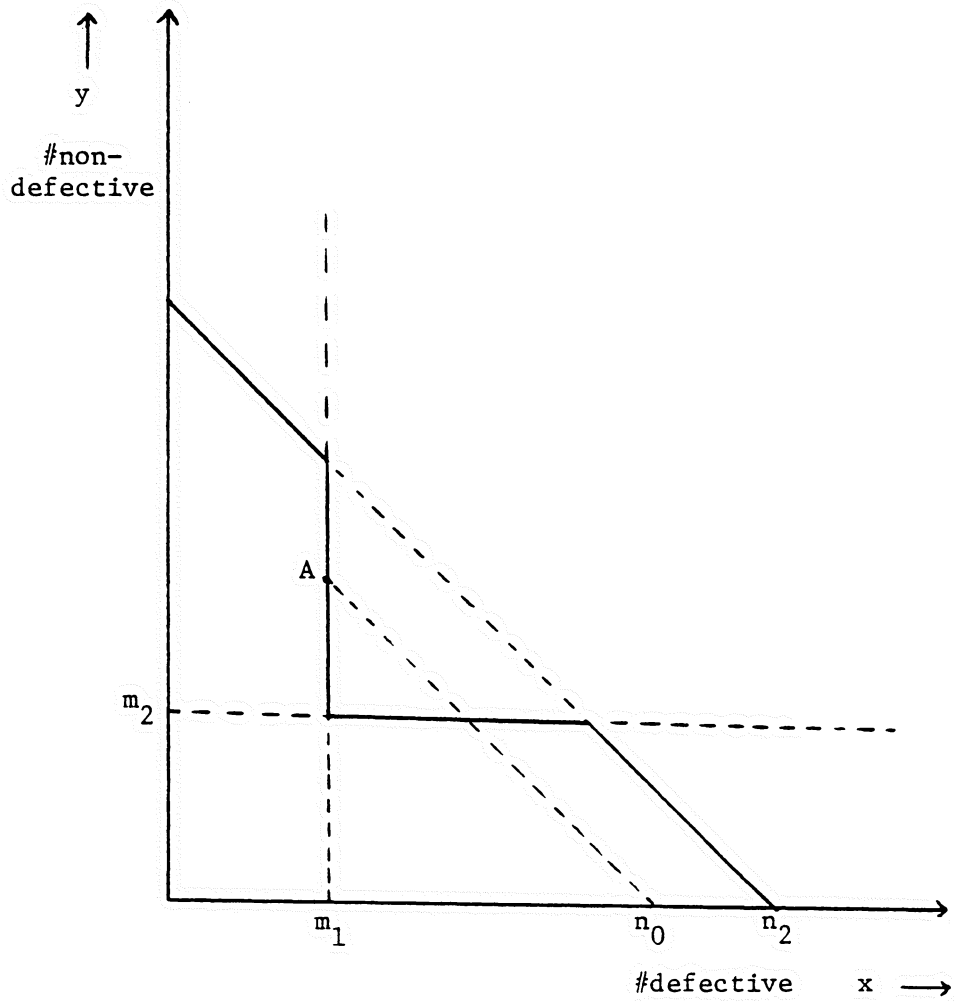


Figure 1. Wald's SPR on a generalized sampling diagram.

Figure 2. S_6 .

Figure 3. S_8 .

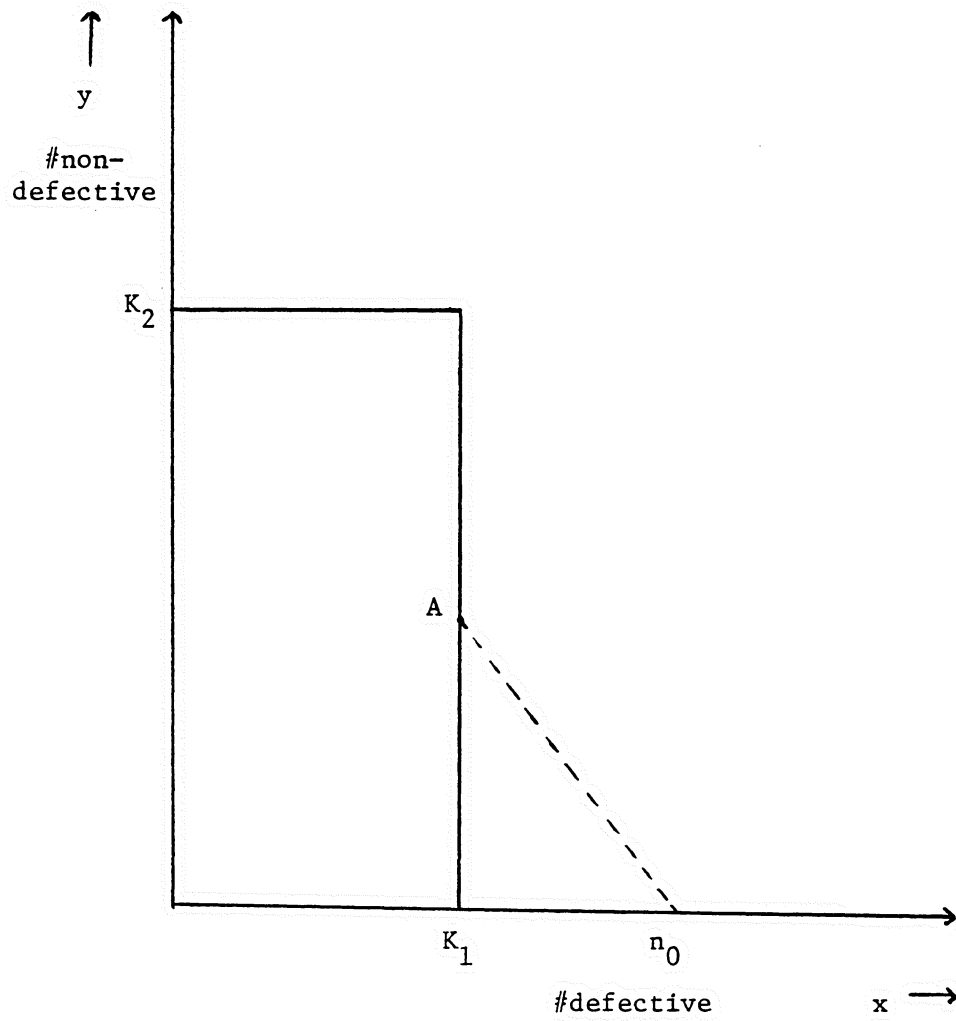


Figure 4. Curtailed single sampling plan $S^*(K_1, K_2)$.

critical point A with associated critical sample size n_0 determines the acceptance and rejection portion of the boundary, that being accept along boundary points at, above, and to the left of A while reject at boundary points strictly below A. The appeal of the curtailed single sampling plan $S^*(K_1, K_2)$ is that like a Wald SPR plan, observation is terminated early when the quality level is near the defined limits of interest and is terminated later at intermediate values of p .

For any given one of these sampling plans, with parameters specified, the Operating Characteristic (O.C.) curve is defined to be the probability of accepting the lot as a function of p , the underlying proportion defective. Thus an O.C. curve is more favorable the higher the value of the curve for acceptable values of p and the lower the value of the curve for unacceptable values of p . If an O.C. curve is specified, the various plans with the same O.C. curve may be compared by using some type of cost criterion. However, it is usually not possible to obtain a plan with the exact O.C. curve which is desired. Given sampling errors α and β as in Duncan [6], parameters may be chosen for the sampling plans so that approximately the desired O.C. curve is obtained. One way of doing this is to fix either α or β so that the sampling plan may "fit" either $(p_1, 1-\alpha)$ or (p_2, β) and not be "too far" from the other, where p_1 is taken to be the desired quality level and p_2 the lot tolerance fraction defective. In this work the parameters are selected so that the O.C. curve for each of the sampling plans under consideration would pass through these two points. (Note that, as seen later, the resulting O.C. curves will

actually be in close agreement over the full range of significant values of p .)

The previously defined estimator $\hat{p} = \frac{x-u_n}{n-1}$, where x is the number of defectives observed, n is the accumulated number of observations, $u_n=1$ if the last observation yields a defective, and $u_n=0$ if the last item is non-defective, is shown to be unbiased for sampling plans S_6 , S_8 , and S^* . For Wald's SPR plans W however this estimator may only be close to being unbiased. Nevertheless, as in Kim and Nachlas [11], the various plans are compared by using

$$Z = \text{ASN} + R \text{Var}(\hat{p})$$

as a cost criterion. The plan with the lower Z value would be the better plan. R should be treated as a scaling factor which would allow ASN and $\text{Var}(\hat{p})$ to be weighted for relative importance. If R is large, $\text{Var}(\hat{p})$ would be more important and if R was small, then ASN would be relatively more important. $R = 5000, 50000, \text{ and } 500000$ are used in this study. While $Z = \text{ASN} + R \text{Var}(\hat{p})$ is not the only sensible cost function, it efficiently reflects the balance between inspection costs and sample precision in defining a sampling plan.

The analysis in this work may be summarized as follows. First the two points $(p_1, 1-\alpha)$ and (p_2, β) are specified and the parameters for each of the sampling plans under consideration are determined so that these two points are on the O.C. curve for each plan. Any parameters remaining undetermined after this "fitting" will be selected so as to minimize Z .

The comparisons are carried out using each of two different sets of values for p_1 , p_2 , α , and β , and each of three different values for R , giving a total of six test cases. Values for $Z(S_6)$ are taken from Kim and Nachlas [11]. The other values are approximated numerically using a digital computer. For plans W , Wald's approximations are used, as well as another approximating procedure for $\text{Var}(\hat{p}(W))$. The formulas and calculations employed are described in more detail in Chapters III, IV, and V.

$S^*(K_1, K_2)$ is shown to be better than Wald's plans in a number of cases. Also it is shown that, depending on whether ASN or $\text{Var}(\hat{p})$ is weighted heavier, $S^*(K_1, K_2)$ or S_6 would be the optimal plan. Ranges of values of the weighting factor R are also obtained in which S_6 is better than S_8 .

CHAPTER II. REVIEW OF LITERATURE

During World War II, Abraham Wald [17] developed the probability theory for sequential testing. He proposed the sequential probability ratio test and studied its properties with particular emphasis on hypothesis testing, a context somewhat different from what is considered here. His work has been extensively studied since the time it was proposed.

Kim and Nachlas [10] outline various sampling plans of sequences of Bernoulli trials based upon four parameters. The parameters m_1 , the minimum number of successes (defectives) before termination; m_2 , the minimum number of failures (non-defectives) before termination; n_1 , the minimum sample size; and n_2 , the maximum sample size, fully determine the general plan $S_0(\cdot; n_1, n_2, m_1, m_2)$ depicted in Figure 5. In the plan S_0 as described in Kim and Nachlas [10], S_6 may be defined by setting $m_2=0$, while S_8 is defined by setting $n_1=m_1+m_2$.

Kim and Nachlas [10] also discuss the previously defined estimator, $\hat{p} = \frac{x-u}{n-1}$, and show that it is unbiased for S_0 .

Kim and Nachlas [11] use $Z = ASN + R \text{Var}(\hat{p})$ as a cost criterion to compare a fixed sample size sampling plan with S_6 in hope of determining which plan would be optimal and under what circumstances.

As mentioned in Kim and Nachlas [10] the equation

$$Z = ASN + R \text{Var}(\hat{p})$$

eliminates the need for three or more cost coefficients, i.e. sampling cost, acceptance cost, and rejection cost as used earlier by Duncan

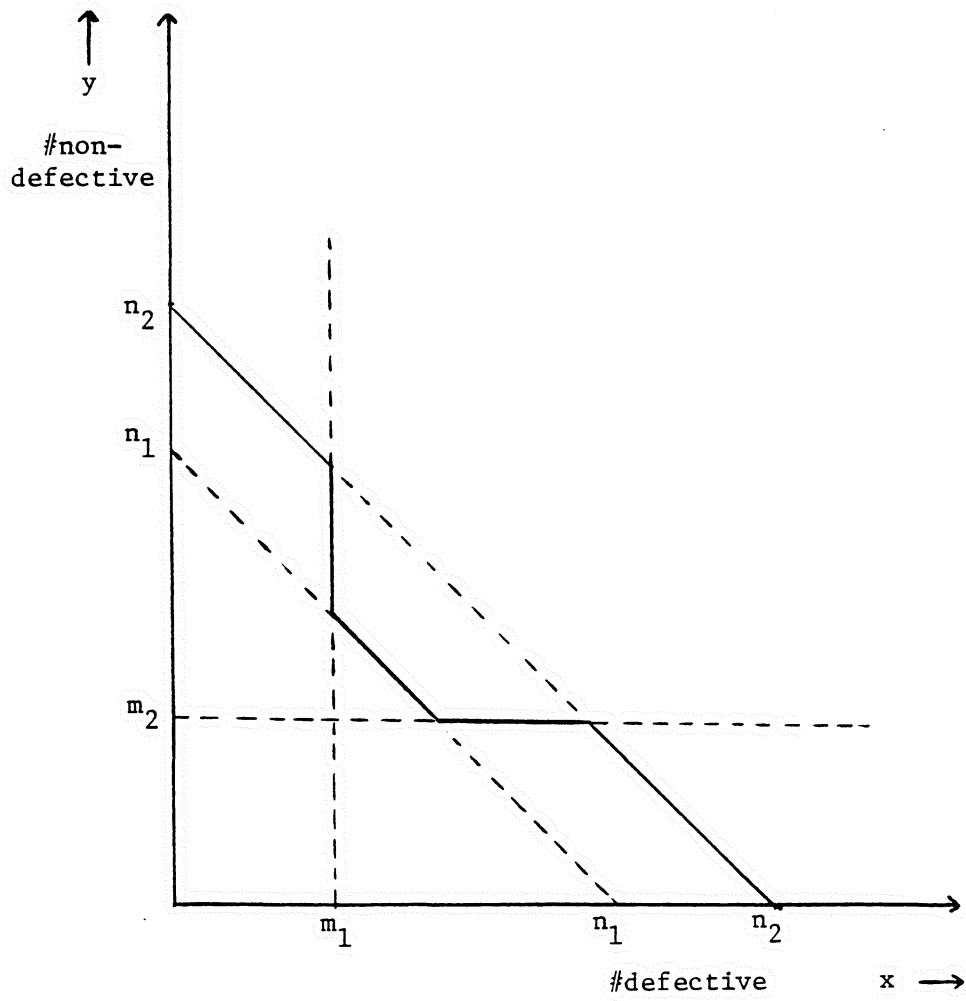


Figure 5. $S_0(\cdot; n_1, n_2, m_1, m_2)$.

