Appendix A: Third Order Laminated Beam Theory

A thin-walled laminated beam theory is developed to calculate deflections and ply-level stresses under transverse loading, following the approach of Barbero et al. [1]. Third-order kinematics are used to model warping and to permit a quadratic variation of shear strain through the thickness of the beam. This feature eliminates the need for a shear correction factor and captures warping-related effects.

In addition to the assumptions/restrictions made by Barbero et al. [1] in their Mechanics of Laminated Beam (MLB) theory, the panels are assumed to be either vertical (web) or horizontal (flanges) to simplify the development. Furthermore, the derivation is presented for a single web panel and two flanges, although the theory could easily be modified for additional panels. Unlike the MLB model, this model accounts for shear deformation in the flanges using first order (Timoshenko) kinematics.

The global stiffness quantities are found by equating the total strain energy of the individual panels expressed in terms of the local panel stiffnesses to the strain energy of the beam expressed in terms of the global stiffnesses. The governing equations are then found using variational principles, and a solution technique employed by Dufort et al. [2] is followed to provide analytical solutions.

1 Kinematics and Strain-Displacement Relations

1.1 Global

The global coordinate system is oriented as shown in Figure A.1. Following the notation and convention of Reddy [3] and Reddy et al. [4], the kinematics for a cross-section of the beam are as follows:

$$\begin{aligned} u(x,z) &= u_o(x) + z f(x) - a z^3 \left(f(x) + w_{o,x}(x) \right) \\ w(x,z) &= w_o(x) \end{aligned} \tag{1-1}$$

where $f(x)$ is the rotation of the cross-section due to mid-plane curvature and shear deformation. The third term accounts for warping and permits a parabolic shear strain distribution. Assuming small deformations, the strain-displacement relationships become

$$\begin{aligned} e_x(x,z) &= e_x^{(0)} + z e_x^{(1)} + z^3 e_x^{(3)} \\ g_{xz}(x,z) &= g_{xz}^{(0)} + z^2 g_{xz}^{(2)} \end{aligned} \tag{1-2}$$

where

$$e_x^{(0)} = u_{o,x}(x)$$
$$e_x^{(1)} = f_{x,x}(x)$$
$$e_x^{(3)} = (-a)\left(f_{x,x}(x) + w_{o,xx}(x)\right)$$
$$a = \frac{4}{3h^2}$$

and

$$g_{xz}^{(0)} = f_x(x) + w_{o,x}(x)$$
$$g_{xz}^{(2)} = (-b)\left(f_x(x) + w_{o,x}(x)\right)$$
$$b = 3a = \frac{4}{h^2}$$

1.2 Flanges

The local coordinate system of the flange panels is oriented such that the local x-axis is parallel to the global x-axis, and the local z-axis is parallel to the global z-axis. The direction of the local y-axis is determined by the orientation angle q and the right hand rule (Figure A.2). In order to capture the shear strain in the flange panels, we assume first order shear deformable kinematics in the flanges. This assumption yields an additional refinement beyond the theory of Barbero et al. [1], which neglects any shear deformation in the flanges. In the case of a moderately thick-walled beam, this strain may not be negligible. The local displacements become

$$\begin{aligned} \overline{u}(x,z') &= \overline{u}_o + z'\overline{f}(x) \\ \overline{w}(x) &= \overline{w}_o \end{aligned} \tag{1-3}$$

where z' is the local z-coordinate:

$$z = \overline{z} + z'\cos q \tag{1-4}$$

The bar notation of Barbero et al. [1] is introduced to denote local panel/laminate values. The mid-surface displacements for the flanges are found from the global kinematics:

$$\begin{aligned} \overline{u}_o(x) &= u(x,\overline{z}) = u_o(x) + \overline{z}f(x) - a\overline{z}^3\left(f(x) + w_{o,x}(x)\right) \\ \overline{w}_o(x) &= w(x)\cos q_i = w_o(x)\cos q_i \end{aligned} \tag{1-5}$$

In order to find the local rotation, $\overline{f}$, the axial displacements at two points within the flange are computed using the global expression for u. The two unknowns of $\overline{u}_o$ and $\overline{f}$ in Equation (1-3) can be solved for using these two known values. In order to force $\overline{u}_o$ to equal the global value of u at the flange mid-plane, we choose the mid-plane $(x, \overline{z})$ as one point. Next, it is convenient to choose either the inner or outer surface (z = $\overline{z} \pm t_f \cos q$). The magnitude of $\overline{f}$ at these locations will differ slightly, depending upon the amount of warping in the cross-section. To ensure continuity of displacements at the flange-web interface, we therefore choose the inner surface as the second point:

$$\bar{f}(x) = \left[f(x) + \frac{1}{4}\left(-t_f^2 \cos^2 q \pm 6t_f\bar{z} \mp 12\bar{z}^2\right)a\left(f(x) + w_o'(x)\right)\right]\cos q \tag{1-6}$$

The ± signs depend upon which flange is considered. However, recognizing that $\bar{z} >> t_f$ for thin-walled beams and substituting for a, these terms are small, and the expression becomes

$$\bar{f}(x) \approx \left[f_x(x) - \bar{z}^2 b\left(f_x(x) + w_{o,x}(x)\right)\right]\cos q \tag{1-7}$$

The local in-plane strain measures, in terms of the global coordinates, are found to be

$$\begin{aligned} &\bar{e}_x(x,z) = \bar{u}_{,x}(x) = \bar{u}_{o,x}(x) + z'\bar{f}_{,x}(x) = \bar{e}_x^{(0)} + z'\bar{e}_x^{(1)} \\ &\text{where} \\ &\bar{e}_x^{(0)}(x) = e_x(x,\bar{z}) = e_x^{(0)} + \bar{z}e_x^{(1)} + \bar{z}^3 e_x^{(3)} \\ &\bar{e}_x^{(1)} = \bar{f}'(x) = \left(f_{x,x}(x) - \bar{z}^2 b\,(f_{x,x}(x) + w_{o,xx}(x))\right)\cos q = \left(e_x^{(1)} + 3\bar{z}^2 e_x^{(3)}\right)\cos q \end{aligned} \tag{1-8}$$

where $\bar{z}$ = distance to the flange's mid-surface. The flange shear strain is found from

$$\bar{g}_{xz}(x,z) = \frac{\partial\bar{u}}{\partial z} + \frac{\partial\bar{w}}{\partial x} \tag{1-9}$$

to be equal to

$$\bar{g}_{xz}(x,z) = \left(1 + \frac{1}{4}(-t_f^2 \cos^2 q \pm 6t_f\bar{z} \mp 12\bar{z}^2)a\right)(f_x(x) + w_{o,x}(x))\cos q \tag{1-10}$$

For thin-walled beams, Equation (1-10) becomes

$$\begin{aligned} &\bar{g}_{xz}(x,z) = (1 - 3\bar{z}^2 a)(f_x(x) + w_{o,x}(x)) = (1 - \frac{4\bar{z}^2}{h^2})(f_x(x) + w_{o,x}(x)) \\ &\bar{g}_{xz}(x,z) = g_{xz}^{(0)} + \bar{z}^2 g_{xz}^{(2)} = g_{xz}(x,\bar{z}) \end{aligned} \tag{1-11}$$

Thus, the shear strain in the flanges is approximately linear through their thickness in accordance with the Timoshenko assumption, and the flange value can be determined from the global shear strain expression.

The factor *cosq* is included to account for the orientation of the panel. For instance, if the top and bottom flanges of a symmetric beam are defined to have exactly the same lay-up, but they are not individually symmetric, then they will be oriented in opposite directions in the beam. Thus, the effect of curvature and warping will be of opposite signs for the k^{th} lamina in each panel. The *cosq* provides a negative sign for panels oriented at 180°. Note that Barbero et al. [1] allow panels to be oriented at any angle θ; in the current development, panels are restricted to vertical or horizontal, i.e. q = 0º, ± 90º and 180º.

