# Optimal Point Sets with Few Distinct Triangles

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**ABSTRACT.** In this thesis we consider the maximum number of points in $\mathbb{R}^d$ which form exactly $t$ distinct triangles, which we denote by $F_d(t)$. We determine the values of $F_d(1)$ for all $d \geq 3$, as well as determining $F_3(2)$. It was known from the work of Epstein et al. [Ep] that $F_2(1) = 4$. Here we show somewhat surprisingly that $F_3(1) = 4$ and $F_d(1) = d + 1$, whenever $d \geq 3$, and characterize the optimal point configurations. We also show that $F_3(2) = 6$ and give one such optimal point configuration. This is a higher dimensional extension of a variant of the distinct distance problem put forward by Erdős and Fishburn [EF].

# Optimal Point Sets with Few Distinct Triangles

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**GENERAL AUDIENCE ABSTRACT.** In this thesis we consider the following question: Given a number of triangles, $t$, where each of these triangles are different, we ask what is the maximum number of points that can be placed in $d$-dimensional space, such that every triplets of these points form the vertices of only the $t$ allowable triangles. We answer this for every dimension, $d$ when the number of triangles is $t = 1$, as well as show that when $t = 2$ triangle are in $d = 3$-dimensional space. This set of questions rises from considering the work of Erdős and Fishburn in higher dimensional space [EF].

*This work is dedicated to my parents, grandfather, and the memory of my late grandmother. You have always supported, guided, and encouraged me, and for this I am eternally grateful.*

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# Chapter 1

# Introduction

## 1.1 Historical Background:

1946 Erdős proposed his distinct distance conjecture, which stated that any set of $n$ points in the plain will define at least $\Omega(n/\sqrt{\log n})$ distinct distances [Er46]. Since that time, optimal points sets have been a heavily studied topic within the field of discrete geometry. Guth and Katz made significant progress towards proving this conjecture when they showed in 2010 that a set of $n$ points in the plane defined at least $\Omega(n/\log n)$ distinct distances. The analogous problems in dimensions 3 and higher remain open.

In 1996 Erdős and Fishburn asked a question related to this: Given a positive integer $k$, what is the maximum number of points which can be embedded in the plane such that precisely $k$ distinct distances are defined, and can all such point configurations be characterized? In their paper, Erdős and Fishburn characterized the optimal configurations for $1 \le k \le 4$, and this was extended by Shinohara for $k = 5$ and by Wei for $k = 6$. Further, Erdős conjectured that for sufficiently large values of $k$, an optimal point configuration exists in the triangular lattice, which continues as an open conjecture. (Figure 1.1 shows the optimal configurations for $k$ distinct distances in the plane when $2 \le k \le 6$.)

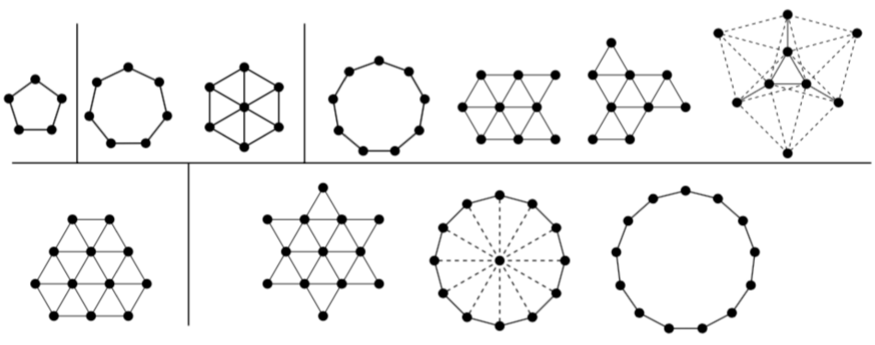


Figure 1.1: Optimal, or maximal, point set configurations determining exactly $k$ distinct distances in the plane, for $2 \le k \le 6$ [BMP]. For all $k$, $2 < k \le 6$, there exists a configuration in the triangular lattice; Erdős conjectured that this is always true when $k$ is sufficiently large.

Erdős' distance problem can be extended to consider triangles in place of distances. Since the set of distances generated by a point set can be thought of as being determined by the collection of 2-point subsets, we may analogously consider the set of triangles formed by a point set as determined by the collection of 3-point subsets. Erdős' distance problem then becomes: What is the minimum number of distinct triangles formed by a collection of $n$ points in the plane? Hence, the analogue of Erdős' and Fishburn's problem is: Given $t$ distinct triangles, with $t$ fixed, what is the maximum number of points, $n$, placed in the plane which define exactly $t$ distinct triangles? Epstein et al. focused on the latter of these analogues and showed that $n = 4$ for $t = 1$, $n = 5$ for $t = 2$, and conjectured that $n = 6$ for $t = 3$ [Ep]. Maximal point sets in the plane remains an open question for higher values of $t$.