[6]; Chiu and Wetherill [3]; and Schmidt, Case, and Bennett [15].

Among other contributions in the literature are the works of Phatak and Bhat [12] and Cohen [3] which seek to reduce sampling costs by using the maximum likelihood estimator for p for curtailment in fixed sample size plans.

Also mention should be made of the somewhat general results of Bertsekas [2]. A different cost function that employed here is used, but general results are obtained.

The plan $S^*(K_1, K_2)$ was first considered by Girshick, Mosteller, and Savage [7], and later by Phatak and Bhat [13] and Cohen [4]. Anderson and Friedman [1] compare Wald's SPR plan with the curtailed single sampling plan $S^*(K_1, K_2)$ for K_1 , the number of defectives, equal to 1. Later Samuel [14] further expanded the work of Anderson and Friedman [1] to a comparison of $S^*(1, K_2)$ with randomized SPR's.

Wasam [18] also mentions a comparison of curtailed single sampling plans $S^*(K_1, K_2)$ with Wald's SPR, using several means to compare SPR and $S^*(K_1, K_2)$. On one hand comparison is based strictly on ASN, but it should also be noted that a minimization of cost per unit times ASN plus Variance is also used. This is equivalent to

$$Z = \text{ASN} + R \text{Var}(\hat{p})$$

which is used in this work.

Also a problem arises since the O.C. curve and ASN formulas for Wald's SPR are only approximate. Page [12], Kemp [9], and Tallis and Vagholkar [16] have been able to give better approximations for these.

Lastly mention must be made of Govindarajulu [8]. This work is as complete in discussing sequential procedures as it could be and still remain in only one volume. The pertinent results from the time of Wald's work and on are discussed with a complete set of references given.

CHAPTER III. ANALYTICAL DEFINITION OF SAMPLING PLANS

In this chapter the necessary mathematical formulas for carrying out the calculations outlined in Chapter I are given.

The Operating Characteristic (O.C.) curve is the probability $Pa(p)$ of acceptance as a function of p . For Wald's Plan the O.C. curve is found in Wald [17] and is given by the approximation

$$Pa(p) \approx \frac{\left(\frac{1-\beta}{\alpha}\right)^h - 1}{\left(\frac{1-\beta}{\alpha}\right)^h - \left(\frac{\beta}{1-\alpha}\right)^h}$$

where h is the solution to

$$1 = \frac{\left(\frac{1-p_2}{1-p_1}\right)^h}{\left(\frac{p_2}{p_1}\right)^h - \left(\frac{1-p_2}{1-p_1}\right)^h}$$

Thus by specifying α , β , p , and p_2 , the above formulas determine how the parameters of Wald's plan are chosen to "fit" the O.C. curve to the points $(p_1, 1-\alpha)$ and (p_2, β) .

For this plan the ASN is given by approximation

$$ASN(p) \approx \frac{Pa(p)\log B + (1-Pa(p))\log A}{p \log \frac{p_2}{p_1} + (1-p)\log \frac{1-p_2}{1-p_1}}$$

where $Pa(p)$ is the O.C. curve for the SPR plan, $A \approx (1-\beta)/\alpha$ and $B \approx \beta/(1-\alpha)$ as stated previously.

To find \hat{p} and $\text{Var}(\hat{p})$ for Wald's plan the number of ways to get to each point along the boundary is needed. This may be computed recursively as follows: Let $f(k, \ell)$ denote the number of paths from $(0, 0)$ to the point (k, ℓ) which stay between the lines $y = \frac{1-s}{s}x + \frac{b_2}{s}$ and $y = \frac{1-s}{s}x - \frac{b_1}{s}$ always, except at the last inspection in which case (k, ℓ) is a stopping boundary point. First define

$$f(0, \ell) = 1 \quad \text{for} \quad 0 \leq \ell \leq \left\lceil \frac{b_2}{s} \right\rceil$$

and

$$f(k, 0) = 1 \quad \text{for} \quad 0 \leq k \leq \left\lceil \frac{b_1}{1-s} \right\rceil$$

where $\lceil x \rceil$ is the smallest integer greater than or equal to x . Next appears the recursion formula to compute the remaining $f(k, \ell)$. First for $0 < k < \frac{b_1}{1-s}$,

$$f(k, \ell) = \begin{cases} f(k-1, \ell) + f(k, \ell-1), & 0 < \ell < \left\lceil \frac{1-s}{s}(k-1) + \frac{b_2}{s} \right\rceil \\ f(k, \ell-1) & , \left\lceil \frac{1-s}{s}(k-1) + \frac{b_2}{s} \right\rceil \leq \ell \leq \left\lceil \frac{1-s}{s}k + \frac{b_2}{s} \right\rceil \end{cases}$$

and for $k \geq \frac{b_1}{1-s}$ the relation becomes

$$f(k, \ell) = \begin{cases} f(k-1, \ell) & , \left\lceil \frac{1-s}{s}(k-1) - \frac{b_1}{s} \right\rceil < \ell \leq \left\lceil \frac{1-s}{s}k - \frac{b_1}{s} \right\rceil + 1 \\ f(k-1, \ell) + f(k, \ell-1), & \left\lceil \frac{1-s}{s}k - \frac{b_1}{s} \right\rceil + 1 < \ell < \left\lceil \frac{1-s}{s}(k-1) + \frac{b_2}{s} \right\rceil \\ f(k, \ell-1) & , \left\lceil \frac{1-s}{s}(k-1) + \frac{b_2}{s} \right\rceil \leq \ell \leq \left\lceil \frac{1-s}{s}k + \frac{b_2}{s} \right\rceil \end{cases}$$

and $f(k, \ell) = 0$ for all other combinations of k and ℓ not heretofore defined, where $[x]$ is the greatest integer less than or equal to x .

Basically these formulas relate the fact that some points interior to the lines $y = \frac{1-s}{s}x + \frac{b_2}{s}$ and $y = \frac{1-s}{s}x - \frac{b_1}{s}$ may be reached by either horizontal or vertical steps, but that others may be reached only by horizontal or only by vertical steps. Also stopping points along and directly above the top line $y = \frac{1-s}{s}x + \frac{b_2}{s}$ may only be reached by a vertical step, i.e. the last inspection must be a non-defective item, but that the stopping points along and to the right of the lower line $y = \frac{1-s}{s}x - \frac{b_1}{s}$ may only be reached by a horizontal step, i.e. a defective item must be the last inspection outcome.

Given the above recursion formulas, the probability of stopping at the termination points along and directly above the upper line may be computed. This probability is the O.C. curve for Wald's Plan W and is given by

$$Pa(p) = \sum_{k=0}^{\infty} f\left(k, \left\lceil \frac{1-s}{s}k + \frac{b_2}{s} \right\rceil\right) p^k q^{\left\lceil \frac{1-s}{s}k + \frac{b_2}{s} \right\rceil}.$$

Also it may be noted here that the probability of terminating along the lower line would be approximated by

$$1 - Pa(p) = \sum_{\ell=0}^{\infty} f\left(\left\lfloor \frac{s}{1-s}\ell + \frac{b_1}{1-s} \right\rfloor, \ell\right) p^{\left\lfloor \frac{s}{1-s}\ell + \frac{b_1}{1-s} \right\rfloor} q^{\ell}.$$

To compute the variance of the estimator $\hat{p} = \frac{x-u}{n-1}$, note that for Wald's SPR this estimator may be written in the form

$$\hat{p}(W) = \begin{cases} \frac{k}{\ell+k-1} & \text{along the top line} \\ \frac{k-1}{\ell+k-1} & \text{along the bottom line} \end{cases}$$

Assuming $\hat{p}(W)$ is close to being unbiased, the variance of $\hat{p}(W)$ may be written as

$$\begin{aligned} \text{Var}(\hat{p}) &= E[\hat{p}^2] - E^2[\hat{p}] \\ &= E[\hat{p}^2] - p^2 \\ &= \sum_{k=1}^{\infty} \left(\frac{k}{k + \left[\frac{1-s}{s} k + \frac{b_2}{s} \right] \left[-1 \right]} \right)^2 f \left(k, \left[\frac{1-s}{s} k + \frac{b_2}{s} \right] \right) p^k q^{\left[\frac{1-s}{s} k + \frac{b_2}{s} \right]} \\ &\quad + \sum_{\ell=0}^{\infty} \left(\frac{\left[\frac{s}{1-s} \ell + \frac{b_1}{1-s} \right] \left[-1 \right]}{\ell + \left[\frac{s}{1-s} \ell + \frac{b_1}{1-s} \right] \left[-1 \right]} \right)^2 f \left(\left[\frac{s}{1-s} \ell + \frac{b_1}{1-s} \right], \ell \right) p^{\left[\frac{s}{1-s} \ell + \frac{b_1}{1-s} \right]} q^{\ell} \\ &= p^2 \end{aligned}$$

Note that there are an infinite number of terms in this formula. Therefore an approximation for $\text{Var}(\hat{p})$ will instead be obtained.

A computer program (See Appendix A) is generated to estimate this variance, by adding the terms in the formula of $\text{Var}(\hat{p}(W))$ up to and including a maximum number of defectives. This effectively truncates Plan W vertically at a maximum number of defectives. This truncation is completed at a sufficiently large value of the maximum number of defectives allowable so as actually to obtain a fairly close approximation of $\text{Var}(\hat{p}(W))$ from below.

Next consider the generalized attributes sampling plans S_6 and S_8 . For S_8 , let n_0 be the critical value defining point A as shown on Figure 3 and let p be the probability of a defective. Then the O.C. curve for sampling plan S_8 would be

$$\begin{aligned} Pa(p) &= 1 - \sum_{i=0}^{n_0-1} \binom{i-1}{m_1-1} p^{m_1} q^{i-m_1} \\ &= 1 - B^{-1}(m_1, n_0-1, p) \end{aligned}$$

where $B^{-1}(i, j, p)$ is the cumulative negative binomial probability that the i^{th} success occurs on or before the j^{th} trial.

The O.C. curve for S_6 as given in Kim and Nachlas [11] is the same as for S_8 , namely

$$Pa(p) = 1 - B^{-1}(m_1, n_0-1, p)$$

where again n_0 is the critical sample size as shown in Figure 2 and p is the probability of a defective.

For each of these plans the average sampling number (ASN) must also be obtained.

The ASN for S_6 may be found in Kim and Nachlas [11] and is given as

$$\begin{aligned} ASN(S_6) &= n_2 B(m_1-1, n_2, p) \\ &\quad + n_1 [1 - B(m_1-1, n_1, p)] \\ &\quad + m_1 \sum_{j=n_1+1}^{n_2} b(m_1, j, p) \end{aligned}$$

where $b(m_1, j, p) = \binom{j}{m_1} p^{m_1} q^{j-m_1}$.