1.3 Web

The web kinematics are identical to the global kinematics, since the centroid of the web is assumed to be located at the neutral axis of the beam:

$$\begin{aligned} \overline{u}(x,z) &= u_o(x) + z f(x) - a z^3 \left(f(x) + w_{o,x}(x) \right) \\ \overline{w}(x,z) &= w_o(x) \end{aligned} \tag{1-12}$$

where

$$\begin{aligned} z(s) &= s + \overline{z} \\ \frac{-b}{2} &\leq s \leq \frac{b}{2} \end{aligned} \tag{1-13}$$

where the coordinate across the width of the panel is represented as *s* to distinguish it from the transverse beam coordinate, *y*. Since the web panel is transversely oriented, the strains at the mid-surface of the web panel vary in the transverse, *s* direction:

$$\begin{aligned} \bar{e}_z^{(0)}(x,z) &= e_x(x,z) = e_x^{(0)} + z e_x^{(1)} + z^3 e_x^{(3)} \\ \bar{g}_{xz}^{(0)}(x,z) &= g_{xz}(x,z) = g_{xz}^{(0)} + z^2 g_{xz}^{(2)} \end{aligned} \tag{1-14}$$

Substituting in for *z*,

$$\begin{aligned} \bar{e}_z^{(0)}(x,s) &= e_x^{(0)} + \left(s + \overline{z}\right) e_x^{(1)} + \left(s + \overline{z}\right)^3 e_x^{(3)} \\ \bar{g}_{xz}^{(0)}(x,s) &= g_{xz}^{(0)} + (s + \overline{z})^2 g_{xz}^{(2)} \end{aligned} \tag{1-15}$$

Out of plane curvature in the web panels is neglected, following Barbero's approach.

2 Constitutive Behavior

2.1 Flanges

From first order shear deformable plate theory, the constitutive behavior of the flange panels, expressed in terms of mid-plane values and resultant quantities is

$$\begin{Bmatrix} \{\overline{N}\} \\ \{\overline{M}\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{Bmatrix} \{\overline{e}^{(0)}\} \\ \{\overline{e}^{(1)}\} \end{Bmatrix} \qquad (2\text{-}1)$$

$$\{\overline{Q}\} = [A]\{\overline{g}^{(0)}\}$$

where $\overline{N}, \overline{M}$, and $\overline{Q}$ are defined as

$$\{\overline{N}\} = \begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \int_{-t/2}^{t/2} \begin{Bmatrix} s_{xx} \\ s_{yy} \\ s_{xy} \end{Bmatrix} dz, \qquad \{\overline{M}\} = \begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \int_{-t/2}^{t/2} \begin{Bmatrix} s_{xx} \\ s_{yy} \\ s_{xy} \end{Bmatrix} z dz,$$

$$\{\overline{Q}\} = \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = k \int_{-t/2}^{t/2} \begin{Bmatrix} s_{xz} \\ s_{yz} \end{Bmatrix} dz \qquad (2\text{-}2)$$

where t is the laminate thickness and [A], [B], and [D] are defined as

$$\left(A_{ij}, B_{ij}, D_{ij}\right) = \sum_{k=1}^{N} \int_{z} \overline{Q}_{ij}^{(k)} \left(1, z, z^2\right) dz \qquad (2\text{-}3)$$

Inverting Equation (2-1),

$$\begin{Bmatrix} \{\overline{e}^{(0)}\} \\ \{\overline{e}^{(1)}\} \end{Bmatrix} = \begin{bmatrix} [a] & [b] \\ [b]^T & [d] \end{bmatrix} \begin{Bmatrix} \{\overline{N}\} \\ \{\overline{M}\} \end{Bmatrix} \qquad (2\text{-}4)$$

$$\{\overline{g}^{(0)}\} = [a]\{\overline{Q}\}$$

It is assumed that $N_y = N_{xy} = M_y = M_{xy} = Q_y = 0$, and following Barbero et al., it is assumed that all off-axis fibers are balanced and symmetric. This restriction decouples shear-extension and shear-bending deformations and leads to $\boldsymbol{a}_{16} = \boldsymbol{b}_{16} = 0$. It is also assumed that $\boldsymbol{d}_{16} = 0$; it can be shown using CLT that $\boldsymbol{d}_{16}$ decreases rapidly as the number of layers increases. This assumption is valid for thick laminates.

Using the bar notation of Barbero et al. to again denote panel strains, the inverted equations reduce to

$$\begin{Bmatrix} \bar{\varepsilon}_x^{(0)} \\ \bar{\varepsilon}_x^{(1)} \end{Bmatrix} = \begin{bmatrix} a_{11} & b_{11} \\ b_{11} & d_{11} \end{bmatrix} \begin{Bmatrix} \bar{N}_x \\ \bar{M}_x \end{Bmatrix} \tag{2-5}$$

$$\bar{\gamma}_{xz}^{(0)} = a_{55}\bar{Q}_x$$

Inverting back, the reduced force resultant-strain equations become

$$\begin{Bmatrix} \bar{N}_x \\ \bar{M}_x \end{Bmatrix} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{B} & \bar{D} \end{bmatrix} \begin{Bmatrix} \bar{\varepsilon}_x^{(0)} \\ \bar{\varepsilon}_x^{(1)} \end{Bmatrix} \tag{2-6}$$

$$\bar{Q}_x = \bar{A}_{55}\bar{\gamma}_{xz}^{(0)}$$

2.2 Web

The web is modeled as an orthotropic panel with moduli E_x and G_{xz} (in global coordinates). E_x is the longitudinal modulus E_1 (local coordinates) and G_{xz} is the in-plane shear modulus G_{12}, as calculated using classical laminate theory (CLT). The web is assumed to carry only axial stress and shear stresses. Curvatures and curvature-related force resultants are neglected, as these forces would act through the thickness of the web, and we are not interested in these deformations. Thus, all higher-order through-the-thickness effects are neglected. However, the shear stress distribution is allowed to vary in the *y*- (or *s*-) direction, so third order constitutive relations are required to describe the shear stiffness. The resulting constitutive equations are

$$\bar{N}_x = A_{11}\varepsilon_x^{(0)} \tag{2-7}$$

$$\begin{Bmatrix} \bar{Q}_x \\ \bar{R}_x \end{Bmatrix} = \begin{bmatrix} \bar{A}_{xz} & \bar{D}_{xz} \\ \bar{D}_{xz} & \bar{F}_{xz} \end{bmatrix} \begin{Bmatrix} \bar{\gamma}_{xz}^{(0)} \\ \gamma_{xz}^{(2)} \end{Bmatrix}$$

where $\bar{R}_x$ is an additional higher order shear resultant defined as

$$\left\{\bar{R}_x\right\} = \int_{-h/2}^{h/2} \sigma_{xz} z^2 dz \tag{2-8}$$

The higher order stiffness quantities become

$$
\begin{aligned}
\overline{A}_{xz} &= \int G_{xz}\, dA = G_{xz} A_{web} \\
\overline{D}_{xz} &= \int G_{xz} z^2 dA = G_{xz} I_{web}^{(2)} \\
\overline{F}_{xz} &= \int G_{xz} z^4 dA = G_{xz} I_{web}^{(4)}
\end{aligned}
\tag{2-9}
$$

following the notation of Reddy et al.

2.3 Beam: Global Relations

The constitutive equations for the beam are expressed in terms of unknown global stiffness quantities following the notation of Reddy et al. [3]:

$$
\begin{Bmatrix} N_{xx} \\ M_{xx} \\ P_{xx} \end{Bmatrix} = \begin{bmatrix} A_{xx} & B_{xx} & E_{xx} \\ B_{xx} & D_{xx} & F_{xx} \\ E_{xx} & F_{xx} & H_{xx} \end{bmatrix} \begin{Bmatrix} e_x^{(0)} \\ e_x^{(1)} \\ e_x^{(3)} \end{Bmatrix}
\tag{2-10}
$$

$$
\begin{Bmatrix} Q_x \\ R_x \end{Bmatrix} = \begin{bmatrix} A_{xz} & D_{xz} \\ D_{xz} & F_{xz} \end{bmatrix} \begin{Bmatrix} g_{xz}^{(0)} \\ g_{xz}^{(2)} \end{Bmatrix}
$$

where P_{xx} is an additional higher order resultant related to warping:

$$
P_{xx} = \int_{-h/2}^{h/2} s_{xx} z^3 dz
\tag{2-11}
$$

Note that for a rectangular orthotropic beam, the stiffness parameters are found from

$$
\begin{aligned}
A_{xx} &= \int E_x dA = E_x A \\
B_{xx} &= \int E_x z dA = 0 \\
D_{xx} &= \int E_x z^2 dA = E_x I_{yy}^{(2)} \\
E_{xx} &= \int E_x z^3 dA = 0 \\
F_{xx} &= \int E_x z^4 dA = E_x I_{yy}^{(4)} \\
H_{xx} &= \int E_x z^6 dA = E_x I_{yy}^{(6)} \\
A_{xz} &= \int G_{xz} dA = G_{xz} A \\
D_{xz} &= \int G_{xz} z^2 dA = G_{xz} I_{yy}^{(2)} \\
F_{xz} &= \int G_{xz} z^4 dA = G_{xz} I_{yy}^{(4)}
\end{aligned}
\tag{2-12}
$$

In order to calculate these quantities for a thin-walled laminated beam, the definition of the beam stiffness coefficients must be derived from energy considerations, as follows.