## 1.2 Main Results:

As mentioned above, the higher dimensional analogues of Erdős' distance problem are as yet open. Here we concern ourselves with the higher dimensional analogue of Erdős' and Fishburn's question, rather than with higher values of $k$. Our main results are the following two theorems:

**Theorem 1.2.1.** *Optimal Point Sets Determining a Single Triangle:*
*Let $F_d(t)$ denote the maximum number of points which can be placed in $\mathbb{R}^d$ to determine exactly $t$ distinct triangles. Then*

1. *$F_3(1) = 4$ and the only configurations which achieve this are when the points lie on the vertices of a square, a rectangle, or a tetrahedron, and*

2. *$F_d(1) = d + 1$ when $d > 3$ and the only configuration which achieves this is the regular $d$-simplex.*

**Theorem 1.2.2.** *Optimal Point Sets determining Two Triangles in $\mathbb{R}^3$:*
*Let $F_d(t)$ denote the maximum number of points which can be placed in $\mathbb{R}^d$ to determine exactly $t$ distinct triangles. Then $F_3(2) = 6$ and the vertices of a regular octahedron achieves this value.*

To prove Theorem 1.2.1(1) we characterize all possible 5 point configurations and demonstrate that each determines at least two distinct triangles, and it quickly follows that $F_3(1) = 4$. For Theorem 1.2.1(2) we show that much of the proof for $d = 3$ generalizes to $d$-dimensional space, and that the only point configuration which does not determine two or more distinct triangles is the $d$-simplex.

Theorem 1.2.2 is proven by showing that $F_3(2) < 7$, and then by presenting a 6-point set, vis. the vertex set of the regular octahedron, defining only two distinct triangles.

# Chapter 2

# Observations & Conjectures:

## 2.1 Observations in Connection with Theorem 1.2.1:

We make three observations in connection with Theorem 1.2.1. First, in $\mathbb{R}^d$, $d > 3$ the $d$-simplex is the unique optimal point configuration, yet in dimensions 2 and 3 this is not so. Second, in addition to the above, the 2-simplex fails to be optimal in $\mathbb{R}^2$. Third, note that in $\mathbb{R}^3$ there are two optimal configurations, vis. the vertex set of the regular tetrahedron (i.e. the 3-simplex) and the rectangle (to include the square), while in $\mathbb{R}^2$ and $\mathbb{R}^d$, $d > 3$, there is a single family of solutions (when considering the square to be a special case of the rectangle.), and hence, the $d$-simplex fails to be unique. This transition that happens in $\mathbb{R}^3$ is both surprising and novel.

## 2.2 Conjecture Regarding Higher Dimensions:

The first conjecture we make deals with point sets determining 2 triangles in higher dimensional space and is motivated by the results of Theorem 1.2.2. It is the following:

**Conjecture 2.2.1.** *Optimal Point Sets Determining Two Triangles in $\mathbb{R}^d$, $d \geq 3$:*
*Let $F_d(t)$ denote the maximum number of points which can be placed in $\mathbb{R}^d$ to determine exactly $t$ distinct triangles. Then for $d \geq 3$, $F_d(2) = 2d$ and the vertices of the $d$-orthoplex, or $d$-dimensional cross-polytope, achieves this maximum.*

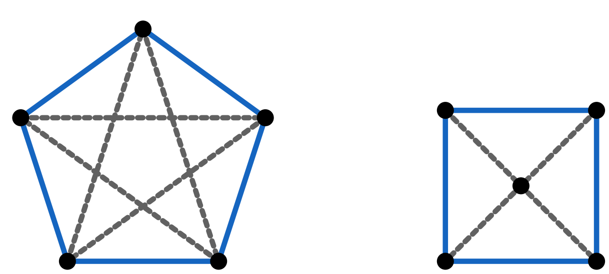


Figure 2.1: The vertices of the regular pentagon and the square with its center point are the only optimal point sets determining exactly two triangles in $\mathbb{R}^2$ [Ep].

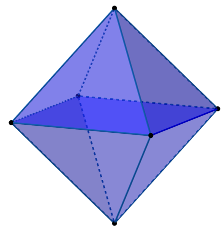


Figure 2.2: Here is the 3-orthoplex, or regular octahedron. It is an optimal point set determining 2 distinct triangles in $\mathbb{R}^3$. It is conjectured that is $\mathbb{R}^d$, $d \geq 3$, the $d$-orthoplex is always optimal.