The ASN for S_8 may be calculated by setting $n_1 = m_1 + m_2$ in the formula given for $ASN(S_0)$ in Kim and Nachlas [10]. Thus,

$$\begin{aligned} ASN(S_8) &= n_2 [1 + B(m_1 - 1, n_2, p) - B(n_2 - m_2, n_2, p)] \\ &\quad + (m_1 + m_2) [B(m_1, m_1 + m_2, p) - B(m_1 - 1, m_1 + m_2, p)] \\ &\quad + \frac{m_1}{p} [B^{-1}(m_1 + 1, n_2 + 1, p) - B^{-1}(m_1 + 1, m_1 + m_2 + 1, p)] \\ &\quad + \frac{m_2}{q} [B^{-1}(m_2 + 1, n_2 + 1, q) - B^{-1}(m_2 + 1, m_1 + m_2 + 1, q)] \end{aligned}$$

$$\text{where } B(y, n, p) = \sum_{i=0}^y b(i, n, p) = \sum_{i=0}^y \binom{n}{i} p^i q^{n-i} .$$

The argument to show that the estimator \hat{p} is unbiased for S_8 follows that of Kim and Nachlas [10]. Let D_0 denote the boundary points under $S_8 = S_0(\cdot; m_1 + m_2, n_2, m_1, m_2)$ and D'_0 the boundary points under $S_8 = S_0(\cdot; m_1 + m_2 - 1, n_2 - 1, m_1 - 1, m_2)$. Then since

$$P[N=n, X=x, U_N = u_n] = \binom{n-1}{x-u_n} p^x q^{n-x} \quad \text{for } (n, x, u_n) \in D_0$$

$$\text{where } u_n = \begin{cases} 1 & p \\ 0 & q=1-p \end{cases} ,$$

$$\begin{aligned} E[\hat{p}(S_8)] &= \sum_{(n, x, u_n) \in D_0} \frac{x - u_n}{n-1} \binom{n-1}{x-u_n} p^x q^{n-x} \\ &= \sum_{(n, x, 1) \in D_0} \frac{x-1}{n-1} \binom{n-1}{x-1} p^x q^{n-x} + \sum_{(n, x, 0) \in D_0} \frac{x}{n-1} \binom{n-1}{x} p^x q^{n-x} \\ &= p \left[\sum_{(n, x, 1) \in D_0} \frac{x-1}{n-1} \binom{n-1}{x-1} p^{x-1} q^{n-x} + \sum_{(n, x, 0) \in D_0} \frac{x}{n-1} \binom{n-1}{x} p^{x-1} q^{n-x} \right] \end{aligned}$$

$$\begin{aligned}
&= p \left[\sum_{(n,x,1) \in D_0} \binom{n-2}{x-2} p^{x-1} q^{n-x} + \sum_{(n,x,0) \in D_0} \binom{n-2}{x-1} p^{x-1} q^{n-x} \right] \\
&= p \left[\sum_{(n,x,u_n) \in D_0} \binom{n-2}{x-1-u_n} p^{x-1} q^{n-x} \right] \\
&= p \left[\sum_{(n',x',u'_n) \in D_0} \binom{n'-1}{x'-u'_n} p^{x'} q^{n'-x'} \right] \quad \text{where } \begin{array}{l} n'=n-1 \\ x'=x-1 \end{array} \\
&= p .
\end{aligned}$$

This same argument will show $E[\hat{p}(S_6)] = p$ if D_0 stands for the set of points on the boundary of $S_6 = S_0(\cdot; n_1, n_2, m_1, 0)$ and D_0' the set of points on the boundary of $S_6 = S_0(\cdot; n_1-1, n_2-1, m_1-1, 0)$.

The variance of the unbiased estimators for S_8 and S_6 are also found in Kim and Nachlas [11]. These are

$$\begin{aligned}
\text{Var}(\hat{p}(S_6)) &= \sum_{K=0}^{m_1-1} (K/n_2)^2 b(K, n_2, p) \\
&+ \sum_{K=n_1}^{n_1} (K/n_1)^2 b(K, n_1, p) \\
&+ \sum_{K=n_1+1}^{n_2} ((m_1-1)/(K-1))^2 b^{-1}(m_1, K, p) \\
&- p^2, \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
\text{Var}(\hat{p}(S_8)) &= \sum_{x=0}^{m_1-1} (x/n_2)^2 b(x, n_2, p) \\
&+ \sum_{x=n_2-m_2+1}^{n_2} (x/n_2)^2 b(x, n_2, p)
\end{aligned}$$

$$\begin{aligned}
& + (m_1 / (m_1 + m_2))^2 \binom{m_1 + m_2}{m_1} p^{m_1} q^{m_2} \\
& + \sum_{K=m_1+m_2+1}^{n_2} ((m_1 - 1) / (K - 1))^2 b^{-1}(m_1, K, p) \\
& + \sum_{K=m_1+m_2+1}^{n_2} ((K - m_2) / (K - 1))^2 b^{-1}(m_2, K, q) \\
& - p^2 .
\end{aligned}$$

One other plan is also to be considered, the curtailed single sampling plan. The O.C. curve for sampling plan S^* can be seen to be

$$Pa(p) = 1 - B^{-1}(K_1, n_0 - 1, p) .$$

Also the ASN for plan S^* is

$$\begin{aligned}
ASN &= \sum_{j=K_1}^{K_1+K_2-1} j b^{-1}(K_1, j, p) + \sum_{j=K_2}^{K_1+K_2-1} j b^{-1}(K_2, j, q) \\
&= \sum_{j=K_1}^{K_1+K_2-1} j \frac{(j-1)!}{(K_1-1)!(j-K_1)!} p^{K_1} q^{j-K_1} + \sum_{j=K_2}^{K_1+K_2-1} j \frac{(j-1)!}{(K_2-1)!(j-K_2)!} q^{K_2} p^{j-K_2} \\
&= K_1 \sum_{j=K_1}^{K_1+K_2-1} \frac{j!}{K_1!(j-K_1)!} p^{K_1} q^{j-K_1} + K_2 \sum_{j=K_2}^{K_1+K_2-1} \frac{j!}{K_2!(j-K_2)!} q^{K_2} p^{j-K_2} \\
&= K_1 \sum_{j=K_1}^{K_1+K_2-1} b(K_1, j, p) + K_2 \sum_{j=K_2}^{K_1+K_2-1} b(K_2, j, q)
\end{aligned}$$

$$\begin{aligned}
&= K_1 \sum_{j=K_1}^{K_1+K_2-1} \frac{1}{p} b^{-1}(K_1+1, j+1, p) + K_2 \sum_{j=K_2}^{K_1+K_2-1} \frac{1}{q} b^{-1}(K_2+1, j+1, q) \\
&= \frac{K_1}{p} \sum_{\ell=K_1+1}^{K_1+K_2} b^{-1}(K_1+1, \ell, p) + \frac{K_2}{q} \sum_{\ell=K_2+1}^{K_1+K_2} b^{-1}(K_2+1, \ell, q) \\
&= \frac{K_1}{p} \left[B^{-1}(K_1+1, K_1+K_2, p) - B^{-1}(K_1+1, K_1, p) \right] \\
&= \frac{K_2}{q} \left[B^{-1}(K_2+1, K_1+K_2, q) - B^{-1}(K_2+1, K_2, q) \right] \\
&= \frac{K_1}{p} B^{-1}(K_1+1, K_1+K_2, p) + \frac{K_2}{q} B^{-1}(K_2+1, K_1+K_2, q)
\end{aligned}$$

To show that the estimator $\hat{p} = \frac{x-u}{n-1}$ is unbiased for $S^*(K_1, K_2)$, this estimator is rewritten in the form

$$\hat{p}(S^*) = \begin{cases} \frac{K_1-1}{n-1} & \text{along the line } K_1 \\ \frac{n-K_2}{n-1} & \text{along the line } K_2 \end{cases},$$

then $E[\hat{p}]$ is computed.

$$\begin{aligned}
E[\hat{p}] &= \sum_{j=1}^{K_1+K_2-1} \frac{K_1-1}{j-1} b^{-1}(K_1, j, p) + \sum_{j=1}^{K_1+K_2-1} \frac{j-K_2}{j-1} b^{-1}(K_2, j, q) \\
&= \sum_{j=K_1-1}^{K_1+K_2-1} \frac{K_1-1}{j-1} \binom{j-1}{K_1-1} p^{K_1} q^{j-K_1} + \sum_{j=K_2-1}^{K_1+K_2-1} \frac{j-K_2}{j-1} \binom{j-1}{K_2-1} q^{K_2} p^{j-K_2}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=K_1}^{K_1+K_2-1} \binom{j-2}{K_1-2} p^{K_1} q^{j-K_1} + \sum_{j=K_2+1}^{K_1+K_2-1} \binom{j-2}{K_2-1} q^{K_2} p^{j-K_2} \\
&= p \left[\sum_{j=K_1}^{K_1+K_2-1} \binom{j-2}{K_1-2} p^{K_1-1} q^{j-K_1} + \sum_{j=K_2+1}^{K_1+K_2-1} \binom{j-2}{K_2-1} q^{K_2} p^{j-K_2-1} \right] \\
&= p \left[\sum_{j=K_1}^{K_1+K_2-1} b^{-1}(K_1-1, j-1, p) + \sum_{j=K_2+1}^{K_1+K_2-1} b^{-1}(K_2, j-1, q) \right] \\
&= p \left[\sum_{\ell=K_1-1}^{K_1+K_2-2} b^{-1}(K_1-1, \ell, p) + \sum_{\ell=K_2}^{K_1+K_2-2} b^{-1}(K_2, \ell, q) \right] \\
&= p ,
\end{aligned}$$

since the last term in the bracket is the probability of reaching the line K_1-1 or the line K_2 . That probability is 1.

The variance of $\hat{p}(S^*)$ is computed as follows:

$$\begin{aligned}
\text{Var}(\hat{p}(S^*)) &= E[\hat{p}^2] - (E[\hat{p}])^2 \\
&= E[\hat{p}^2] - p^2 \\
&= \sum_{j=K_1}^{K_1+K_2-1} \left(\frac{K_1-1}{j-1} \right)^2 b^{-1}(K_1, j, p) + \sum_{j=K_2}^{K_1+K_2-1} \left(\frac{j-K_2}{j-1} \right)^2 b^{-1}(K_2, j, q) - p^2
\end{aligned}$$

CHAPTER IV. DEFINITION OF TEST CASES AND RESULTS

As mentioned previously in Chapter I the points $(p_1, 1-\alpha)$ and (p_2, β) are the same for each of the O.C. curves for the various plans. Two cases are used $\alpha \approx .051$, $\beta \approx .216$, $p_1 = .01$, $p_2 = .05$, and $\alpha \approx .075$, $\beta \approx .238$, $p_1 = .02$, $p_2 = .08$. To line up the plans S^* , W , and S_8 with S_6 take $K_1 = m_1 = c+1$ and $n_0 = n+1$ as parameters in the O.C. curves of S^* , S_6 , and S_8 , where the values of c and n are given in Kim and Nachlas [11]. Then, for the case $\alpha \approx .051$, $\beta \approx .216$, n is 82 and c is 2; whereas for the case $\alpha \approx .075$, $\beta \approx .238$, n is 49 and c is 2. Note that this leaves only K_2 as a free parameter in plans S^* , and m_2 and n_2 as free parameters in plans S_8 . For Wald's plan given α , β , p_1 , p_2 the approximate values on the O.C. curve may be computed by the formulas given earlier.

Although only the two points $(p_1, 1-\alpha)$ and (p_2, β) are "fitted" on these O.C. curves by consulting Table 1 in Appendix B and Figures 6 and 7 for the graphs of the O.C. curves for p , the underlying proportion defective, being between 0.00 to 0.10, it can be seen that the curves do lie very close to one another.

Also as mentioned earlier the plans will be compared under a minimization of $Z = ASN + R \text{Var}(\hat{p})$. If R is about 5000, the ASN is weighted heavier. If R is about 50,000, then ASN and $\text{Var}(\hat{p})$ are weighted approximately equal. Last if R is about 500,000 then $\text{Var}(\hat{p})$ is weighted heavier.

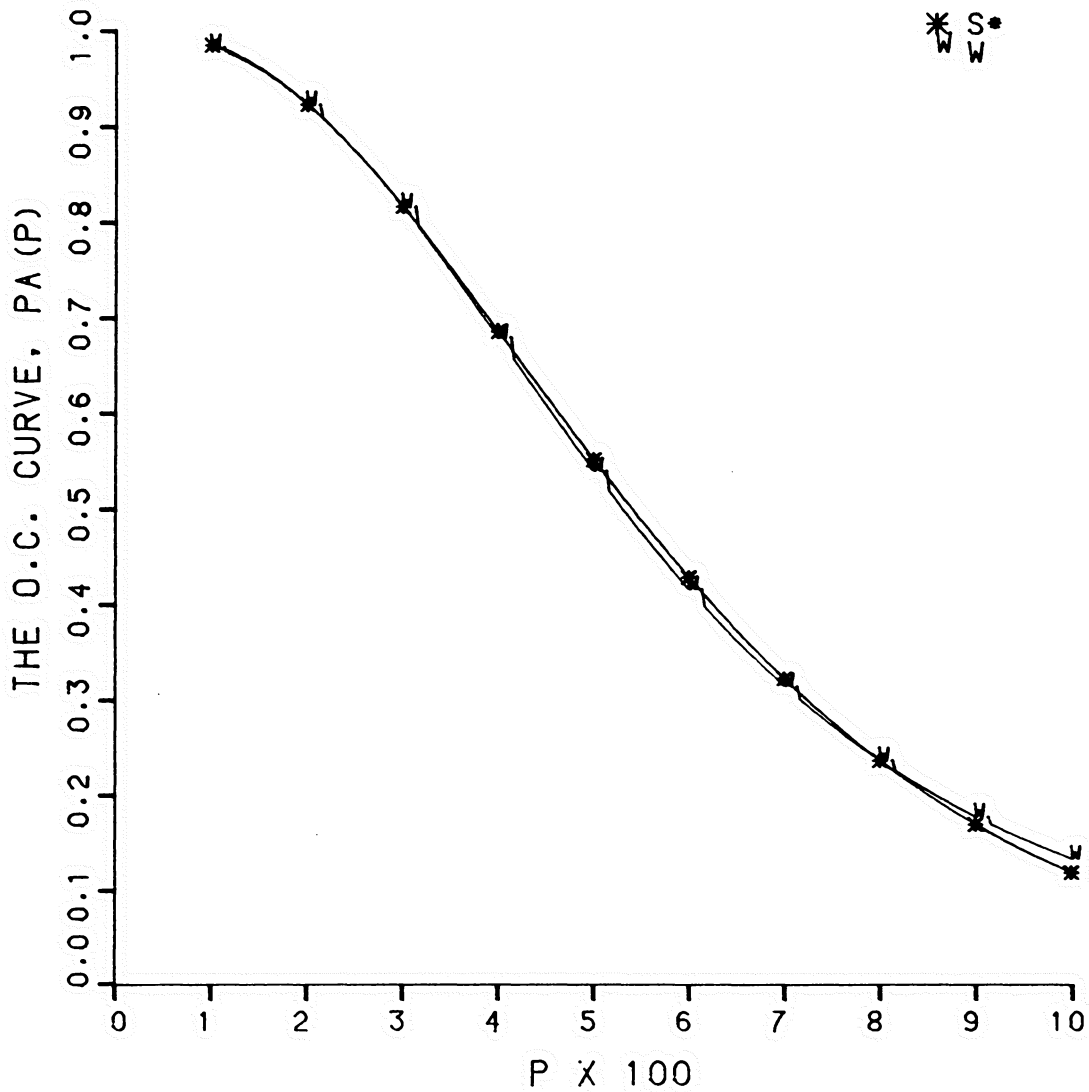


Figure 6. A comparison of the O.C. curves. $\alpha \approx .075$, $\beta \approx .238$, $p_1 = .02$, $p_2 = .08$. Note S^* , S_6 , and S_8 have exactly the same O.C. curve, so only S^* and W are drawn.