3 Strain Energy Expressions

The general expression for the strain energy in a laminate can be expressed as

$$U = \frac{1}{2}\int_V \{e\}^T[\overline{Q}]\{e\}dV = \frac{1}{2}\int_V \{e\}^T\left[\overline{Q}\right]\{e\}dV = \frac{1}{2}\int_x\int_y\left(\sum_k\int_z \{e\}^T\left[\overline{Q}\right]\{e\}dz\right)dydx \tag{3-1}$$

For the moment, we assume third order kinematics. Substituting in the plate kinematics, the strain energy becomes

$$U = \frac{1}{2}\int_x\int_y\left(\sum_k\int_z\left[\begin{array}{l}\left\{e^{(0)} + ze^{(1)} + z^3e^{(3)}\right\}^T\left[\overline{Q}\right]_k\left\{e^{(0)} + ze^{(1)} + z^3e^{(3)}\right\} + \\ \left\{g^{(0)} + z^2g^{(2)}\right\}^T\left[\overline{Q}\right]_k\left\{g^{(0)} + z^2g^{(2)}\right\}\end{array}\right]dz\right)dydx \tag{3-2}$$

Performing the matrix multiplication and grouping like terms, we have

$$\begin{aligned}U = {} & \frac{1}{2}\int_x\int_y\left[\sum_k\int_z\left(\left\{e^{(0)}\right\}^T\left[\overline{Q}\right]_k\left\{e^{(0)}\right\} + 2z\left\{e^{(0)}\right\}^T\left[\overline{Q}\right]_k\left\{e^{(1)}\right\} + z^2\left\{e^{(1)}\right\}^T\left[\overline{Q}\right]_k\left\{e^{(1)}\right\} + 2z^3\left\{e^{(0)}\right\}^T\left[\overline{Q}\right]_k\left\{e^{(3)}\right\}\right)dz\right]dydx \\ & + \frac{1}{2}\int_x\int_y\left[\sum_k\int_z\left(2z^4\left\{e^{(1)}\right\}^T\left[\overline{Q}\right]_k\left\{e^{(3)}\right\} + z^6\left\{e^{(3)}\right\}^T\left[\overline{Q}\right]_k\left\{e^{(3)}\right\}\right)dz\right]dydx \\ & + \frac{1}{2}\int_x\int_y\left[\sum_k\int_z\left(\left\{g^{(0)}\right\}^T\left[\overline{Q}\right]_k\left\{g^{(0)}\right\} + 2z^2\left\{g^{(0)}\right\}^T\left[\overline{Q}\right]_k\left\{g^{(2)}\right\} + z^4\left\{g^{(2)}\right\}^T\left[\overline{Q}\right]_k\left\{g^{(2)}\right\}\right)dz\right]dydx\end{aligned}$$

Equation (3-3)

Then using the definition of A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, and H_{ij} for plates (see Reddy [3]), which are similar to Equation (2-9) for a beam,

$$U = \frac{1}{2}\int_x\int_y\left[\begin{array}{l}\left\{e^{(0)}\right\}^T\left[A_{ij}\right]\left\{e^{(0)}\right\} + 2\left\{e^{(0)}\right\}^T\left[B_{ij}\right]\left\{e^{(1)}\right\} + \left\{e^{(1)}\right\}^T\left[D_{ij}\right]\left\{e^{(1)}\right\} + 2\left\{e^{(0)}\right\}^T\left[E_{ij}\right]\left\{e^{(3)}\right\} + \\ 2\left\{e^{(1)}\right\}^T\left[F_{ij}\right]\left\{e^{(3)}\right\} + \left\{e^{(3)}\right\}^T\left[H_{ij}\right]\left\{e^{(3)}\right\} + \\ \left\{g^{(0)}\right\}^T\left[A_{ab}\right]\left\{g^{(0)}\right\} + 2\left\{g^{(0)}\right\}^T\left[D_{ab}\right]\left\{g^{(2)}\right\} + \left\{g^{(2)}\right\}^T\left[F_{ab}\right]\left\{g^{(2)}\right\}\end{array}\right]dydx$$

Equation (3-4)

where i,j = 1,2,6, and a,b = 4,5. For convenience, we rewrite Equation (3-4) in terms of the stress resultants:

$$U=\frac{1}{2}\int_x\int_y\left[\left\{e^{(0)}\right\}^T[N]+\left\{e^{(1)}\right\}^T[M]+\left\{e^{(3)}\right\}^T[P]+\left\{g^{(0)}\right\}^T[Q]+\left\{g^{(2)}\right\}^T[R]\right]dydx \qquad (3\text{-}5)$$

This is the general expression of the strain energy for a laminated plate modeled using third order kinematics.

3.1 Flanges

For the 1-dimensional beam assumption, we assumed previously that $N_y = N_{xy} = M_y = M_{xy} = Q_y = 0$ in the flanges, allowing only first order shear deformation in the flanges. The reduced energy expression for a flange panel becomes

$$U=\frac{1}{2}\int_x\int_y\left[\overline{A}\left(\overline{e}_x^{(0)}\right)^2+2\overline{B}\left(\overline{e}_x^{(0)}\right)\left(\overline{e}_x^{(1)}\right)+\overline{D}\left(\overline{e}_x^{(1)}\right)^2+\overline{A}_{xz}\left(\overline{g}_{xz}^{(0)}\right)^2\right]dydx \qquad (3\text{-}6)$$

where the bar notation is again used to denote panel values. Integrating in the *y*-direction from -*b*/2 to *b*/2, where *b* is the width of the flange panels,

$$U=\frac{1}{2}\int_x b\left[\overline{A}\left(\overline{e}_x^{(0)}\right)^2+2\overline{B}\left(\overline{e}_x^{(0)}\right)\left(\overline{e}_x^{(1)}\right)+\overline{D}\left(\overline{e}_x^{(1)}\right)^2+k\overline{A}_{xz}\left(\overline{g}_{xz}^{(0)}\right)^2\right]dx \qquad (3\text{-}7)$$

A shear correction factor for the flange, k_f, has been introduced to account for the error in assuming a constant shear strain through the thickness. Substituting the flange kinematics into this expression,

$$U=\frac{1}{2}\int_x b\begin{bmatrix}\overline{A}\left(e_x^{(0)}+\overline{z}e_x^{(1)}+\overline{z}^3e_x^{(3)}\right)^2+2\overline{B}\left(e_x^{(0)}+\overline{z}e_x^{(1)}+\overline{z}^3e_x^{(3)}\right)\left(e_x^{(1)}+3\overline{z}^2e_x^{(3)}\right)\cos q+\\ \overline{D}\left(e_x^{(1)}+3\overline{z}^2e_x^{(3)}\right)^2+k_f\overline{A}_{xz}\left(g_{xz}^{(0)}+\overline{z}^2g_{xz}^{(2)}\right)^2\end{bmatrix}dx$$