If Conjecture 2.2.1 holds, then the geometry of optimal 2 triangle point sets in $\mathbb{R}^2$ and those in $\mathbb{R}^d$, with $d \geq 3$, differ. In $\mathbb{R}^2$ we have the vertices of the regular pentagon and the square with its center, see Figure 2.1. But in $\mathbb{R}^d$, $d \geq 3$, the, possibly unique, solutions take on the new geometric structure of the $d$-orthoplex. See Figure 2.2 for a representation of the 3-orthoplex or the regular octahedron. We again note that the 2-orthoplex (i.e. the square) is not an optimal point set determining 2 triangles in $\mathbb{R}^2$, in much the same fashion as the 2-simplex fails to be optimal in determining a single triangle in $\mathbb{R}^2$. Hence, the phase shift in the geometry of optimal points in $\mathbb{R}^2$ and those in $\mathbb{R}^d$, $d \geq 3$, is again present.

## 2.3 Conjecture Regarding Three Distinct Triangles:

We also make a conjecture about the maximum number of points in $\mathbb{R}^3$ which determine exactly 3 triangles. This conjecture is motivated by observations made while constructing the proof of Theorem 1.2.2. It is formally stated as follows:

**Conjecture 2.3.1.** *Optimal Point Sets Determining Three Triangles in* $\mathbb{R}^3$*:*
*Let* $F_d(t)$ *denote the maximum number of points which can be placed in* $\mathbb{R}^d$ *to determine exactly* $t$ *distinct triangles. Then* $F_3(3) = 8$ *and the vertices of a cube achieve this maximum.*

Conjecture 2.3.1, if true, would seem to be a very natural occurrence. In $\mathbb{R}^3$, we have seen that Platonic solids have appeared as optimal point sets before, vis. for $F_3(1) = 4$ the vertices of the regular tetrahedron and for $F_3(2) = 6$ the vertices of the regular octahedron. See Figure 2.3 for a representation of the cube.

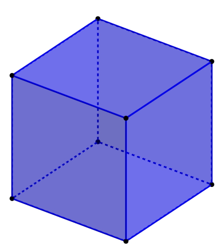


Figure 2.3: It is conjectured that the cube is an optimal point set determining exactly 3 distinct triangles in $\mathbb{R}^3$.

# Chapter 3

# Proof of Theorem 1.2.1: Optimal point sets determining a single triangle

## 3.1 Definitions and Lemmas

We formalize the concepts of a triangle and set out our notation with the following definitions:

**Definition 3.1.1.** *Given a finite point set $P \subset \mathbb{R}^d$, $d \geq 3$, we say two triples $(a, b, c)$, $(a', b', c') \in P^3$ are equivalent if there is an isometry mapping one to the other, and we denote this as $(a, b, c) \sim (a', b', c')$.*

**Definition 3.1.2.** *Given a finite point set $P \subset \mathbb{R}^d$, $d \geq 3$, we denote by $P^3_{nc}$ the set of non-collinear triples $(a, b, c) \in P^3$.*

**Definition 3.1.3.** *Given a finite point set $P \subset \mathbb{R}^2$, we define the set of distinct triangles determined by $P$ as*

$$T(P) := P^3_{nc} / \sim . \tag{3.1}$$

In this paper when we discuss and count the number of distinct triangles of a finite point set $P \in \mathbb{R}^d$ we are precisely working with the set $T(P)$. Note that this excludes degenerate triangles where all three points lie on a line.

**Definition 3.1.4.** *Let $p$ and $q$ be points in $\mathbb{R}^d$ for $d \geq 1$. We denote the Euclidean distance between $p$ and $q$ by $d(p, q)$.*

Our proof of Theorem 1.2.1 relies on the following lemma. The proof of this lemma is located in the chapter on proofs of lemmas.

**Lemma 3.1.5.** *Let $S$ be a set of points in $\mathbb{R}^d$, $d \geq 3$, which defines a single distinct equilateral triangle. Then $S$ has at most $d + 1$ points.*

## 3.2 Proof of Theorem 1.2.1

Below is the proof of Theorem 1.2.1. The structure of the proof is to show first that Theorem 1.2.1(1) holds, and then to generalize many of these arguments to prove Theorem 1.2.1(2), noting only where the arguments differ from the 3-dimensional case.

### 3.2.1 Proof of Theorem 1.2.1(1)

We believe that the clearest proof of Theorem 1.2.1(1) is to characterize all of the 5 point configurations and show that these lead to 2 or more distinct triangles. Consider the following cases:

**Case 1:** All 5 points lie in the same plane, $P$.

**Case 2:** 4 points lie in the same plane, $P$, while the $5^{\text{th}}$ is out of the plane $P$.

**Case 3:** No 4 point subset of the 5 points lie in the same plane.

*Proof of Theorem 1.2.1(1).*
Let $S = \{A, B, C, D, E\}$ be a set of 5 points in $\mathbb{R}^3$.