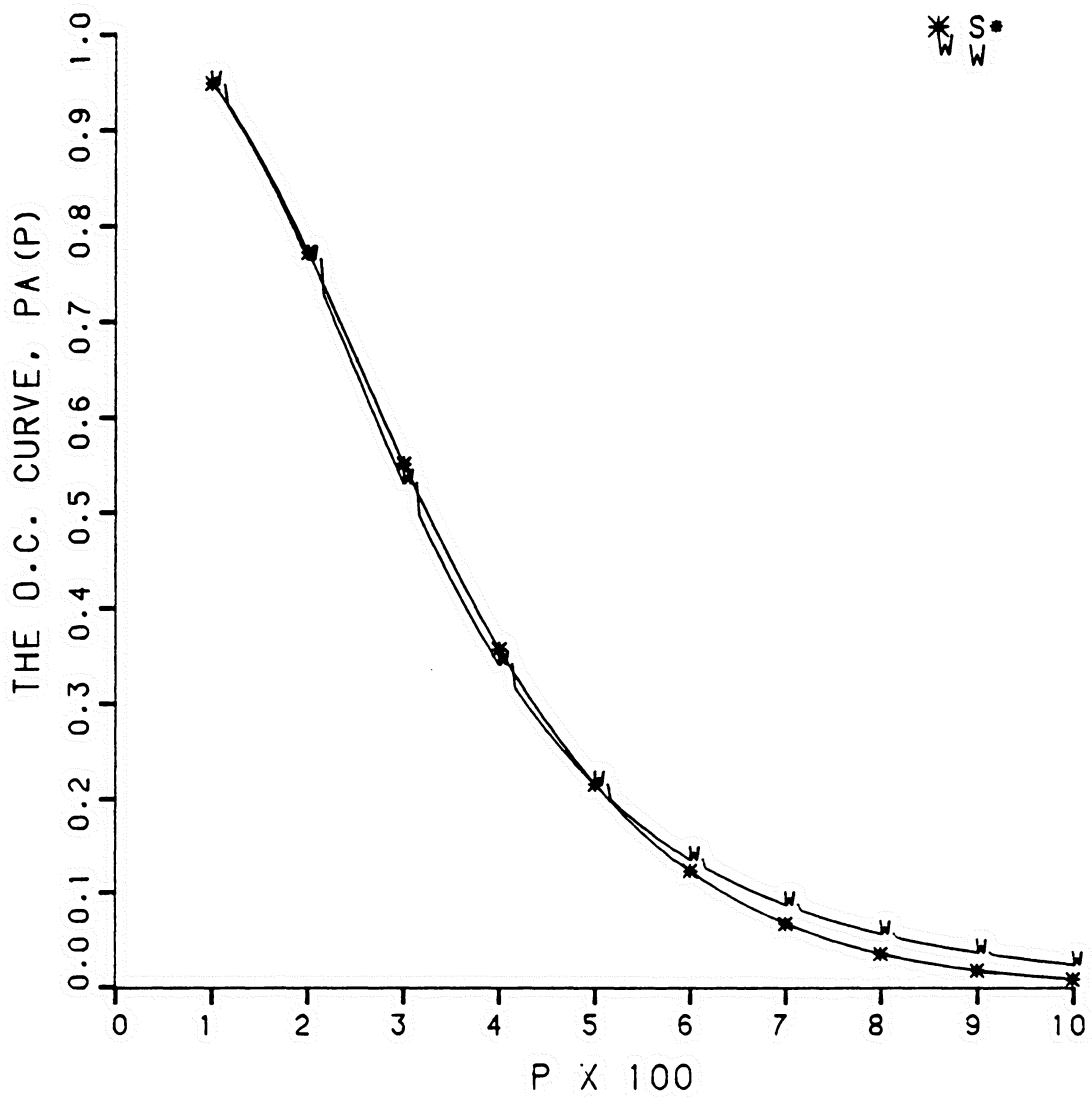


Figure 7. A comparison of the O.C. curves. $\alpha \approx .051$, $\beta \approx .216$, $p_1 = .01$, $p_2 = .05$. Note S^* , S_6 , and S_8 have exactly the same O.C. curve, so only S^* and W are drawn.

To compare S_8 and S_6 the algebraic condition

$$0 \geq \text{ASN}(S_6) + R \text{Var}(\hat{p}(S_6)) - \text{ASN}(S_8) - R \text{Var}(\hat{p}(S_8))$$

is examined to obtain values of R for which S_6 or S_8 is better. In doing this comparison n_1 in S_6 was taken to be m_1+m_2 , so that the boundaries would line up as in Figure 8.

To compare ASN and $\text{Var}(\hat{p})$ for plans S_6 and S_8 refer to Figure 8. The probability of crossing the line segment between $(n_1, 0)$ and (m_1, m_2) is equal to the probability of crossing the line segment between (m_1, m_2) and (n_2-m_2, m_2) plus the probability of crossing the line segment between $(n_2, 0)$ and (n_2-m_2, m_2) . Thus

$$\begin{aligned} & \sum_{k=m_1}^{m_1+m_2} b(k, m_1+m_2, p) \\ &= \sum_{x=0}^{n_2-m_1-m_2-1} b^{-1}(m_2, m_1+m_2+x+1, q) \\ &+ \sum_{k=m_1}^{m_1+m_2} b(k+n_2-m_1-m_2, n_2, p) \end{aligned}$$

It may be seen from Figure 8 that $\text{ASN}(S_6) \leq \text{ASN}(S_8)$ by observing that, for every sample path, the number of observations required to reach the boundary for S_8 is always greater than the number of observations required to reach the boundary of S_6 .

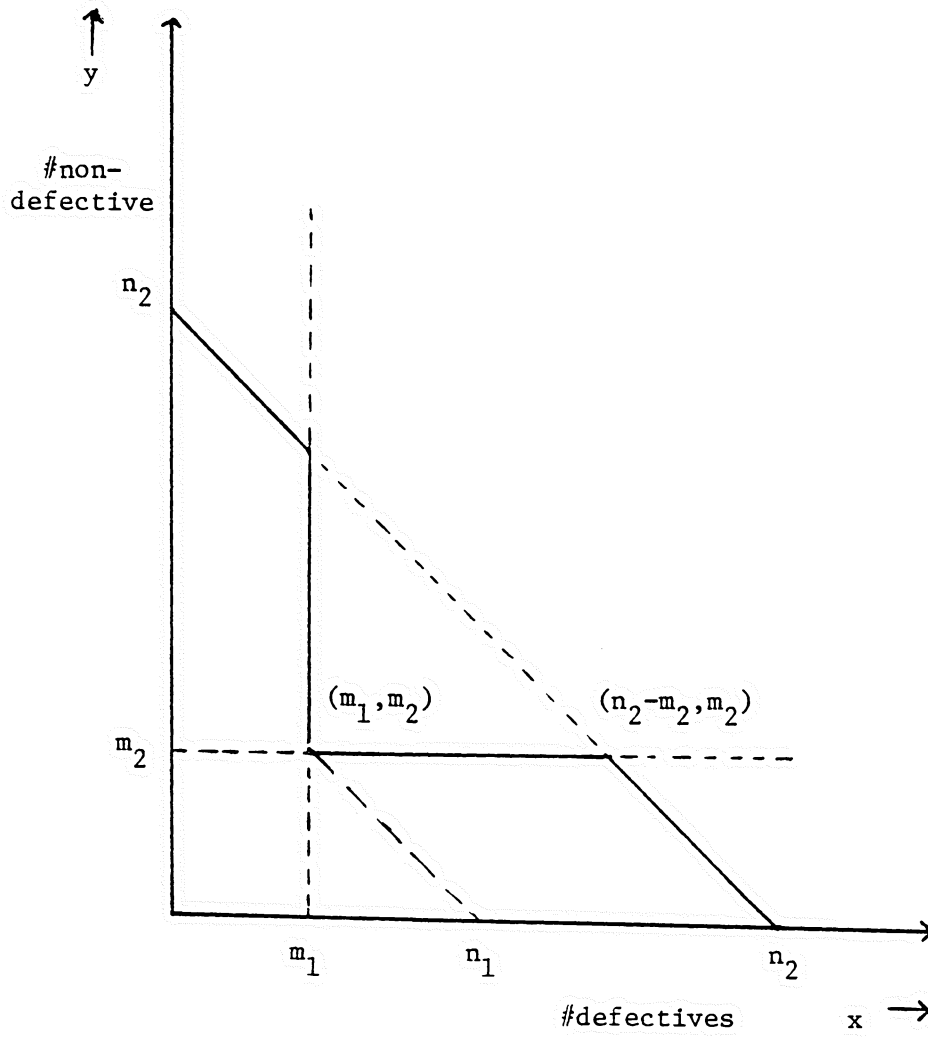


Figure 8. Comparison of S_6 and S_8 .

Next compare $\text{Var}(\hat{p}(S_6))$ and $\text{Var}(\hat{p}(S_8))$. Now

$$\begin{aligned} \text{Var}(\hat{p}(S_6)) - \text{Var}(\hat{p}(S_8)) &= \sum_{k=m_1}^{m_1+m_2} \left(\frac{k}{m_1+m_2} \right)^2 b(k, m_1+m_2, p) \\ &\quad - \sum_{x=0}^{n_2-m_1-m_2-1} \left(\frac{x+m_1+1}{x+m_1+m_2} \right)^2 b^{-1}(m_2, m_1+m_2+x+1, q) \\ &\quad - \sum_{k=m_1}^{m_1+m_2} \left(\frac{k-m_1-m_2+n_2}{n_2} \right)^2 b(k+n_2-m_1-m_2, n_2, p) \end{aligned}$$

If $n_2 = m_1+m_2$, $\text{Var}(\hat{p}(S_6)) = \text{Var}(\hat{p}(S_8))$. Otherwise suppose $n_2 > m_1+m_2$. Then, $\text{Var}(\hat{p}(S_6)) > \text{Var}(\hat{p}(S_8))$.

Thus S_6 is better than S_8 if

$$0 > \text{ASN}(S_6) - \text{ASN}(S_8) + R(\text{Var}(\hat{p}(S_6)) - \text{Var}(\hat{p}(S_8)))$$

or if

$$R < \frac{\text{ASN}(S_8) - \text{ASN}(S_6)}{\text{Var}(\hat{p}(S_6)) - \text{Var}(\hat{p}(S_8))} .$$

To compare plans S^* with S_6 and W , a computer routine is generated (See Appendix C) to compute the $Z(S^*) = \text{ASN}(S^*) + R \text{Var}(\hat{p}(S^*))$ values for $R = 5000, 50000, \text{ and } 500000$. In doing this the one free parameter K_2 is determined so as to minimize $Z(S^*)$.

To compare plans S_6 with S^* and W , the values of $Z(S_6) = \text{ASN}(S_6) + R \text{Var}(\hat{p}(S_6))$ are taken from Kim and Nachlas [11]. Note that n_1 and n_2 were the free parameters determined in S_6 by a minimization of $Z(S_6)$.

ASN and $\text{Var}(\hat{p}(W))$ are computed using the approximate equations described in Chapter III. Tables 8 and 9 in Appendix D contain the computed values for the underlying probability $p=.01$ to $.10$ (in steps of $.01$), along with the values of the parameter h appearing in the approximate equations. Thus $Z(W) = \text{ASN}(W) + R \text{Var}(\hat{p}(W))$ may be approximated. Also it may be noted that the computations to give an approximation to $\text{Var}(\hat{p}(W))$ give an approximation valid for 7 decimal places.