Equation (3-8)

where the strain measures shown are now the global beam values. Substituting for the mid-plane strains measures,

$$U=\frac{1}{2}\int_x b\begin{bmatrix}\overline{A}\left(u_{o,x}+\overline{z}f_{x,x}+\overline{z}^3(-a)\left(f_{x,x}+w_{o,xx}\right)\right)^2+\\ 2\overline{B}\left(u_{o,x}+\overline{z}f_{x,x}+\overline{z}^3(-a)\left(f_{x,x}+w_{o,xx}\right)\right)\left(f_{x,x}+\overline{z}^2b\left(f_{x,x}+w_{o,xx}\right)\right)\cos q+\\ \overline{D}\left(f_{x,x}+\overline{z}^2b\left(f_{x,x}+w_{o,xx}\right)\cos q\right)^2+k_f\overline{A}_{xz}\left(\left(f_x+w_{o,x}\right)+\overline{z}_i^2(-b)\left(f_x+w_{o,x}\right)\right)^2\end{bmatrix}dx$$

Equation (3-9)

After multiplying the terms in parentheses, the expression finally becomes

$$U = \frac{1}{2}\int_x b \left[\begin{array}{l} \overline{A}\left[\begin{array}{l} (u_{o,x})^2 + 2\overline{z}u_{o,x}f_{x,x} - 2a\overline{z}^3 u_{o,x}\left(f_{x,x} + w_{o,xx}\right) \\ -2\overline{z}^4 a f_{x,x}\left(f_{x,x} + w_{o,xx}\right) + \overline{z}^6 a^2\left(f_{x,x} + w_{o,xx}\right)^2 + \overline{z}^2\left(f_{x,x}\right)^2 \end{array} \right] + \\ 2\overline{B}\left[\left(\begin{array}{l} u_{o,x}f_{x,x} + \overline{z}f_{x,x}^2 - a\overline{z}^3 f_{x,x}\left(f_{x,x} + w_{o,xx}\right) + u_{o,x}\overline{z}^2 b\left(f_{x,x} + w_{o,xx}\right) + \\ f_{x,x}\overline{z}^3 b\left(f_{x,x} + w_{o,xx}\right) - \overline{z}^5 a b\left(f_{x,x} + w_{o,xx}\right)^2 \end{array} \right) \cos q \right] + \\ \overline{D}\left[(f_{x,x})^2 + 2\overline{z}^2 b f_{x,x}\left(f_{x,x} + w_{o,xx}\right) + \overline{z}^4 b^2\left(f_{x,x} + w_{o,xx}\right)^2 \right] \cos^2 q + \\ k\overline{A}_{xz}\left[(f_x + w_{o,x})^2 - 2\overline{z}^2 b\left(f_x + w_{o,x}\right)^2 + \overline{z}^4 b^2\left(f_x + w_{o,x}\right)^2 \right] \end{array} \right] dx$$

Equation (3-10)

Now group like terms:

$$U = \frac{1}{2}\int_x b \left[\begin{array}{l} \overline{A}(u_{o,x})^2 \\ + 2\left[\overline{A}\overline{z} + \overline{B}\cos q\right] u_{o,x} f_{x,x} \\ + \left[\overline{A}\overline{z}^2 + 2\overline{B}\overline{z}\cos q + \overline{D}\cos^2 q\right] (f_{x,x})^2 \\ - 2\left[\overline{A}\overline{z}^3 - 2\overline{B}\overline{z}^2 \cos q\right] u_{o,x} a\left(f_{x,x} + w_{o,xx}\right) \\ - 2\left[\overline{A}\overline{z}^4 - 2\overline{B}\overline{z}^3 \cos q - 3\overline{D}\overline{z}^2 \cos^2 q\right] f_{x,x} a\left(f_{x,x} + w_{o,xx}\right) \\ + \left[\overline{A}\overline{z}^6 - 6\overline{B}\overline{z}^5 \cos q + 9\overline{D}\overline{z}^4 \cos^2 q\right] a^2\left(f_{x,x} + w_{o,xx}\right)^2 \\ + \left[k\overline{A}_{xz}\left(1 - b\overline{z}^2\right)^2\right] (f_x + w_{o,x})^2 \end{array} \right] dx \qquad (3\text{-}11)$$

3.2 Web

Based on our previous assumptions for a web panel, the energy expression becomes

$$U = \frac{1}{2}\int_x\int_s \overline{A}\left(\overline{e}_x^{(0)}\right)^2 ds dx + \frac{1}{2} b \int_x \left[\overline{A}_{xz}\left(g_{xz}^{(0)}\right)^2 + 2\overline{D}_{xz}\left(g_{xz}^{(0)}\right)\left(g_{xz}^{(2)}\right) + \overline{F}_{xz}\left(g_{xz}^{(2)}\right)^2\right] dx \qquad (3\text{-}12)$$

Now substitute in for the mid-surface strains in the first term. The integration along the *y*-direction of the web (vertical) is slightly more complicated as the location of the mid-surface is given as $z(s) = s + \overline{z}$.

$$U = \frac{1}{2}\int_x\int_s \overline{A}\left(e_x^{(0)} + z e_x^{(1)} + z^3 e_x^{(3)}\right)^2 ds dx + \frac{1}{2} b \int_x \left[\overline{A}_{xz}\left(g_{xz}^{(0)}\right)^2 + 2\overline{D}_{xz}\left(g_{xz}^{(0)}\right)\left(g_{xz}^{(2)}\right) + \overline{F}_{xz}\left(g_{xz}^{(2)}\right)^2\right] dx$$

Equation (3-13)

or

$$U=\frac{1}{2}\int_x\int_s \overline{A}\left(u_{o,x}+zf_{x,x}+z^3(-a)\left(f_{x,x}+w_{o,xx}\right)\right)^2 dsdx$$
$$+\frac{1}{2}b\int_x\left[\overline{A}_{xz}\left(g_{xz}^{(0)}\right)^2+2\overline{D}_{xz}\left(g_{xz}^{(0)}\right)\left(g_{xz}^{(2)}\right)+\overline{F}_{xz}\left(g_{xz}^{(2)}\right)^2\right]dx \tag{3-14}$$

or

$$U=\frac{1}{2}\int_x\int_s \overline{A}\begin{pmatrix}\left(u_{o,x}\right)^2+2zu_{o,x}f_{x,x}-2az^3u_{o,x}\left(f_{x,x}+w_{o,xx}\right)+z^2\left(f_{x,x}\right)^2-az^4f_{x,x}\left(f_{x,x}+w_{o,xx}\right)+\\ a^2z^6\left(f_{x,x}+w_{o,xx}\right)^2\end{pmatrix}dsdx+$$
$$\frac{1}{2}b\int_x\left[\overline{A}_{xz}\left(g_{xz}^{(0)}\right)^2+2\overline{D}_{xz}\left(g_{xz}^{(0)}\right)\left(g_{xz}^{(2)}\right)+\overline{F}_{xz}\left(g_{xz}^{(2)}\right)^2\right]dx$$

Equation (3-15)

Substituting for *z*,

$$U=\frac{1}{2}\int_x\int_s \overline{A}\begin{pmatrix}\left(u_{o,x}\right)^2+2(\overline{z}+s)u_{o,x}f_{x,x}-2a(\overline{z}+s)^3u_{o,x}\left(f_{x,x}+w_{o,xx}\right)+(\overline{z}+s)^2\left(f_{x,x}\right)^2\\ -a(\overline{z}+s)^4f_{x,x}\left(f_{x,x}+w_{o,xx}\right)+a^2(\overline{z}+s)^6\left(f_{x,x}+w_{o,xx}\right)^2\end{pmatrix}dsdx+$$
$$\frac{1}{2}b\int_x\left[\overline{A}_{xz}\left(g_{xz}^{(0)}\right)^2+2\overline{D}_{xz}\left(g_{xz}^{(0)}\right)\left(g_{xz}^{(2)}\right)+\overline{F}_{xz}\left(g_{xz}^{(2)}\right)^2\right]dx$$

Equation (3-16)