**Case 1:** Suppose that the points of $S = \{A, B, C, D, E\}$ all lie in the plane $P$. Then by the work of Epstein et al. [Ep], we see that this must define at least 2 distinct triangles, so we are done.

**Case 2:** Suppose that 4 points of $S$ lie in the same plane, $P$.
WLOG, suppose that $\{A, B, C, D\}$ lie in the plane $P$, and $E$ lies outside of $P$.
Drawing once again from the work of Epstein et al. [Ep], it follows that $\{A, B, C, D\}$ must lie at the vertices of either a square of a rectangle, else the configuration will define at least 2 distinct triangles. Hence, we consider two cases here:

**Subcase 1.** Suppose that $\{A, B, C, D\}$ form a non-square rectangle in the plane $P$. Then the single distinct triangle that is formed by these points, call it $T$, has 3 distinct edge lengths, call these $d_1$, $d_2$, and $d_3$.
Choose a point in $\{A, B, C, D\}$, WLOG, suppose we choose $A$.
Note that each distance $d(A, q)$, where $q \in \{B, C, D\}$, is in $\{d_1, d_2, d_3\}$ and these are distinct. That is, the distances $d(A, B)$, $d(A, C)$, and $d(A, D)$ are precisely $d_1$, $d_2$, and $d_3$, up to some permutation. WLOG, suppose $d(A, B) = d_1$, $d(A, C) = d_2$, and $d(A, D) = d_3$.
Now, consider the distance $d = d(A, E)$:
Suppose first that $d \notin \{d_1, d_2, d_3\}$. Then we note that the triangle formed by $A$, $E$, and any of the other 3 points clearly defines a triangle distinct from $T$. Thus, we see that $d \in \{d_1, d_2, d_3\}$.
WLOG, suppose that $d = d_1$.
Then consider the triangle $AEB$. This triangle cannot be congruent to $T$ since edges $AE$ and $AB$ are equal, while in $T$ all the edge lengths are distinct.

Thus, we have that if $\{A, B, C, D\}$ lie at the vertices of a non-square rectangle in the plane $P$, the addition of a point $E$ out of the plane $P$ necessarily defines a second, triangle distinct from $T$.

**Subcase 2.** Suppose that $\{A, B, C, D\}$ lie at the vertices of a square in the plane $P$.
By the work of Epstein et al. [Ep], we know that this defines a single unique triangle, call it $T$, and further that $T$ is an isosceles triangle. Call the repeated edge length of $T$ $d_1$ and the unique edge length of $T$ $d_2$.
Choose a point in $\{A, B, C, D\}$, WLOG, suppose we choose $A$.
Note that two of the distances $d(A, B)$, $d(A, C)$, and $d(A, D)$ equal $d_1$ and the third is $d_2$. WLOG, suppose that $d(A, B) = d(A, D) = d_1$ and $d(A, C) = d_2$.
Consider the distance $D = d(A, E)$:
We see that $D \in \{d_1, d_2\}$, since if it were not, the triangle formed by $A$, $E$, and any other point would clearly be distinct from $T$.
Suppose that $D = d_2$. Then consider the triangle formed by $A$, $E$, and $C$. We note that edges $AE$ and $AC$ are both of length $d_2$ and so we obtain a triangle which is not congruent to $T$.
The last possibility to consider is $D = d_1$. In this situation we make 3 observations:

**1)** Since $AB$ and $AE$ are of length $d_1$, $BE$ must be of length $d_2$, else we obtain a triangle distinct from $T$.

**2)** Since $AB$ and $AD$ are of length $d_1$, $BD$ is of length $d_2$ by the same reasoning as above.

**3)** Since $AD$ and $AE$ are of length $d_1$, $DE$ is of length $d_2$ again by the same reasoning as above.

Noting these observations, we see that the triangle $BDE$ is an equilateral triangle of side length $d_2$ and is therefore trivially non-congruent to $T$.

Hence, a 5 point configuration with 4 co-planer points defines at least 2 distinct triangles.

**Case 3:** Suppose the set $S$ is such that no 4 point subset is co-planer.
Assume that this configuration defines only 1 unique triangle, $T$.
Consider the 3 subcases:

**Subcase 1.** Suppose that $T$ is an equilateral triangle.
Then, by Lemma 3.1.5 we immediately have a contradiction since the 3-simplex has at most $3 + 1 = 4$ points.