The comparisons between S^* , S_6 , and W may be found in Figures 9 to 14 and also numerically in Appendix B, Tables 2 to 7. These comparisons are based upon $Z = \text{ASN} + R \text{Var}$ for $R = 5000, 50000,$ and 500000 . Recall that the plan with the lower value of Z would be the better plan.

The question may arise as to which plan would be better by strictly comparing ASN or $\text{Var}(\hat{p})$. In the case of ASN two additional figures are drawn, namely Figures 15 and 16 along with in Appendix D the corresponding Tables 8 and 9 which show the relationship between $\text{ASN}(W)$ and $\text{ASN}(S^*)$. $\text{ASN}(S_6)$ was not drawn, since

$$\text{ASN}(S^*) \leq \text{ASN}(S_6) .$$

This can be easily seen by noting that every path reaching the boundary of S_6 would have to pass through the boundary of S^* previously, so that every path reaching S^* would be shorter than when it reached S_6 . Note also that $\text{Var}(S^*) \geq \text{Var}(S_6)$.

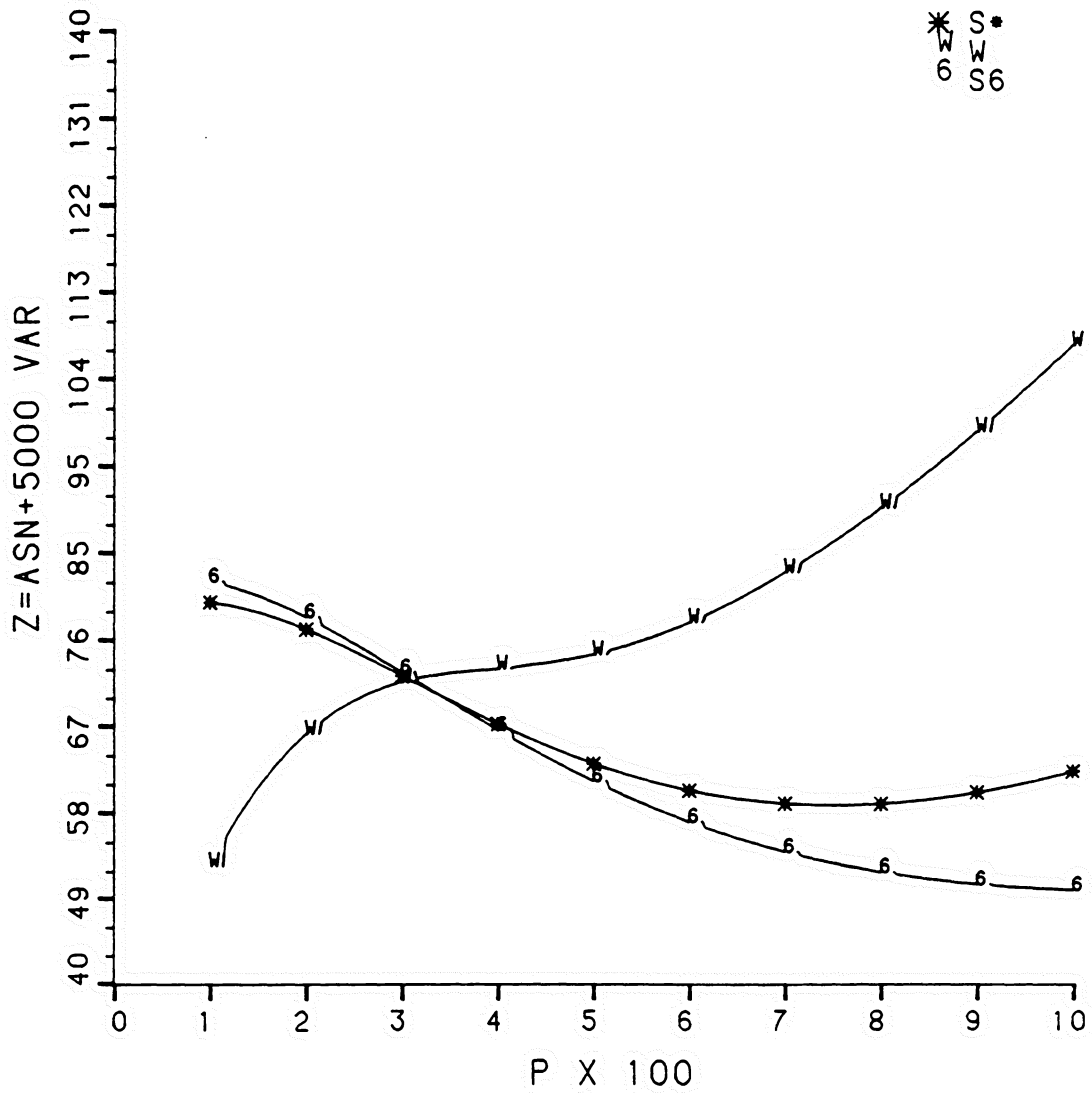


Figure 9. Comparison of S^* , W , and S_6 . $\alpha \approx .051$, $\beta \approx .216$, $p_1 = .01$, and $p_2 = .05$.

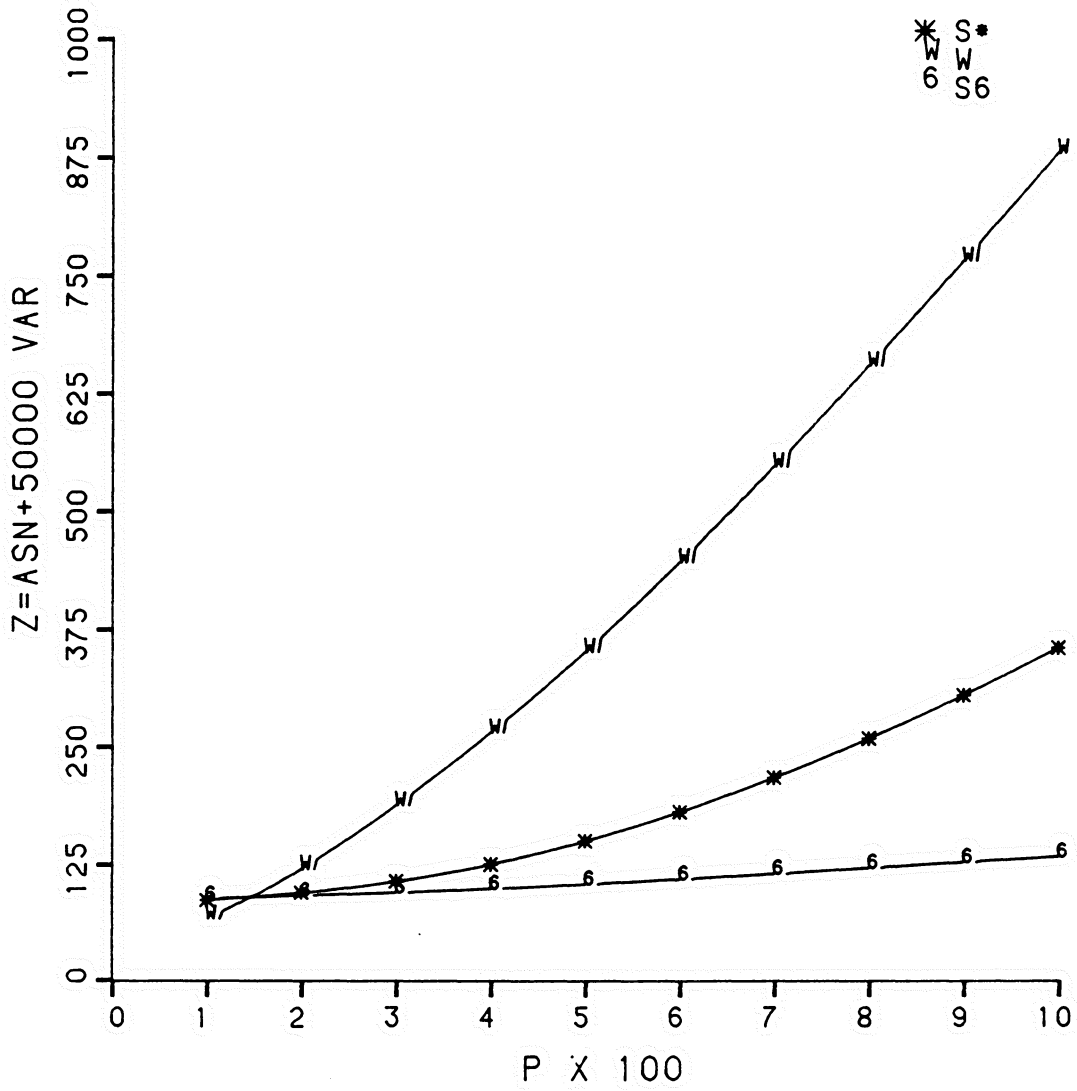


Figure 10. Comparison of S^* , W , and S_6 . $\alpha \approx .051$, $\beta \approx .216$, $p_1 = .01$, and $p_2 = .05$.

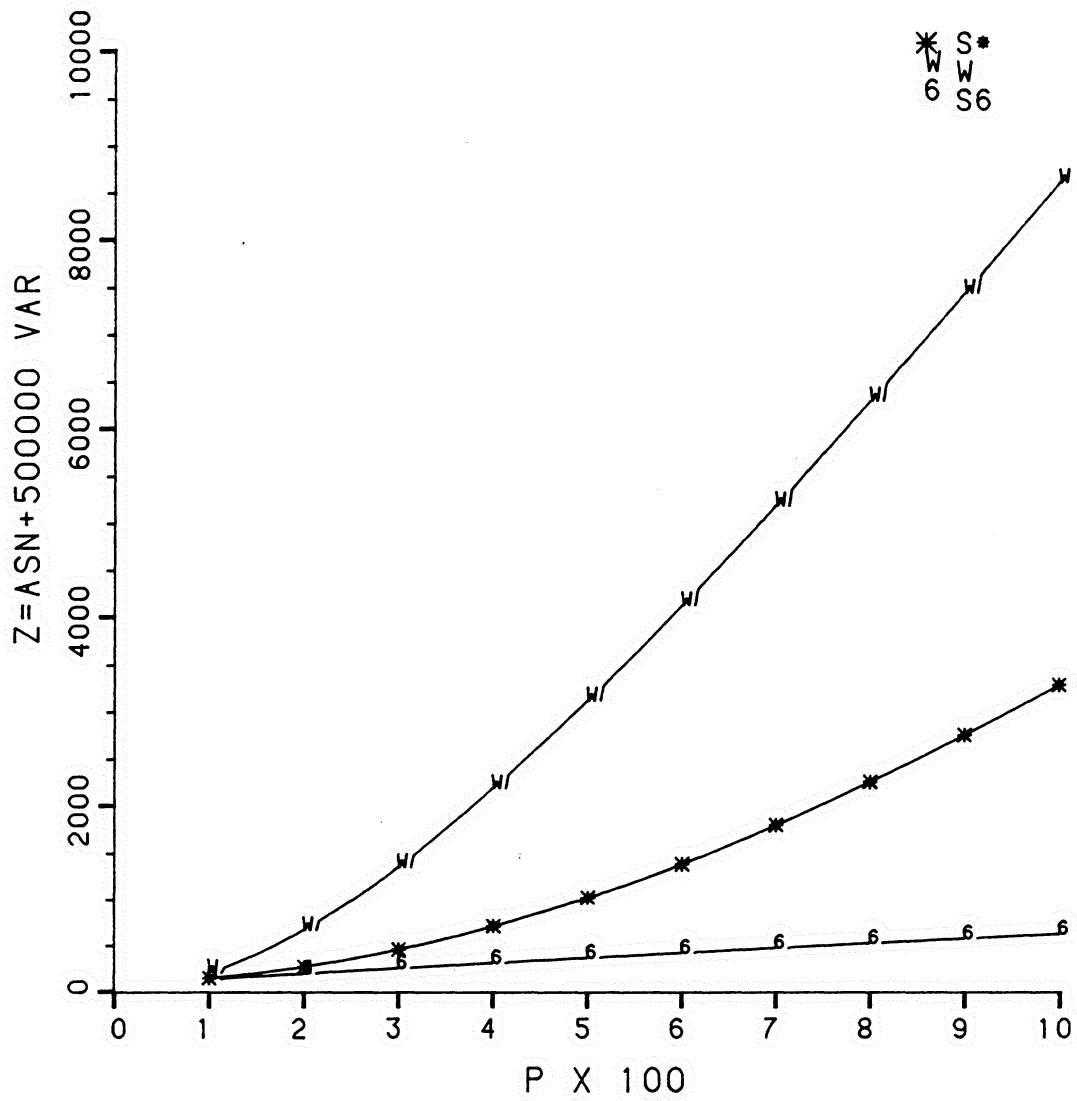


Figure 11. Comparison of S^* , W , and S_6 . $\alpha \approx .051$, $\beta \approx .216$, $p_1 = .01$, and $p_2 = .05$.