Expanding the *z* + *s* terms,

$$U=\frac{1}{2}\int_x\int_s\left[\overline{A}\begin{pmatrix}\left(u_{o,x}\right)^2+2(\overline{z}+s)u_{o,x}f_{x,x}-2a\left(\overline{z}^2+2s\overline{z}+s^2\right)(\overline{z}+s)u_{o,x}\left(f_{x,x}+w_{o,xx}\right)\\ +\left(\overline{z}^2+2s\overline{z}+s^2\right)\left(f_{x,x}\right)^2-a\left(\overline{z}^2+2s\overline{z}+s^2\right)\left(\overline{z}^2+2s\overline{z}+s^2\right)f_{x,x}\left(f_{x,x}+w_{o,xx}\right)\\ +a^2\left(\overline{z}^2+2s\overline{z}+s^2\right)\left(\overline{z}^2+2s\overline{z}+s^2\right)\left(\overline{z}^2+2s\overline{z}+s^2\right)\left(f_{x,x}+w_{o,xx}\right)^2\end{pmatrix}\right]dsdx$$
$$+\frac{1}{2}b\int_x\left[\overline{A}_{xz}\left(g_{xz}^{(0)}\right)^2+2\overline{D}_{xz}\left(g_{xz}^{(0)}\right)\left(g_{xz}^{(2)}\right)+\overline{F}_{xz}\left(g_{xz}^{(2)}\right)^2\right]dx$$

Equation (3-17)

or

$$U = \frac{1}{2}\int_x\int_s \left[\overline{A} \begin{pmatrix} (u_{o,x})^2 + 2(\overline{z}+s)u_{o,x} f_{x,x} - 2a\left(\overline{z}^3 + 3s\overline{z}^2 + 3s^2\overline{z} + s^3\right)u_{o,x}\left(f_{x,x} + w_{o,xx}\right) \\ + \left(\overline{z}^2 + 2s\overline{z} + s^2\right)(f_{x,x})^2 - a\left(\overline{z}^4 + 4s\overline{z}^3 + 6s^2\overline{z}^2 + 4s^3\overline{z} + s^4\right)f_{x,x}\left(f_{x,x} + w_{o,xx}\right) + \\ a^2\left(\overline{z}^6 + 6s\overline{z}^5 + 15s^2\overline{z}^4 + 20s^3\overline{z}^4 + 15s^4\overline{z}^2 + 6s^5\overline{z} + s^6\right)\left(f_{x,x} + w_{o,xx}\right)^2 \end{pmatrix} \right] ds dx$$
$$+ \frac{1}{2} b \int_x \left[\overline{A}_{xz}\left(g_{xz}^{(0)}\right)^2 + 2\overline{D}_{xz}\left(g_{xz}^{(0)}\right)\left(g_{xz}^{(2)}\right) + \overline{F}_{xz}\left(g_{xz}^{(2)}\right)^2 \right] dx$$

Equation (3-18)

Integrating in *s*, the strain energy of a web panel becomes

$$U = \frac{1}{2}\int_x \left[\overline{A} \begin{pmatrix} b(u_{o,x})^2 + 2(\overline{z}b)u_{o,x} f_{x,x} - 2a\left(\overline{z}^3 b + \overline{z}\frac{b^3}{4}\right)u_{o,x}\left(f_{x,x} + w_{o,xx}\right) + \left(\overline{z}^2 b + \frac{b^3}{12}\right)(f_{x,x})^2 \\ - a\left(\overline{z}^4 b + \overline{z}^2\frac{b^3}{2} + \frac{b^5}{80}\right)f_{x,x}\left(f_{x,x} + w_{o,xx}\right) + \\ a^2\left(\overline{z}^6 b + \overline{z}^4\frac{15b^3}{12} + \frac{3b^5}{16}\overline{z}^2 + \frac{b^7}{448}\right)\left(f_{x,x} + w_{o,xx}\right)^2 \end{pmatrix} \right] dx$$
$$+ \frac{1}{2} b \int_x \left[\overline{A}_{xz}\left(g_{xz}^{(0)}\right)^2 + 2\overline{D}_{xz}\left(g_{xz}^{(0)}\right)\left(g_{xz}^{(2)}\right) + \overline{F}_{xz}\left(g_{xz}^{(2)}\right)^2 \right] dx$$

Equation (3-19)

Recognizing that $\overline{z}$ = 0 for the web, the expression reduces to

$$U = \frac{1}{2}\int_x \left[\overline{A}\left(b(u_{o,x})^2 + \left(\frac{b^3}{12}\right)(f_{x,x})^2 - a\left(\frac{b^5}{80}\right)f_{x,x}\left(f_{x,x} + w_{o,xx}\right) + a^2\left(\frac{b^7}{448}\right)\left(f_{x,x} + w_{o,xx}\right)^2 \right) \right] dx$$
$$+ \frac{1}{2} b \int_x \left[\overline{A}_{xz}\left(g_{xz}^{(0)}\right)^2 + 2\overline{D}_{xz}\left(g_{xz}^{(0)}\right)\left(g_{xz}^{(2)}\right) + \overline{F}_{xz}\left(g_{xz}^{(2)}\right)^2 \right] dx$$

Equation (3-20)

Substituting in for the shear strain measures, we finally have

$$U = \frac{1}{2}\int_x \left[\overline{A}\left(b(u_{o,x})^2 + \left(\frac{b^3}{12}\right)(f_{x,x})^2 - a\left(\frac{b^5}{80}\right) f_{x,x}(f_{x,x} + w_{o,xx}) + a^2\left(\frac{b^7}{448}\right)(f_{x,x} + w_{o,xx})^2 \right)\right] dx$$

$$+ \frac{1}{2} b \int_x \left[\overline{A}_{xz}(f_x + w_{o,x})^2 - 2\overline{D}_{xz}\, b\,(f_x + w_{o,x})^2 + \overline{F}_{xz}\, b^2 (f_x + w_{o,x})^2 \right] dx$$

Equation (3-21)

Note that Equation (3-19) could also be used to establish the strain energy for vertical panels which are not symmetric about $z = 0$.

4 Determination of Global Stiffness Coefficients

The strain energy of the beam in terms of the global kinematics and stiffness values is

$$U = \frac{1}{2}\int_x \begin{bmatrix} A_{xx}(u_{o,x})^2 + 2B_{xx}u_{o,x}f_{x,x} + D_{xx}(f_{x,x})^2 - 2E_{xx}u_{o,x}a(f_{x,x} + w_{o,xx}) - 2F_{xx}f_{x,x}a(f_{x,x} + w_{o,xx}) \\ + H_{xx}a^2(f_{x,x} + w_{o,x})^2 + A_{xz}(f_x + w_{o,x})^2 - 2D_{xz}\,b\,(f_x + w_{o,x})^2 + F_{xz}\,b^2(f_x + w_{o,x})^2 \end{bmatrix} dx$$

Equation (4-1)

The beam stiffness coefficients are defined by equating like terms in the global energy expression to the sum of the flange and web energy expressions:

$$
\begin{aligned}
A_{xx} &= \sum_{i=1}^{nflanges} \overline{A}_i b_i + \sum_{i=1}^{nwebs} \overline{A}_i b_i \\
B_{xx} &= \sum_{i=1}^{nflanges} \left(\overline{A}_i \overline{z}_i + \overline{B}_i \cos q_i \right) b_i \\
D_{xx} &= \sum_{i=1}^{nflanges} \left(\overline{A}_i \overline{z}_i^2 + 2\overline{B}_i \overline{z}_i \cos q_i + \overline{D}_i \cos^2 q_i \right) b_i + \sum_{i=1}^{nwebs} \overline{A}_i \left(\frac{b_i^3}{12} \right) \\
E_{xx} &= \sum_{i=1}^{nflanges} \left(\overline{A}\overline{z}^3 - 2\overline{B}\overline{z}^2 \cos q \right) b_i \\
F_{xx} &= \sum_{i=1}^{nflanges} \left(\overline{A}\overline{z}^4 - 2\overline{B}\overline{z}^3 \cos q - 3\overline{D}\overline{z}^2 \cos^2 q \right) b_i + \sum_{i=1}^{nwebs} \overline{A}_i \left(\frac{b_i^5}{80} \right) \\
H_{xx} &= \sum_{i=1}^{nflanges} \left(\overline{A}\overline{z}^6 - 6\overline{B}\overline{z}^5 \cos q + 9\overline{D}\overline{z}^4 \cos^2 q \right) b_i + \sum_{i=1}^{nwebs} \overline{A}_i \left(\frac{b_i^7}{448} \right) \\
A_{xz} &= \sum_{i=1}^{nwebs} \overline{A}_{xz} b_i + \sum_{i=1}^{nflanges} k_f \overline{A}_{xz,i} b_i \left(1 - \frac{4\overline{z}_i^2}{h^2} \right)^2 \\
D_{xz} &= \sum_{i=1}^{nwebs} \overline{D}_{xz} \\
F_{xz} &= \sum_{i=1}^{nwebs} \overline{F}_{xz}
\end{aligned}
\qquad (4\text{-}2)
$$

The significance of each stiffness coefficient is given in the table below:

A_{xx}	Extensional stiffness
B_{xx}	Bending-extension coupling stiffness
D_{xx}	Bending stiffness
E_{xx}	Warping-extension coupling stiffness
F_{xx}	Warping-bending coupling stiffness
H_{xx}	Warping - higher order bending coupling stiffness
A_{xz}	Shear stiffness
D_{xz}	Shear-warping coupling stiffness
F_{xz}	Warping stiffness

Note that if the beam is symmetric, the stiffness quantities B_{xx} and E_{xx} will be zero.