**Subcase 2.** Suppose that $T$ is an isosceles triangle with repeated edge length $d_1$ and unique edge length $d_2$.
Choose a point is $S$, WLOG, let this point be $A$.
Note that there are 4 edges exiting $A$.
If two of these edges are of length $d_2$, then the triangle formed by $A$ and the 2

points at distance $d_2$ from $A$ form a triangle not congruent to $T$. Hence, at most one edge coming out of $A$ is of length $d_2$, WLOG, suppose $d(A,C) = d_2$.
Then the other three edges exiting $A$ must be of length $d_1$. Consider the 3 triangles which are formed by taking the point $A$ and two points at distance $d_1$ from $A$, i.e. 2 points from $\{B, D, E\}$.
All three of these must be congruent to $T$ and so the edges $BD$, $BE$, and $DE$ must be of length $d_2$. But then the triangle $BDE$ is a equilateral triangle and so not congruent to $T$.

**Subcase 3.** Suppose that $T$ is such that all three edges are of different lengths, call them $d_1$, $d_2$, and $d_3$.
Choose a point is $S$, WLOG, let this point be $A$.
Again note that there are 4 edges exiting $A$.
Since each of these must be of length $d_1$, $d_2$, or $d_3$, it follows clearly that at least 2 of these edges must be of equal length. WLOG, suppose that $d(A,B) = d(A,C)$.
Then we note that the triangle $ABC$ is an isosceles triangle (and possibly equilateral if $d(B,C) = d(A,B)$ as well) and therefore clearly non-congruent to $T$.

Hence we see that in all three subcases there are at least 2 distinct triangles.

Therefore, every configuration of 5 points in $\mathbb{R}^3$ defines at least 2 distinct triangles, and it follows, as desired, that $F_3(1) = 4$. □

### 3.2.2 Proof of Theorem 1.2.1(2)

To prove Theorem 1.2.1(2) we follow the same structure as in dimension 3, differing only where the ambient dimension of the space comes into play within the argument. Hence, we once again proceed by cases:

*Proof of Theorem 1.2.1(2).*
Fix $d \geq 4$ and let $S = \{A_1, \ldots, A_{d+2}\}$ be a set of $d+2$ points in $\mathbb{R}^d$ which determine a single distinct triangle, $T$.

**Case 1:** Suppose that 5 or more of the points of $S$ lie in the same plane. Then we see that from the work of Epstein et al. [Ep] that $S$ defines at least 2 distinct triangles, a contradiction.

**Case 2:** Suppose that a 4 point subset of $S$, call it $X = \{A_1, A_2, A_3, A_4\}$, lies in the same plane, $P$. We note here that the argument in the corresponding case of 1.2.1(1) relies only on the fact that the dimension of the space is at least one larger than that of the plane $P$. Since $\dim(P) = 2$ and we are in $\mathbb{R}^d$, $d \geq 4$, this assumption holds here and an analogous contradiction arises here in this case.

**Case 3:** Suppose that $S$ is such that no 4 points lie in the same plane. We again consider 3 subcases:

**Subcase 1:** Suppose that $T$ is an equilateral triangle. Then, by Lemma 3.1.5, we have a simplex, but this is a contradiction since the $d$-simplex has $d+1$ points, but $S$ has $d+2$ points.

**Subcase 2:** Suppose in this case that $T$ is an isosceles triangle. Here we note that the argument in 1.2.1(1) for the corresponding subcase relies again only on the ambient dimension being at least one larger that that of a plane, hence, the argument transfers over, and this case leads to a contradiction.

**Subcase 3:** Suppose that $T$ is such that all the edge lengths are unique. Here, we again note that the argument from the corresponding subcase of the proof of 1.2.1(1) applies in this setting, and we obtain a contradiction.

Since every case leads to a contradiction, it follows that $F_d(1) = d+1$, for $d > 3$, as desired. $\square$

# Chapter 4

# Proof of Theorem 1.2.2: Optimal point sets determining two triangles in $\mathbb{R}^3$

## 4.1 Additional Definitions and Lemmas:

We give the following additional definitions and lemmas in order to streamline to proof of Theorem 1.2.2.

**Definition 4.1.1.** *We denote a triangle with edge lengths* $e_1, e_2$*, and* $e_3$ *by* $\Delta\{e_1, e_2, e_3\}$*.*

**Lemma 4.1.2.** *A collection of points lying on the surface of a sphere in* $\mathbb{R}^3$ *determining only two distinct triangles and two distinct distances, one of which is the radius of the sphere can have at most 4 points.*

**Definition 4.1.3.** *Let* $t_1$ *and* $t_2$ *be triangles. We say that* $t_1$ *and* $t_2$ *associate in a if either the two triangles share an edge length in common, or if there is a finite sequence of allowable triangles,* $T_0, \ldots, T_k$*, which are such that* $t_1 = T_0$*,* $t_2 = T_k$*, and for every pair of triangles,* $(T_{i-1}, T_i)$ *with* $1 \leq i \leq k$*, share an edge length in common. Further, we call the sequence* $T_0, \ldots, T_k$ *an associating sequence of triangles.*