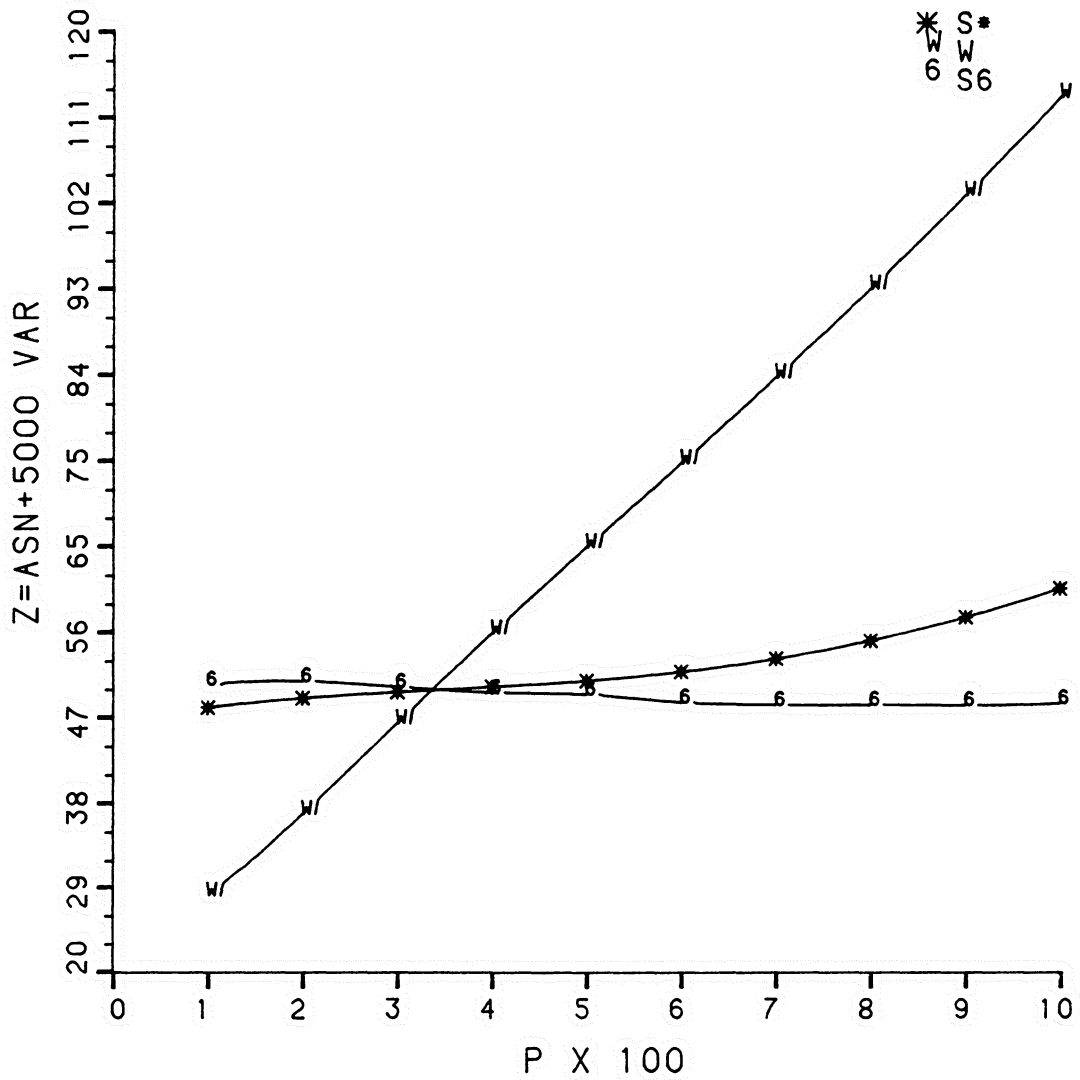


Figure 12. Comparison of S^* , W , and S_6 . $\alpha \approx .075$, $\beta \approx .238$, $p_1 = .02$, and $p_2 = .08$.

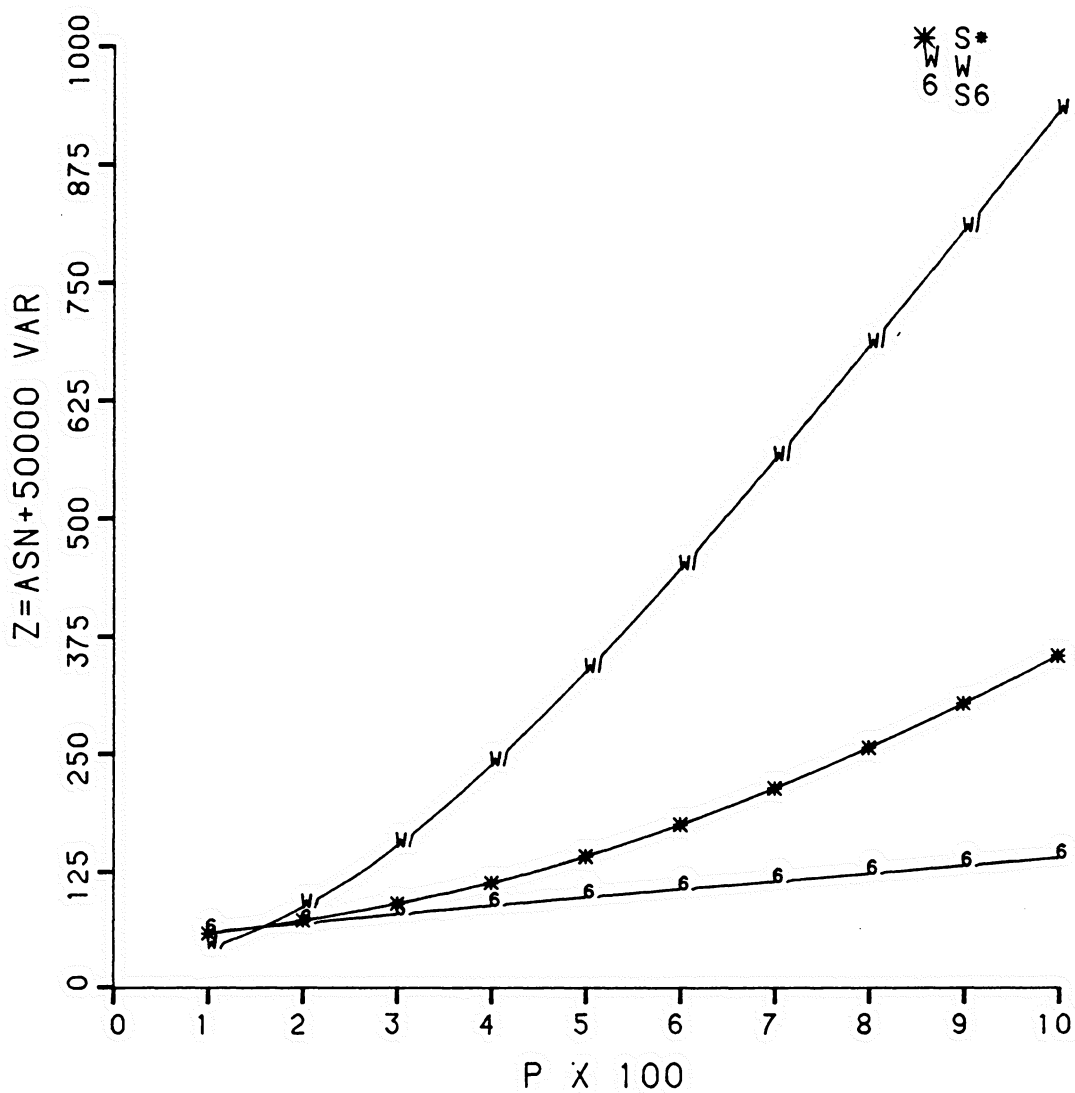


Figure 13. Comparison of S^* , W , and S_6 . $\alpha \approx .075$, $\beta \approx .238$, $p_1 = .02$, and $p_2 = .08$.

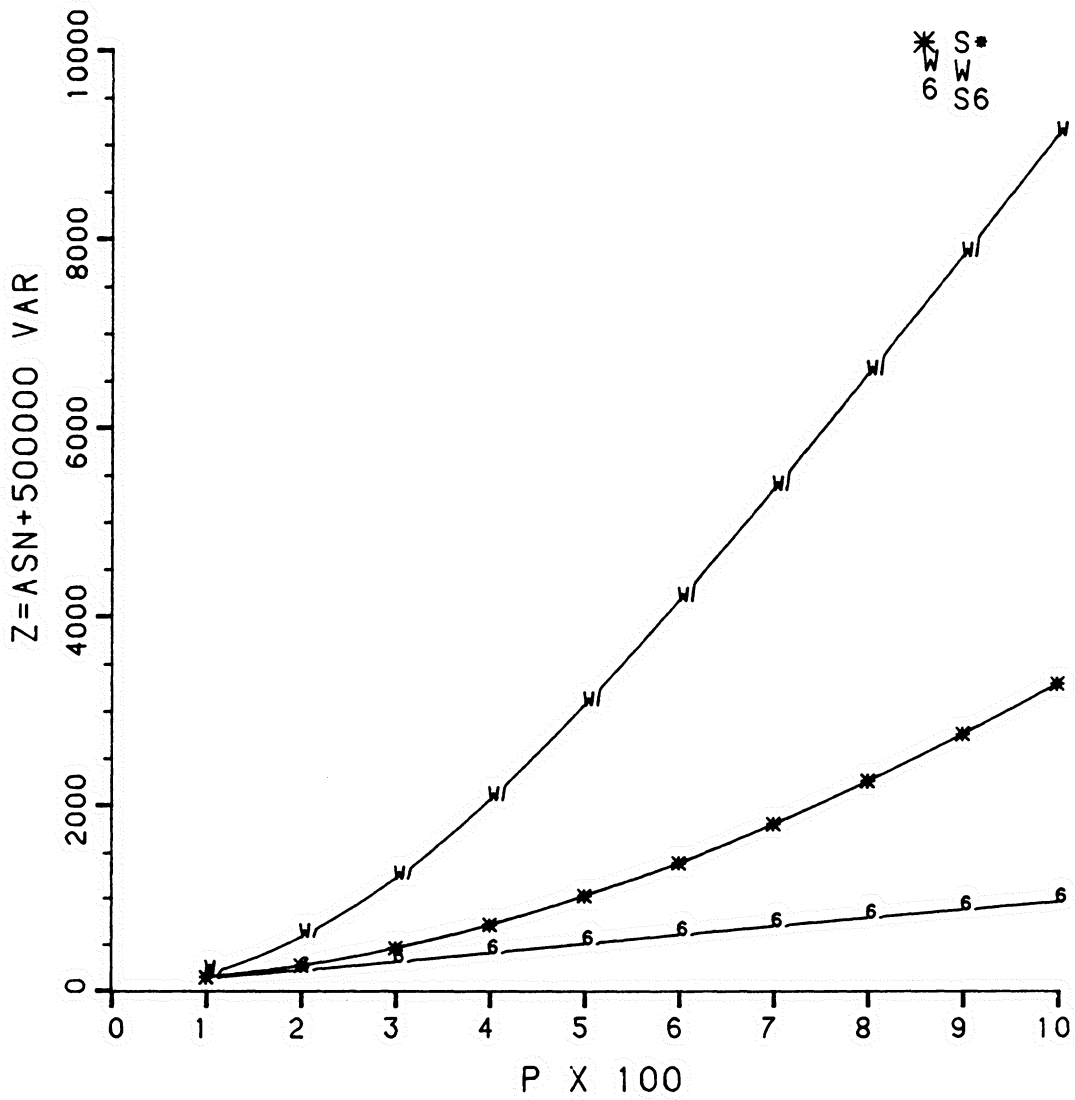


Figure 14. Comparison of S^* , W , and S_6 . $\alpha \approx .075$, $\beta \approx .238$, $p_1 = .02$, and $p_2 = .08$.