5 Stress Resultant Definitions

The strain energy of the beam can be expressed in terms of global stress resultants:

$$
U_{beam} = \frac{1}{2} \int_0^L \left[N_{xx} e_x^{(0)} + M_{xx} e_x^{(1)} + P_{xx} e_x^{(3)} + Q_x g_{xz}^{(0)} + R_x g_{xz}^{(2)} \right] dx \qquad (5\text{-}1)
$$

where N_{xx}, M_{xx}, P_{xx}, Q_x, and R_x are the stress resultants as defined by Reddy et al. [4] Similarly, the total strain energy of the flange and web panels can be expressed in terms of wall stress resultants:

$$
\begin{aligned}
U_{flange} &= \frac{1}{2}\int_x\int_y \left[\overline{N}\overline{e}_x^{(0)} + \overline{M}\overline{e}_x^{(1)} + \overline{Q}_x g_{xz}^{(0)}\right] dydx \\
U_{web} &= \frac{1}{2}\int_x\int_s \left[\overline{N}e_x^{(0)} + \overline{Q}_x g_{xz}^{(0)} + \overline{R}_x g_{xz}^{(2)}\right] dydx
\end{aligned}
\qquad (5\text{-}2)
$$

Substituting into the flange energy expression for the local strain measures to express the flange strain energy in terms of global strain measures,

$$U_{flange} = \frac{1}{2}\int_x\int_y \left[\overline{N}\left(e_x^{(0)} + \overline{z}e_x^{(1)} + \overline{z}^3 e_x^{(3)}\right) + \overline{M}\left(\left(e_x^{(1)} + 3\overline{z}^2 e_x^{(3)}\right)\cos q\right) + \overline{Q}_x\left(g_{xz}^{(0)} + \overline{z}^2 g_{xz}^{(2)}\right)\right] dydx$$

Equation (5-3)

or grouping like terms

$$U_{flange} = \frac{1}{2}\int_x\int_y \left[\left(\overline{N}\right)e_x^{(0)} + \left(\overline{N}\overline{z} + \overline{M}\cos q\right)e_x^{(1)} + \left(\overline{N}\overline{z}^3 + 3\overline{M}\overline{z}^2\cos q\right)e_x^{(3)} + \left(\overline{Q}\right)_x g_{xz}^{(0)} + \left(\overline{Q}_x\overline{z}^2\right)g_{xz}^{(2)}\right] dydx$$

Equation (5-4)

Equating coefficients of the strain measures, the global stress resultants can be found in terms of the panel stress resultants:

$$
\begin{aligned}
N_{xx} &= \sum_{i=1}^{npanels} \overline{N}_i b_i \\
M_{xx} &= \sum_{i=1}^{nflanges} \left(\overline{N}_i\overline{z}_i + \overline{M}_i\cos q_i\right)b_i \\
P_{xx} &= \sum_{i=1}^{nflanges} \left(\overline{N}_i\overline{z}_i^3 + 3\overline{M}\overline{z}^2\cos q_i\right)b_i \\
Q_x &= \sum_{i=1}^{nwebs} \overline{Q}_i b_i \\
R_x &= \sum_{i=1}^{nwebs} \overline{Q}_i\overline{z}^2 b_i
\end{aligned}
\qquad (5\text{-}5)
$$

6 Governing Equations

Recall the strain energy expression for the beam in terms of the global stiffness coefficients, and add the energy due to the external loading to form the total potential energy:

$$p = \frac{1}{2}\int_0^L \begin{bmatrix} A_{xx}\left(u_{o,x}\right)^2 + 2B_{xx}u_{o,x}f_{x,x} - 2D_{xx}\left(f_{x,x}\right)^2 - 2E_{xx}au_{o,x}\left(f_{x,x} + w_{o,xx}\right) - 2F_{xx}a\,f_{x,x}\left(f_{x,x} + w_{o,xx}\right) \\ + H_{xx}a^2\left(f_{x,x} + w_{o,x}\right)^2 + A_{xz}\left(f_x + w_{o,x}\right)^2 - 2D_{xz}b\left(f_x + w_{o,x}\right)^2 + F_{xz}b^2\left(f_x + w_{o,x}\right)^2 \end{bmatrix}dx$$
$$- \int_0^L q w_o(x)dx - Pw_o(x_P)$$

Equation (6-1)

where q_o is a uniform load, P is a point load located at $x = x_p$ (x_p a boundary, i.e. either $x = 0$ or $x = L$). To find the governing equations, we minimize the total potential energy by taking the first variation and setting it equal to zero:

$$dp = \int_0^L \begin{bmatrix} A_{xx}\left(u_{o,x}\right)du_{o,x} + B_{xx}\left(du_{o,x}f_{x,x} + u_{o,x}df_{x,x}\right) - 2D_{xx}\left(f_{x,x}\right)df_{x,x} - \\ E_{xx}a\left(du_{o,x}f_{x,x} + du_{o,x}w_{o,xx} + u_{o,x}df_{x,x} + u_{o,x}dw_{o,xx}\right) \\ - F_{xx}a\left(2f_{x,x}df_{x,x} + df_{x,x}w_{o,xx} + f_{x,x}dw_{o,xx}\right) + H_{xx}a^2\left(df_{x,x} + dw_{o,x}\right) \\ + A_{xz}\left(df_x + dw_{o,x}\right) - 2D_{xz}b\left(df_x + dw_{o,x}\right) + F_{xz}b^2\left(df_x + dw_{o,x}\right) \end{bmatrix}dx \qquad (6\text{-}2)$$
$$- \int_0^L q dw_o(x)dx - Pdw_o(x_P) = 0$$

Using the definitions of the global stress resultants from Section 2.3,

$$dp = \int_0^L \left[N_{xx}du_{o,x} + M_{xx}df_{x,x} + P_{xx}(-a)d\left(f_{x,x} + w_{o,xx}\right) + Q_x d\left(f_x + w_{o,x}\right) + R_x(-b)d\left(f_x + w_{o,x}\right)\right]dx$$
$$- \int_0^L q dw_o(x)dx - Pdw_o(x_P) = 0$$

Equation (6-3)

Note that this form is equivalent to the Principal of Virtual Work. Next, we apply integration by parts to relieve the differentiation on the virtual displacements and set the first variation of the potential energy to zero:

$$
dp = \frac{1}{2}\int_0^L \left[-N_{xx,x} du_o - M_{xx,x} df_x + P_{xx,x} a df_x - P_{xx,xx} a dw_o + Q_x df_x - Q_{x,x} dw_o - R_x b df_x + R_{x,x} b dw_o \right] dx
$$
$$
-\int_0^L q dw_o(x) dx - P dw_o(x_P) + \left[N_{xx} du_o + M_{xx} df_x - P_{xx} a d\left(f_x + w_{o,x}\right) + P_{xx,x} a dw_o + Q_x dw_o - R_x b dw_o \right]_0^L
$$
$$
= 0
$$

Equation (6-4)

The governing equations are found from the coefficients of the virtual displacements:

$$
\begin{aligned}
&N_{xx,x} = 0 \\
&Q_{x,x} - bR_{x,x} + aP_{xx,xx} + q = 0 \\
&M_{xx,x} - aP_{xx,x} - Q_x + bR_x = 0
\end{aligned}
\qquad (6\text{-}5)
$$

These equations match those given by Reddy [3], when his equations are reduced to the case of 1-D beam analysis. The boundary conditions are found in the bracketed terms:

$$
\text{Either} \quad
\begin{aligned}
&u_o \text{ specified} \\
&w_o \text{ specified} \\
&f_x \text{ specified} \\
&w_{o,x} \text{ specified}
\end{aligned}
\quad \text{or} \quad
\begin{aligned}
&N_{xx} = 0 \\
&Q_x - bR_x + aP_{xx,x} + (-P) = 0 \\
&M_{xx} - aP_{xx} = 0 \\
&-aP_{xx} = 0
\end{aligned}
\qquad (6\text{-}6)
$$

The *P* terms in parentheses is added for the case of a point load at either of the boundaries.