**Lemma 4.1.4.** *Suppose that* $S$ *is a collection of* $n$ *points determining* $t$ *distinct triangles. Then the triangles are such that given any 2, an associating sequence of triangles may be constructed such that all of the triangles in the sequence are congruent to some of the* $t$ *triangles determined by* $S$*.*

**Lemma 4.1.5.** *Suppose that* $S$ *is a collection of* $n$ *points determining* $t$ *distinct triangles. Then the number of distinct distances is at most* $2t + 1$*.*

## 4.2 Proof of Theorem 1.2.2

We next prove Theorem 1.2.2. The structure of this proof is to consider all possible configurations of 7 points in $\mathbb{R}^3$, and to show that all such configuartions must determine at least 3 distinct triangles. This shows that $F_3(2) \leq 6$. To show equality, we present a 6 point set which determines only 2 distinct triangles, vis. the vertices of the 3-orthoplex, or regular octahedron.

### 4.2.1 Proof of Theorem 1.2.2

*Proof.* Suppose that $S$ is a 7 point set in $\mathbb{R}^3$ which determines exactly $t = 2$ distinct triangles, call these $t_1$ and $t_2$.

Since $t_1$ and $t_2$ must associate, and they are the only allowable triangles, they must share a common edge length. This is because no sequence of other allowable triangles may lie between $t_1$ and $t_2$ in an associating sequence, which pairwise share edges since there are no further allowable triangles.

By Lemma 4.1.5 we know that $S$ determines at most $2t + 1 = 2(2) + 1 = 5$ distinct distances. Hence, we can divide the proof into cases based upon the number of distinct distances determined by $S$.

**Case 1:** Suppose that $S$ determines 5 distinct distances. Denote these by $d_1, d_2, d_3, d_4$, and $d_5$.
Since there are only two distinct triangles, by assumption, it follows that for all 5 distinct distances to appear, both of these triangles must be scalene, having but a single edge length in common.
Fix one point in $S$, call it $\mathcal{O}$, and note that there are 6 distances from $\mathcal{O}$ to each of the other points in $S$. Since there can be at most 5 distinct distances present, it follows that 2 of these other points are of the same distance from $\mathcal{O}$. Denote these points by $A$ and $B$, and WLOG, suppose the distance is $d_1$. Then we have that, $\overline{\mathcal{O}A} = \overline{\mathcal{O}B} = d_1$. Note that the triangle $\Delta\mathcal{O}AB$ is isosceles (and equilateral if $\overline{AB} = d_1$), and is therefore non-congruent to either of the scalene triangles, which which implies that $S$ determines a third distinct triangle, a contradiction.

**Case 2:** Suppose that $S$ determines 4 distinct distances. Denote these by $d_1, d_2, d_3$, and $d_4$. Note that neither of the two triangles determined by $S$ can be equilateral, and at most one can be isosceles. This will be argued by contradiction.

**Proof of Note:** Suppose one of the triangles determined by $S$, $t_1$, is equilateral. The second triangle, $t_2$, must associate with $t_1$, and since there are no other allowable triangles in $S$, $t_2$ must share at least one edge in common with $t_1$. This leaves at most 2 edges (of $t_2$) with edge lengths not yet assigned, but by assumption, there are 3 remaining distinct distances that must be present. Thus, Neither triangle may be equilateral.
Now suppose that both $t_1$ and $t_2$ are isosceles. This means that each of the 2 triangles determined by $S$ contribute 2 distinct edge lengths for a maximum total of 4 distinct edge lengths. Since there must be, by assumption, 4 distinct distances determined by $S$, it follows that this maximum is achieved. However, this implies that no edges are shared between $t_1$ and $t_2$, and since there are no other allowable

triangles in $S$, this implies that $t_1$ and $t_2$ do not associate. Hence, at most one of the two triangles is isosceles. This concludes the proof of the note.

As in the previous case, fix one point in $S$, call it $\mathcal{O}$, and note that there are 6 distances from this point to the other 6 points in $S$, which can be labeled by $\{A, B, C, D, E, F\}$. Since there are 4 distinct distances in $S$, at least 2 of the those 6 points must be equidistant from $\mathcal{O}$. WLOG, suppose it is $A$ and $B$, and label this distance by $d_1$.
Consider the remaining 4 distances from $\mathcal{O}$ to $\{C, D, E, F\}$. Suppose first that one of these is $d_1$, say $\overline{\mathcal{O}C}$. Then, since the triangles $\Delta\mathcal{O}AB$, $\Delta\mathcal{O}CB$, and $\Delta\mathcal{O}AC$ all have two edges of length $d_1$, it follows from the note above that these must all be congruent triangles, since there can be at most one isosceles triangle, and hence $\overline{AB} = \overline{BC} = \overline{AC}$. But this implies that $\Delta ABC$ is equilateral, a contradiction of the note. Hence, none of the 4 distances from $\mathcal{O}$ to $\{C, D, E, F\}$ is $d_1$.
But since there are 4 such distances and only 3 allowable distances, it follows that two of these must be the same, WLOG, suppose $\overline{\mathcal{O}C} = \overline{\mathcal{O}D} = d_2$. However, this implies that $\Delta\mathcal{O}AB$ and $\Delta\mathcal{O}CD$ are non-congruent isosceles triangles, a contradiction of the note proven above.
Since every case leads to a contradiction, it follows that $S$ cannot determine 4 distinct distances.