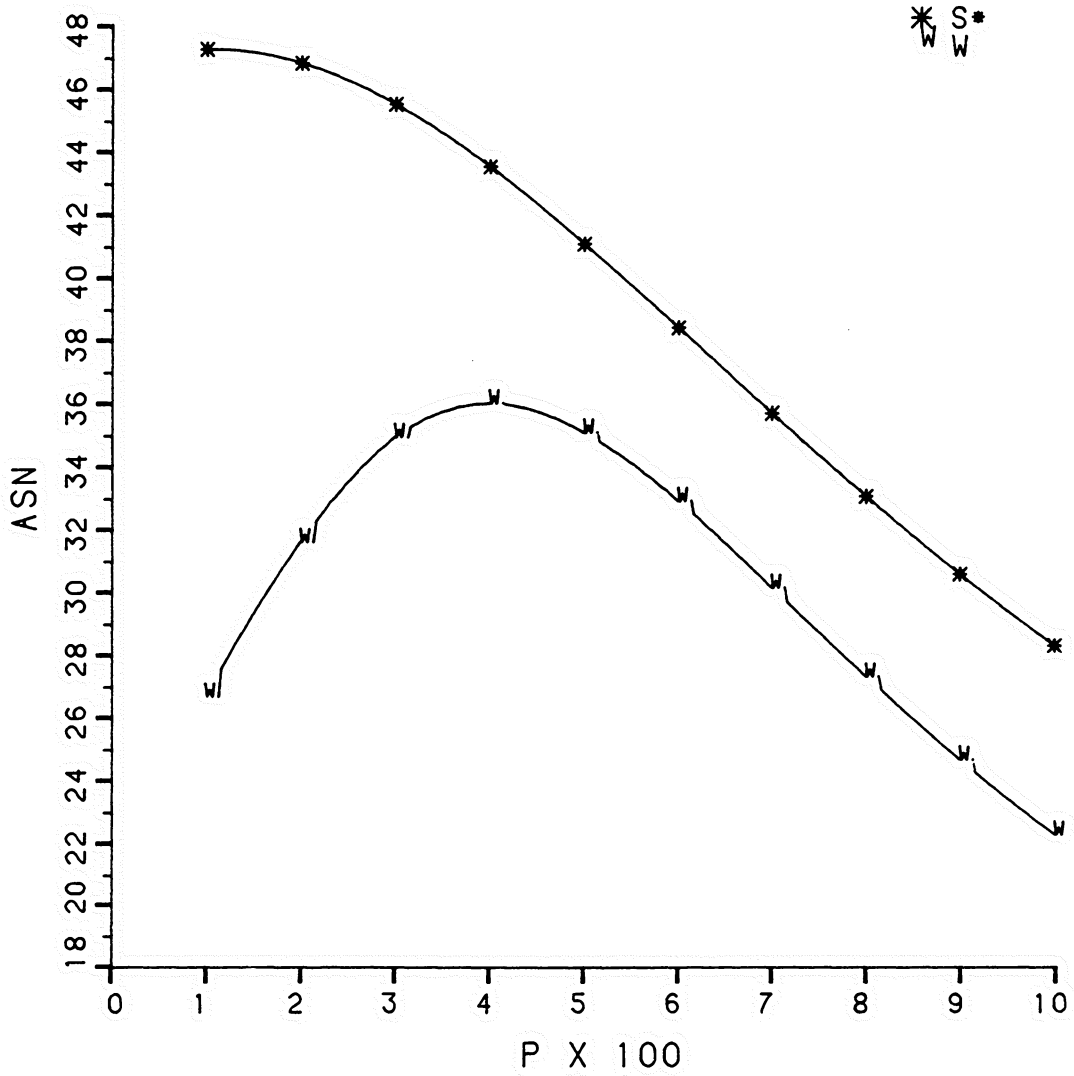


Figure 15. Comparison of ASN for S^* and W . $\alpha \approx .075$, $\beta \approx .238$, $p_1 = .02$, $p_2 = .08$, $n_0 = 50$, $k_1 = 3$, and $k_2 = 47$.

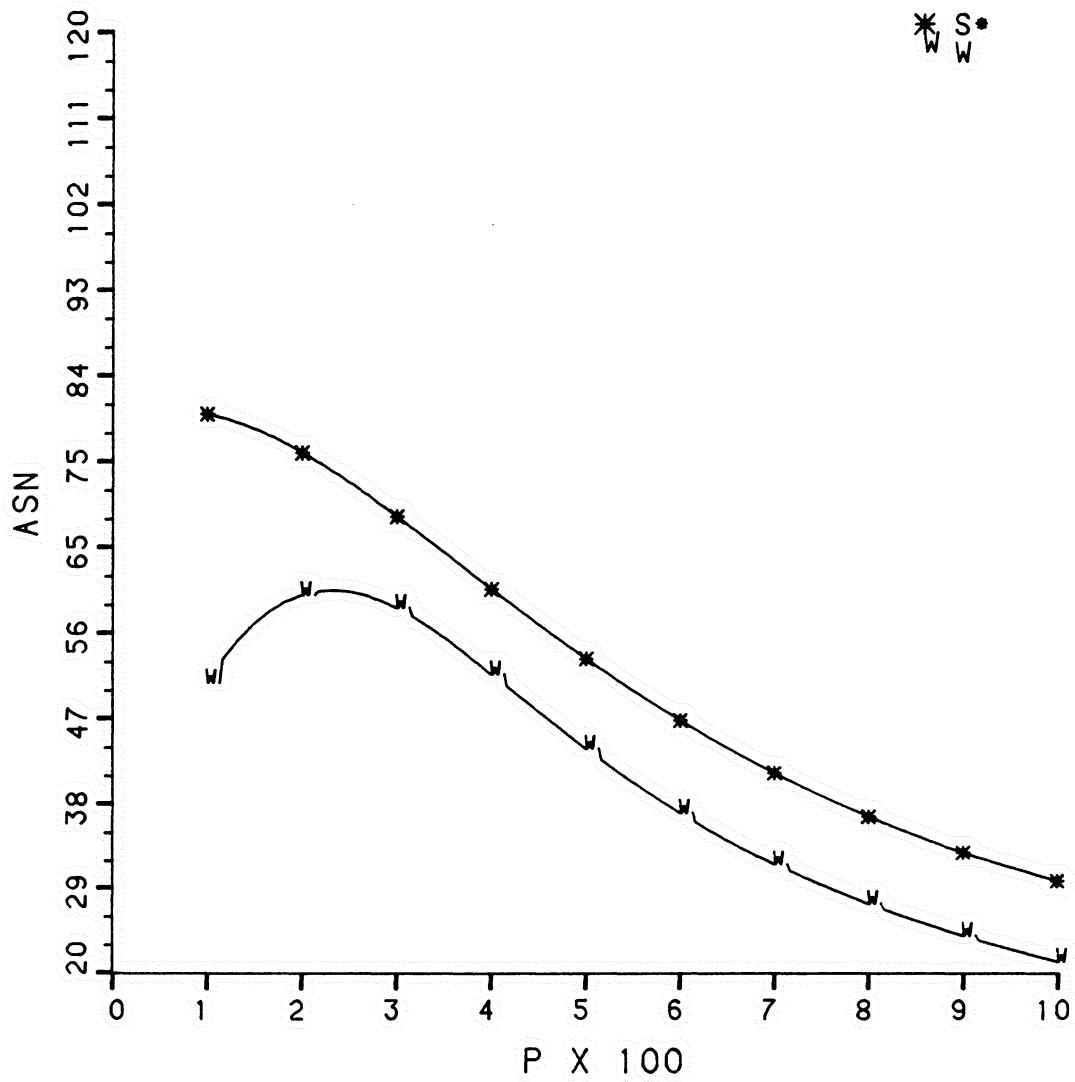


Figure 16. Comparison of ASN for S^* and W . $\alpha \approx .051$, $\beta \approx .216$, $p_1 = .01$, $p_2 = .05$, $n_0 = 83$, $k_1 = 3$, and $k_2 = 80$.

CHAPTER V. CONCLUSIONS

Stated previously is the fact that S_6 is better than S_8 under the condition

$$R < \frac{ASN(S_8) - ASN(S_6)}{\text{Var}(\hat{p}(S_6)) - \text{Var}(\hat{p}(S_8))}$$

Concerning the comparison of S^* , W , and S_6 under the criterion of the minimization of $Z = ASN + R \text{Var}(\hat{p})$ for $R = 5000$, 50000 , and 500000 .

For $R=5000$ it can be seen from Figures 9 and 11 or from the tables in Appendix B that Wald's SPR plans W are better for underlying probabilities p of less than .03 or .04, and S_6 is better for p greater than .04. It should be noted here that if the scaling factor R were even smaller than 5000 even more emphasis would be placed upon ASN , and so by the previous path arguments S^* would end up better than S_6 for p greater than .04.

For $R=50000$, Figures 10 and 13 and the tables in Appendix B show that for all intents and purposes S_6 may be considered to be better than S^* or W with the exception of p less than .02.

In the case the variance of \hat{p} is emphasized, i.e. $R=500000$, Figures 11 and 14 and the tables in Appendix B shows that S_6 is always better than plans S^* and Wald SPR plans W for all cases of p , the underlying proportion defective, considered.

The implication with respect to Wald's plans W is that since S_6 proves better when using the criterion $Z = ASN + R \text{Var}(\hat{p})$ in a number

of cases, especially $R=500000$, that plan S_6 and for the same reasons S^* in the $R=500000$ cases are worthwhile plans and should be considered candidates when selecting a plan.

Also a question may arise as why sampling should continue if the outcome as to whether to accept or reject is known at some point in the process. At one time the reason given was that an unbiased estimate for p cannot be found if a complete inspection of the sample is not taken, but Girshick, Mosteller, and Savage [7] state that this reason is no longer valid, as they give a formula to compute the unbiased estimator \hat{p} , for certain shaped regions of which S^* is one. However, for other shaped regions the reason remains valid, and sampling must continue to the boundary.

If strictly ASN is considered, Wald's SPR plans W turn out better, but on the other hand, if only $\text{Var}(\hat{p})$ is being considered (or at least weighted heavier), S_6 again turns out to be a worthwhile sampling plan.

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APPENDIX A

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C  COMPUTATIONS FOR WALD'S PLAN
    DIMENSION F(47,1100),P(10),VAR(10),Q(10)
    READ(5,400)ALPA,BETA,P1,P2,KMAX
    A=(1.-BETA)/ALPA
    B=BETA/(1.-ALPA)
    D=ALOG(P2/P1)+ALOG((1.-P1)/(1.-P2))
    S=ALOG((1.-P1)/(1.-P2))/D
    B1=ALOG(A)/D
    B2=-ALOG(B)/D
C  THE ACCEPTANCE SLOPE IS
    ASL=(1.-S)/S
C  THE Y INTERCEPT OF THE TOP LINE IS
    B2S=B2/S
C  THE Y INTERCEPT OF THE BOTTOM LINE IS MINUS
    B1S=B1/S
C  BECAUSE OF THE RANGE ON SUBSCRIPT VALUES IF THE ACTUAL VALUE
C  IS AT THE POINT (K,L) THIS WILL BE STORED IN LOCATION
C  F(K+1,L+1).
    F(1,1)=1.
    IB2S=B2S
    III=IB2S+1
    DO 10 L=1,III
        J=L+1
10  F(1,J)=1.
    IST=B1/(1.-S)
    LLL=IST+1
    DO 15 K=1,LLL
        J=K+1
15  F(J,1)=1.
C  AT THIS POINT THE X AND Y AXES HAVE THE NUMBER OF PATHS
C  REACHING THOSE POINTS ASSIGNED.
    BR=B1/(1.-S)
    DO 100 K=1,KMAX
        IF(K.GT.BR) GO TO 42
        IF(K.EQ.BR) GO TO 72
        N=K+1
        TIL=ASL*FLOAT(K-1)+B2S
        IL=IL+1
30  MMM=IL-1
    DO 35 L=1,MMM
        M=L+1
35  F(N,M)=F(N-1,M)+F(N,M-1)
    TIT=ASL*FLOAT(K)+B2S
    IT=TIT
    IF(IT.EQ.TIT) GO TO 37
    IT=IT+1

```

```
37 DO 40 L=IL, IT
    M=L+1
40 F(N,M)=F(N,M-1)
    GO TO 100
42 N=K+1
    IB=ASL*FLOAT(K-1)-B1S
    IN=ASL*FLOAT(K)-B1S
    LOO=IB+1
    IF(IB.LE.0) LOO=1
    LOW=IN+1
    DO 55 L=LOO, LOW
    M=L+1
55 F(N,M)=F(N-1,M)
    TIM=ASL*FLOAT(K-1)+B2S
    IM=TIM
    IF(IM.EQ.TIM) GO TO 57
    IM=IM+1
57 KOP=IN+2
    KEEP=IM-1
    DO 60 L=KOP, KEEP
    M=L+1
60 F(N,M)=F(N-1,M)+F(N,M-1)
    TIT=ASL*FLOAT(K)+B2S
    IT=TIT
    IF(IT.EQ.TIT) GO TO 62
    IT=IT+1
62 DO 70 L=IM, IT
    M=L+1
70 F(N,M)=F(N,M-1)
    GO TO 100
72 N=K+1
    L=1
    M=L+1
    F(N,M)=F(N-1,M)
    TIM=ASL*FLOAT(K-1)+B2S
    IM=TIM
    IF(IM.EQ.TIM) GO TO 74
    IM=IM+1
74 KEEP=IM-1
    DO 76 I=2, KEEP
    M=L+1
76 F(N,M)=F(N-1,M)+F(N,M-1)
    TIT=ASL*FLOAT(K)+B2S
    IT=TIT
    IF(IT.EQ.TIT) GO TO 78
    IT=IT+1
78 DO 80 L=IM, IT
    M=L+1
80 F(N,M)=F(N,M-1)
100 CONTINUE
```