7 Solution of the Governing Equations

Following the approach of Dufort et al. [2], we will rewrite the governing equations in terms of the displacement quantities and then reduce them to one equation in one displacement variable. Because the beam is assumed to be symmetric and the bending and stretching problems are decoupled, the bending problem is considered separately. Therefore, using the definitions of the stress resultants, the last two governing equations are rewritten in terms of the displacement quantities:

$$
A_{xz} g^{(0)}_{xz,x} + D_{xz} g^{(2)}_{xz,x} - b\left(D_{xz} g^{(0)}_{xz,x} + F_{xz} g^{(2)}_{xz,x}\right) + a\left(E_{xx} e^{(0)}_{x,xx} + F_{xx} e^{(1)}_{x,xx} + H_{xx} e^{(3)}_{x,xx}\right) + q = 0
$$
$$
B_{xx} e^{(0)}_{x,x} + D_{xx} e^{(1)}_{x,x} + F_{xx} e^{(3)}_{x,x} - a\left(E_{xx} e^{(0)}_{x,x} + F_{xx} e^{(1)}_{x,x} + H_{xx} e^{(3)}_{x,x}\right) - \left(A_{xz} g^{(0)}_{xz} + D_{xz} g^{(2)}_{xz}\right) +
$$
$$
b\left(D_{xz} g^{(0)}_{xz} + F_{xz} g^{(2)}_{xz}\right) = 0
$$

Equation (7-1)

Grouping like terms,

$$
\begin{aligned}
&(aE_{xx})e^{(0)}_{x,xx} + (aF_{xx})e^{(1)}_{x,xx} + (aH_{xx})e^{(3)}_{x,xx} + (A_{xz} - bD_{xz})g^{(0)}_{xz,x} + (D_{xz} - bF_{xz})g^{(2)}_{xz,x} + q = 0 \\
&(B_{xx} - aE_{xx})e^{(0)}_{x,x} + (D_{xx} - aF_{xx}e^{(1)}_{x,x})e^{(1)}_{x,x} + (F_{xx} - aH_{xx})e^{(3)}_{x,x} + (-A_{xz} + bD_{xz})g^{(0)}_{xz} \\
&+ (-D_{xz} + bF_{xz})g^{(2)}_{xz} = 0
\end{aligned}
$$

Equation (7-2)

Now, substituting in the displacement quantities, but leaving $g_{xz}^{(0)}$ as a variable:

$$
\begin{aligned}
&(aE_{xx})u_{o,xxx} + (aF_{xx})f_{x,xxx} + (aH_{xx})(-a)(f_{x,xxx} + w_{o,xxxx}) + (A_{xz} - bD_{xz})g^{(0)}_{xz,x} + \\
&(D_{xz} - bF_{xz})(-b)(f_{x,x} + w_{o,xx}) + q = 0 \\
&(B_{xx} - aE_{xx})u_{o,xx} + (D_{xx} - aF_{xx})f_{x,xx} + (F_{xx} - aH_{xx})(-a)(f_{x,xx} + w_{o,xxx}) + (-A_{xz} + bD_{xz})g^{(0)}_{xz} \\
&+ (-D_{xz} + bF_{xz})(-b)(f_{x} + w_{o,x}) = 0
\end{aligned}
$$

Equation (7-3)

Assuming the u_o terms are negligible (see Dufort et al. [2]),

$$
\begin{aligned}
&(aF_{xx})f_{x,xxx} + (aH_{xx})(-a)(f_{x,xxx} + w_{o,xxxx}) + (A_{xz} - bD_{xz})g^{(0)}_{xz,x} \\
&+ (D_{xz} - bF_{xz})(-b)(f_{x,x} + w_{o,xx}) + q = 0 \\
&(D_{xx} - aF_{xx})f_{x,xx} + (F_{xx} - aH_{xx})(-a)(f_{x,xx} + w_{o,xxx}) + (-A_{xz} + bD_{xz})g^{(0)}_{xz} \\
&+ (-D_{xz} + bF_{xz})(-b)(f_{x} + w_{o,x}) = 0
\end{aligned}
$$

Equation (7-4)

Noting that $e^{(3)}_{x} = (-a)g^{(0)}_{xz,x}$ and $g^{(2)}_{xz} = (-b)g^{(0)}_{xz}$, then we have

$$
\begin{aligned}
&(aF_{xx})f_{x,xxx} + (aH_{xx})(-a)g^{(0)}_{xz,xxx} + (A_{xz} - bD_{xz})g^{(0)}_{xz,x} + (D_{xz} - bF_{xz})(-b)g^{(0)}_{xz,x} + q = 0 \\
&(D_{xx} - aF_{xx})f_{x,xxx} + (F_{xx} - aH_{xx})(-a)g^{(0)}_{xz,xxx} + (-A_{xz} + bD_{xz})g^{(0)}_{xz} + (-D_{xz} + bF_{xz})(-b)g^{(0)}_{xz,x} = 0
\end{aligned}
$$

Equation (7-5)

or, regrouping,

$$
\begin{aligned}
&(aF_{xx})f_{x,xxx} + (aH_{xx})(-a)g^{(0)}_{xz,xxx} + (A_{xz} - 2bD_{xz} + b^2F_{xz})g^{(0)}_{xz,x} + q = 0 \\
&(D_{xx} - aF_{xx})f_{x,xx} + (F_{xx} - aH_{xx})(-a)g^{(0)}_{xz,xx} + (-A_{xz} + 2bD_{xz} - b^2F_{xz})g^{(0)}_{xz} = 0
\end{aligned}
\qquad (7\text{-}6)
$$

Integrating the first equation and adding to the second, yields the following system:

$$
\begin{aligned}
&-F_{xx}ag^{0}_{xz,xx} + D_{xx}f_{x,xx} + A1 + qx = 0 \\
&a(-F_{xx} + aH_{xx})g^{0}_{xz,xx} - (A_{xz} - 2bD_{xz} + b^2F_{xz})g^{0}_{xz} + (D_{xx} - aF_{xx})f_{x,xx} = 0
\end{aligned}
\qquad (7\text{-}7)
$$

Eliminating $f_{x,xx}$, an ordinary differential equation in $g_{xz}^{(0)}$ is obtained:

$$-a^2\left(D_{xx}H_{xx}-F_{xx}^2\right)g_{xz,xx}^0+D_{xx}\left(A_{xz}-2bD_{xz}+b^2F_{xz}\right)g_{xz}^0+\left(D_{xx}-aF_{xx}\right)\left(A1+qx\right)=0$$

Equation (7-8)

or
$$g_{xz,xx}^0-w^2g_{xz}^0=\frac{\left(D_{xx}-aF_{xx}\right)}{a^2\left(D_{xx}H_{xx}-F_{xx}^2\right)}\left(A1+qx\right) \tag{7-9}$$

where

$$w=\sqrt{\frac{D_{xx}\left(A_{xz}-2bD_{xz}+b^2F_{xz}\right)}{a^2\left(D_{xx}H_{xx}-F_{xx}^2\right)}} \tag{7-10}$$