**Case 3:** Suppose that $S$ determines 3 distinct distances. Denote these by $d_1, d_2$, and $d_3$. Up to a relabeling of the distances, there are 5 possible pairs of triangles in this case. These pairs are given in Table 4.1.

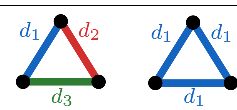
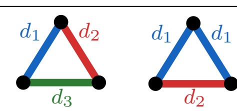
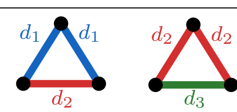
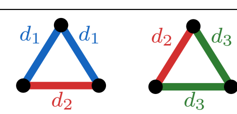
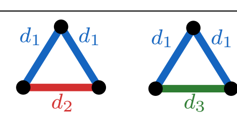
| Pair # | $t_1$ | $t_2$ | Coloured Graph |
|---|---|---|---|
| 1 | $\Delta\{d_1, d_2, d_3\}$ | $\Delta\{d_1, d_1, d_1\}$ |   |
| 2 | $\Delta\{d_1, d_2, d_3\}$ | $\Delta\{d_1, d_1, d_2\}$ |   |
| 3 | $\Delta\{d_1, d_1, d_2\}$ | $\Delta\{d_2, d_2, d_3\}$ |   |
| 4 | $\Delta\{d_1, d_1, d_2\}$ | $\Delta\{d_2, d_3, d_3\}$ |   |
| 5 | $\Delta\{d_1, d_1, d_2\}$ | $\Delta\{d_1, d_1, d_3\}$ |   |

Table 4.1: Possible pairs of triangles in a point set with 2 triangles and 3 distances, up to relabeling edges.

We shall proceed by showing that each of these pairs of triangles lead to a contradiction.

**Pair #1:** Consider when $\{t_1, t_2\}$ are as in Pair # 1 in Table 4.1.

Consider 3 points in $S$, call these $A, B$, and $C$, which are such that $\Delta ABC \cong t_1$. WLOG, suppose that $\overline{AB} = d_1$, $\overline{AC} = d_2$, and $\overline{BC} = d_3$. This triplet of points exist since $S$ determines $t_1$. Let $D \in S$, be distinct from the points $\{A, B, C\}$. Since $S$ determines only the triangles $t_1$ and $t_2$ and $\overline{BC} = d_3$, it follows that $\Delta BCD \cong t_1$. Clearly, $\overline{CD} \neq d_3$.

If $\overline{CD} = d_2$, then we note that the triangle $\Delta ACD$ has two edges of length $d_2$, vis. $\overline{AC}$ and $\overline{CD}$, which is a contradiction, since this would imply that $S$ determines a third distinct triangle.

If, on the other hand, $\overline{CD} = d_1$, then we note that Given another distinct point, $E \in S$, $\overline{CE} = d_1$, since else we could relabel $E$ by $D$ and proceed as before. However, this means that the triangle $\Delta BDE$ has two edges of length $d_2$, a contradiction since this would imply, as before, that $S$ determines a third distinct triangle.

**Pair # 2:** Consider when $\{t_1, t_2\}$ are as in Pair # 1 in Table 4.1.

Here note that the argument from Pair # 1 relied solely on $t_1$ being scalene, and showed that in such cases at least 2 distinct isosceles triangles can be constructed, implying that $S$, determines at least 3 distinct triangles. Since $t_1$ here is scalene, the argument from Pair # 1 applies precisely in this case as well, meaning that Pair # 2 is also not possible.

**Pair # 3:** Since both $t_1$ and $t_2$ are determined by $S$, we know that both must appear, and since the shared edge length appears once on the first tirangle, and twice on the second, there exist 4 points in $S$, call them $A, B, C$, and $D$, such that $\Delta ABC \cong t_1$, $\Delta BCD \cong t_2$, and $\overline{CD} = d_3$. However, then the triangle $\Delta ACD$ has an edge of length $d_1$ and an edge of length $d_3$, which implies that $S$ determines a third distinct triangle, a contradiction. Hence, Pair #3 is not possible.