```

C AT THIS POINT ALL THE WAYS OF GETTING TO EACH POINT
C SHOULD HAVE BEEN GENERATED AND STORED. NEXT AN
C ESTIMATE OF THE VARIANCE IS COMPUTED.
  DO 110 I=1,10
    P(I)=.01*I
    VAR(I)=-P(I)**2
110 Q(I)=1.-P(I)
    KOO=KMAX+1
    DO 190 JJ=1,KOO
      K=JJ-1
      TIN=ASL*FLOAT(K)+B2S
      IN=TIN
      IF(IN.EQ.TIN) GO TO 120
      IN=IN+1
120 XK=K
      XIN=IN-1
      DO 160 I=1,10
160 VAR(I)=VAR(I)+(XK/(XK+XIN))**2*F(K+1,IN+1)*P(I)**K*Q(I)**IN
190 CONTINUE
      LBOT=ASL*FLOAT(KMAX-1)-B1S
      LIP=LBOT+2
      DO 230 L=1,LIP
        XL=L-1
        TIN=XL/ASL+B1/(1.-S)
        IN=TIN
        IF(IN.EQ.TIN) GO TO 195
        IN=IN+1
195 XIN=IN-1
      DO 200 I=1,10
200 VAR(I)=VAR(I)+(XIN/(XL+XIN))**2*F(IN+1,L)*P(I)**IN*Q(I)**(L-1)
230 CONTINUE
      WRITE(6,450)(VAR(I),I=1,10)
400 FORMAT(4F4.2,1I2)
450 FORMAT(' THE VALUES OF VARIANCE IN ORDER ARE',/10(F15.8/))
      END

```

APPENDIX B

Table 1. A comparison of O.C. curves. $\alpha \approx .075$, $\beta \approx .238$, $p_1 = .02$, $p_2 = .08$.

p	$\alpha \approx .075$, $\beta \approx .238$, $p_1 = .02$, $p_2 = .08$		$\alpha \approx .051$, $\beta \approx .216$, $p_1 = .01$, $p_2 = .05$	
	S^* , S_6 , S_8	W	S^* , S_6 , S_8	W
.01	.9869	.9845	.9505	.9490
.02	.9252	.9250	.7739	.7668
.03	.8184	.8181	.5523	.5325
.04	.6878	.6817	.3582	.3425
.05	.5537	.5419	.2163	.2160
.06	.4301	.4175	.1237	.1368
.07	.3241	.3164	.0677	.0876
.08	.2379	.2380	.0357	.0567
.09	.1707	.1788	.0182	.0370
.10	.1200	.1345	.0090	.0243

Table 2. Comparison of S^* , W , and S_6 . $\alpha \approx .051$, $\beta \approx .216$, $p_1 = .01$,
 $p_2 = .05$, $R = 5000$, $n_0 = 83$, and $K_1 = m_1 = 3$.

p	$Z(S^*)$	$Z(W)$	$Z(S_6)$	K_2	n_1	n_2
.01	80.38	52.75	82.55	80	25	83
.02	77.55	66.58	78.85	80	25	83
.03	72.70	72.11	73.09	80	25	83
.04	67.61	73.49	66.99	80	25	83
.05	63.41	74.99	61.59	80	25	83
.06	60.58	78.31	57.31	80	25	83
.07	59.22	83.51	54.18	80	26	83
.08	59.21	90.31	52.07	80	26	83
.09	60.39	98.31	50.79	80	26	83
.10	62.57	107.29	50.16	80	26	83

Table 3. Comparison of S^* , W , and S_6 . $\alpha \approx .051$, $\beta \approx .216$, $p_1 = .01$,
 $p_2 = .05$, $R = 50000$, $n_0 = 83$, and $K_1 = m_1 = 3$.

p	$Z(S^*)$	$Z(W)$	$Z(S_6)$	K_2	n_1	n_2
.01	87.04	67.15	88.48	80	54	83
.02	95.56	121.48	92.61	80	55	83
.03	108.10	189.11	96.28	80	56	83
.04	126.34	266.54	100.33	80	57	83
.05	150.86	352.64	105.10	80	58	83
.06	181.59	447.31	110.53	80	59	83
.07	218.11	548.81	116.40	80	61	83
.08	259.84	655.96	122.46	80	63	83
.09	306.23	767.01	128.50	80	65	83
.10	356.73	881.74	134.36	80	68	83

Table 4. Comparison of S^* , W , and S_6 . $\alpha \approx .051$, $\beta \approx .216$, $p_1 = .01$,
 $p_2 = .05$, $R = 500000$, $n_0 = 83$, and $K_1 = m_1 = 3$.

p	$Z(S^*)$	$Z(W)$	$Z(S_6)$	K_2	n_1	n_2
.01	153.67	211.15	142.72	80	82	83
.02	275.64	670.48	201.58	81	82	83
.03	461.74	1359.11	259.30	87	82	90
.04	713.06	2197.04	315.77	90	82	90
.05	1024.91	3129.14	371.24	91	82	90
.06	1391.38	4137.31	425.53	94	82	90
.07	1806.79	5201.81	478.68	94	82	90
.08	2266.01	6312.46	530.58	97	82	90
.09	2764.53	7454.01	581.08	99	82	90
.10	3298.31	8625.24	630.32	108	82	90

Table 5. Comparison of S^* , W , and S_6 . $\alpha \approx .075$, $\beta \approx .238$, $p_1 = .02$,
 $p_2 = .08$, $R = 5000$, $n_0 = 50$, and $K_1 = m_1 = 3$.

p	$Z(S^*)$	$Z(W)$	$Z(S_6)$	K_2	n_1	n_2
.01	48.42	28.31	50.87	47	25	50
.02	49.43	37.14	51.24	47	25	50
.03	50.08	46.78	50.62	47	25	50
.04	50.61	56.35	50.00	47	25	50
.05	51.25	65.47	49.80	47	25	50
.06	52.21	74.40	48.94	47	25	50
.07	53.61	83.48	48.68	47	26	50
.08	55.55	92.90	48.68	47	26	50
.09	58.04	102.79	48.63	47	26	50
.10	61.08	113.21	48.81	47	26	50

Table 6. Comparison of S^* , W , and S_6 . $\alpha \approx .075$, $\beta \approx .238$, $p_1 = .02$,
 $p_2 = .08$, $R = 50000$, $n_0 = 50$, and $K_1 = m_1 = 3$.

p	$Z(S^*)$	$Z(W)$	$Z(S_6)$	K_2	n_1	n_2
.01	58.49	42.71	59.90	47	49	50
.02	72.52	86.64	69.62	47	49	50
.03	90.76	152.98	79.17	47	49	50
.04	113.99	239.05	88.54	47	49	50
.05	142.44	338.62	97.73	47	49	50
.06	176.04	447.45	106.74	47	49	50
.07	214.54	563.18	115.56	47	49	50
.08	257.60	682.85	124.20	47	49	50
.09	304.84	805.69	132.63	47	49	50
.10	355.89	931.31	140.87	47	49	50

Table 7. Comparison of S^* , W , and S_6 . $\alpha \approx .075$, $\beta \approx .238$, $p_1 = .02$,
 $p_2 = .08$, $R = 500000$, $n_0 = 50$, and $K_1 = m_1 = 3$.

p	$Z(S^*)$	$Z(W)$	$Z(S_6)$	K_2	n_1	n_2
.01	151.04	186.71	143.61	66	49	63
.02	275.64	581.64	225.76	81	49	72
.03	461.74	1214.98	314.31	87	49	90
.04	713.06	2066.05	410.18	90	49	90
.05	1024.91	3070.12	507.92	91	49	90
.06	1391.38	4177.95	604.56	94	49	90
.07	1806.79	5360.18	698.68	94	49	90
.08	2266.01	6582.35	789.78	97	49	90
.09	2764.53	7834.69	877.66	99	49	90
.10	3298.31	9112.31	962.54	108	49	108

APPENDIX C

```

C MAIN PROGRAM FOR MINIMIZING Z=ASN+R VAR FOR PLAN S*.
  DO 45 LL=1,30
    READ(5,100) IR1,NO,P,R
    IR2M=NO-IR1
    CALL AFIV(IR1,IR2M,P,ASN)
    ASNO=ASN
    CALL VFIV(IR1,IR2M,P,VAR)
    VARO=VAR
  83 FORMAT(' THIS IS THE VALUE OF ASN ',1F15.8)
  84 FORMAT(' THIS IS THE VALUE OF VAR ',1F15.8)
    ZOLD=ASN+R*VAR
    IR2=IR2M
    KK=IR2M+1
    DO 5 J=KK , 250
      CALL AFIV(IR1,J,P,ASN)
      CALL VFIV(IR1,J,P,VAR)
      ZNEW=ASN+R*VAR
      IF(ZOLD.LT.ZNEW) GO TO 10
      IR2=J
      ASNO=ASN
      VARO=VAR
    5 ZOLD=ZNEW
  10 WRITE(6,200) ZOLD,IR2,P,R
    WRITE(6,83) ASNO
    WRITE(6,84) VARO
  45 CONTINUE
  100 FORMAT(2I3,1F3.2,1F7.0)
  200 FORMAT(' Z=',1F15.2,' R2=',1I5,' P=',1F3.2,' R=',1F7.0)
  END
  SUBROUTINE AFIV(IR1,IR2,P,ASN)
C COMPUTES ASN FOR PLAN S*, BASED UPON THE PARAMETERS R1,R2,AND P.
  ASN=0.
  Q=1.-P
  ITOP=IR1+IR2-1
  DO 10 J=IR1,ITOP
    CALL COMB(J-1,IR1-1,NN)
    XJ=J
    X=NN
  10 ASN=ASN+XJ*X*P**(IR1)*Q**(J-IR1)
    DO 20 J=IR2,ITOP
      CALL COMB(J-1,IR2-1,NN)
      XJ=J
      X=NN
  20 ASN=ASN+XJ*X*P**(J-IR2)*Q**IR2
    RETURN
  END
  SUBROUTINE VFIV(IR1,IR2,P,VAR)

```

C COMPUTES VARIANCE, VAR, OF THE UNBIASED ESTIMATOR FOR PLAN S* GIVEN
 C PARAMETERS R1,R2, AND P.

```

    Q=1.-P
    TRM1=0.
    TRM2=0.
    III=IR1
    KKK=IR1+IR2-1
    DO 5 J=III , KKK
    R1=IR1
    R2=IR2
    XJ=J
    CALL COMB(J-2,IR1-2,NCOM)
    XCOM=NCOM
  5  TRM1=TRM1+(R1-1.)/(XJ-1.)*XCOM*P**IR1*Q**(J-IR1)
    LLL=IR2
    NNN=IR1+IR2-1
    DO 10 J=LLL , NNN
    R1=IR1
    R2=IR2
    XJ=J
    CALL COMB(J-2,IR2-1,NCOM)
    XCOM=NCOM
 10  TRM2=TRM2+(XJ-R2)/(XJ-1.)*XCOM*Q**IR2*P**(J-IR2)
    VAR=TRM1+TRM2-P**2
    RETURN
  END
  SUBROUTINE COMB(N,M,NC)

```

C COMBINATIONS OF N THINGS TAKEN M AT A TIME WITH THE TOTAL NC
 C RETURNED.

```

    IF(N.LT.M) GOTO 20
    IF(M.EQ.0) GOTO 19
    IF(N,EQ.M) GOTO 19
    NC=1
    IF(M.GE.N-M) GOTO 13
    KKK=N-M+1
    DO 6 I=KKK , N
  6  NC=NC*I
    DO 7 I=1 , M
  7  NC=NC/I
    GOTO 99
 13  LL=M+1
    DO 4 I=LL , N
  4  NC=NC*I
    JJJ=N-M
    DO 5 I=1 , JJJ
  5  NC=NC/I
    GOTO 99
 19  NC=1
    GOTO 99

```

```
20 NC=0  
99 RETURN  
END
```

APPENDIX D

Table 8. Comparison of ASN for plans W and S^* . $\alpha \approx .051$, $\beta \approx .216$,
 $p_1 = .01$, $p_2 = .05$, $n_0 = 83$, $K_1 = 3$, and $K_2 = 80$.

p	h	Pa(p)-Wald	Var($\hat{p}(W)$)	ASN(W)	ASN(S^*)
.01	1.0000	.9490	.00032	51.15	79.64
.02	.2666	.7668	.00122	60.48	75.55
.03	-.2351	.5325	.00260	59.11	68.76
.04	-.6424	.3425	.00429	52.04	61.08
.05	-1.0000	.2160	.00617	44.14	53.69
.06	-1.3281	.1368	.00820	37.31	47.14
.07	-1.6377	.0876	.01034	31.81	41.56
.08	-1.9355	.0567	.01257	27.46	36.92
.09	-2.2260	.0370	.01486	24.01	33.08
.10	-2.5121	.0243	.01721	21.24	29.89

Table 9. Comparison of ASN for plans W and S^* . $\alpha \approx .075$, $\beta \approx .238$,
 $p_1 = .02$, $p_2 = .08$, $n_0 = 50$, $K_1 = 3$, and $K_2 = 47$.

p	h	Pa(p)-Wald	Var($\hat{p}(W)$)	ASN(W)	ASN(S^*)
.01	1.7549	.9845	.00032	26.71	47.30
.02	1.0000	.9250	.00110	31.64	46.86
.03	.5067	.8181	.00236	34.98	45.56
.04	.1220	.6817	.00406	36.05	43.57
.05	-.2032	.5419	.00607	35.12	41.12
.06	-.4914	.4175	.00829	32.95	38.45
.07	-.7545	.3164	.01066	30.18	35.73
.08	-1.0000	.2380	.01311	27.35	33.10
.09	-1.2327	.1788	.01562	24.69	30.61
.10	-1.4559	.1345	.01818	22.31	28.33

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