The solution to Equation (7-9) is found following the procedure in Dufort et al. [2][1]:

$$g_{xz}^0(x)=\left(A_1+qx\right)\frac{D_{xx}-aF_{xx}}{D_{xx}\left(A_{xz}-2bD_{xz}+b^2F_{xz}\right)}+A_2Cosh(wx)+A_3Sinh(wx)$$

Equation (7-11)

The expression for f_x is found by substituting for $g_{xz}^{(0)}$ in the second of Equation (7-7) and then integrating:

$$f_x(x)=A_4+A_5x-\frac{1}{6}\frac{x^2}{D_{xx}}\left(3A_1+qx\right)+A_2\frac{F_{xx}}{D_{xx}}aCosh(wx)+A_3\frac{F_{xx}}{D_{xx}}aSinh(wx)$$

Equation (7-12)

Similarly, an expression for w_o is found by using $g_{xz}^{(0)}=f_x+w_{o,x}$:

$$w_o(x)=A_6-\frac{1}{2}A_5x^2-\left[A_4+\frac{\left(D_{xx}-aF_{xx}\right)}{D_{xx}\left(A_{xz}-2bD_{xz}+b^2F_{xz}\right)}A_1\right]x+\frac{D_{xx}-aF_{xx}}{D_{xx}w}\left[A_3Cosh(wx)+A_2Sinh(wx)\right]+$$
$$\frac{A_1}{6D_{xx}}x^3+q\left[\frac{x^4}{24D_{xx}}+\frac{\left(D_{xx}-aF_{xx}\right)}{D_{xx}\left(A_{xz}-2bD_{xz}+b^2F_{xz}\right)}x^2\right]$$

Equation (7-13)

[1] Note that the solution obtained here has a slightly different form that that obtained by Dufort et al. due to the different kinematic definitions.

This procedure generates 6 unknown constants which can be found using the 6 boundary conditions given in Equation (6-6).

8 Stress and Strain Calculations

Once the solution in Equations (7-11), (7-12), and (7-13) is found, local panel strains can be found using the flange and web kinematic expressions in Equations (1-8) and (1-15). Ply-level stresses can then be obtained using usual constitutive relations for plane stress employed in CLT:

$$\begin{Bmatrix} s_x \\ s_y \\ s_{xy} \end{Bmatrix} = \left[\overline{Q}\right] \begin{Bmatrix} e_x \\ 0 \\ 0 \end{Bmatrix} \tag{8-1}$$

This theory predicts a parabolic shear stress/strain distribution through the beam depth. The method of Barbero et al. in which the axial stress equilibrium equation is integrated *a posteriori* can be used to estimate the actual shear flow. The reader is referred to Reference [1] for additional information.

Figures

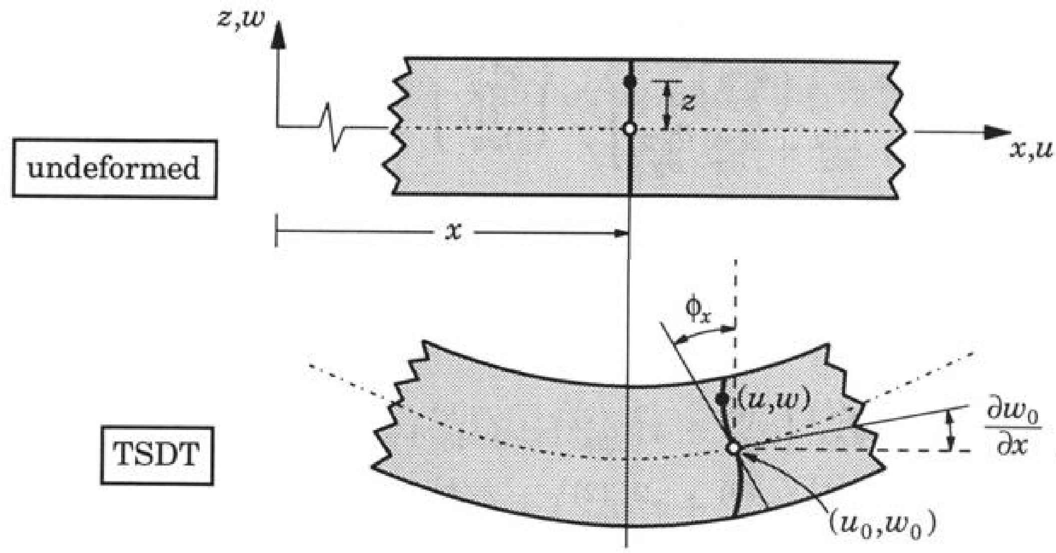

Figure A.1. Third order beam/plate kinematics. (*Figure reproduced from Reference* [3] *with permission from CRC Press.*)

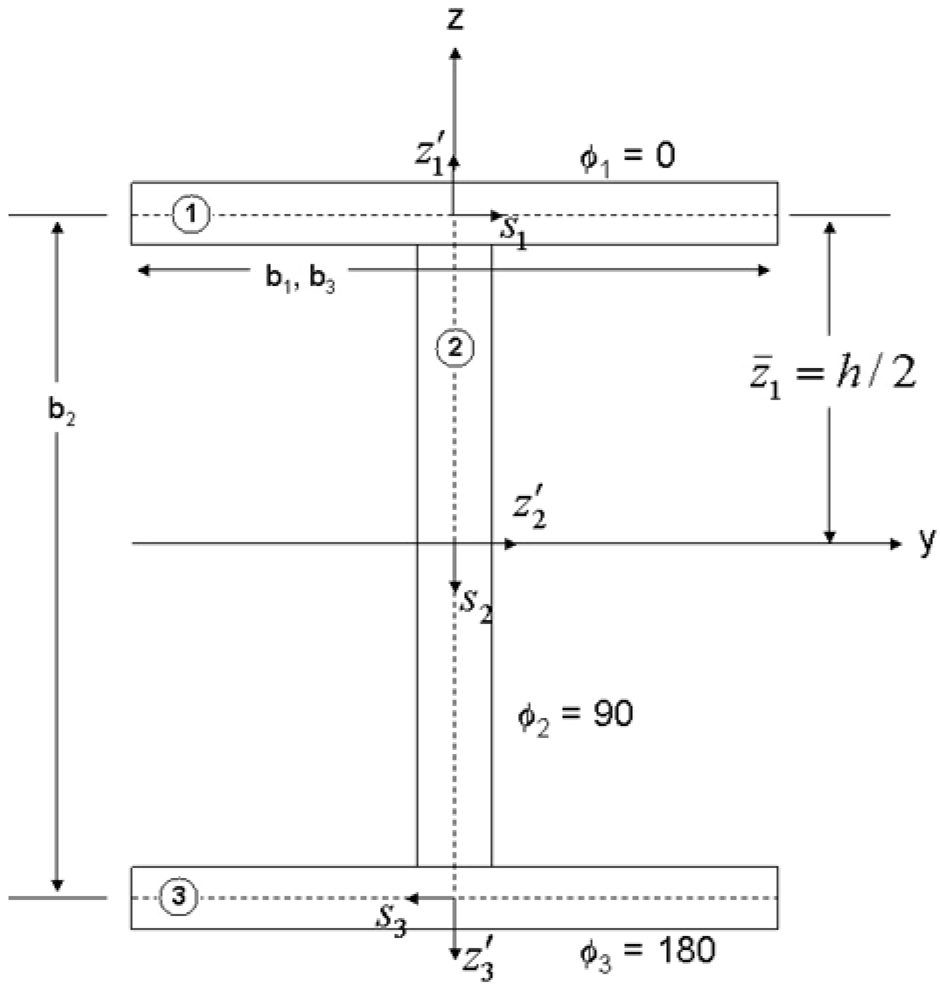

Figure A.2. Cross-section geometry and panel level coordinate system.

References

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3. Reddy, J.N., *Mechanics of Laminated Composite Plates: Theory and Analysis*. 1997, New York: CRC Press, Inc.
4. Reddy, J.N., C.M. Wang, and K.H. Lee, *Relationships between bending solutions of classical and shear deformation beam theories.* International Journal of Solids & Structures, 1997. **34**(26 Sep): p. 3373-3384.