**Pair # 4:** Since both $t_1$ and $t_2$ are determined by $S$, we know that both must appear, and since they share only a single edge in common, vis. the edge of length $d_2$, it follows that there must four points in $S$, call them $A, B, C$, and $D$, such that $\overline{BC} = d_2$, $\Delta ABC \cong t_1$, and $\Delta BCD \cong t_2$. But then $\Delta ABD$ has one edge of length $d_1$ and another of length $d_3$, which means that is is congruent to neither $t_1$ nor $t_2$, but this imples that $S$ determines a third distinct triangle, a contradiction. Hence, Pair # 4 is not possible.

**Pair # 5:** Since both $t_1$ and $t_2$ are determined by $S$, we know that both must appear and share an edge of length $d_1$. If they lie such that the unique edge of each triangle (note both $t_1$ and $t_2$ are isosceles and so have a unique edge and a repeated edge) share a vertex, then the triangle formed by this shared vertex point and the points at the other ends of these edges has an edge of length $d_2$ (the unique edge of $t_1$) and an edge of length $d_3$ (the unique edge of $t_2$), meaning it is congruent to neither $t_1$ nor $t_2$, a contradiction. Hence, Pair #5 is not possible.

Since every possible pair of triangles determining exactly 3 distinct distances results in

$S$ determining a third distinct triangle, it follows that $S$ cannot determine 3 distinct distances.

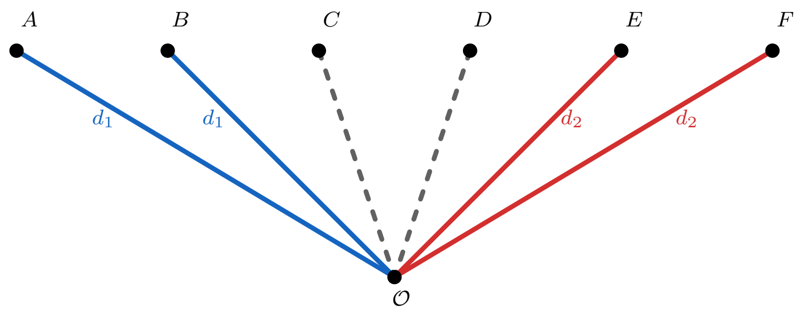



Figure 4.1: Given any choice of $\mathcal{O} \in S$, there are 2 points in $S$ at a distance of $d_1$ and 2 at a distance of $d_2$.

**Case 4:** Suppose that $S$ determines only 2 distinct distances. Denote these by $d_1$ and $d_2$. Suppose that there exists a point in $S$, call it $\mathcal{O}$, which is such that at least 5 of the other points in $S$ are at a distance $d_1$. Then these 5 points lie on the surface of a sphere of radius $d_1$, and determine exactly one other distances (vis. $d_2$). This is not possible, by Lemma 4.1.2. Hence, it follows that for any point in $S$, call it $\mathcal{O}$, there are two points in $S$, $A$ and $B$, such that $\overline{\mathcal{O}A} = \overline{\mathcal{O}B} = d_1$, and 2 other points in $S$, call them $E$ and $F$, such that $\overline{\mathcal{O}E} = \overline{\mathcal{O}F} = d_2$. Hence, for any choice of $\mathcal{O} \in S$, we get Figure 4.1.

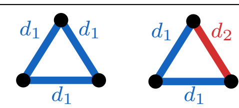
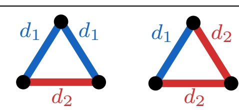
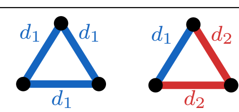
| Pair # | $t_1$ | $t_2$ | Coloured Graph |
|---|---|---|---|
| 1 | $\Delta\{d_1, d_1, d_1\}$ | $\Delta\{d_1, d_1, d_2\}$ |  |
| 2 | $\Delta\{d_1, d_1, d_2\}$ | $\Delta\{d_1, d_2, d_2\}$ |  |
| 3 | $\Delta\{d_1, d_1, d_1\}$ | $\Delta\{d_1, d_2, d_2\}$ |  |

Table 4.2: Possible pairs of triangles in a point set with 2 triangles and 2 distinct distances, up to relabeling edges.

We now note that with exactly 2 distinct triangles and 2 distinct distances, there are (up to relabeling) 3 possible pairs of triangles. These are given in Table 4.2. We shall proceed by showing that each of these pairs cannot occur.

**Pair # 1** Suppose that $\Delta\mathcal{O}AB \cong t_1$. (Note that if this is not the case, then $\Delta\mathcal{O}EF \cong t_1$ and we may exchange the labels on $d_1$ and $d_2$ and the labels on $A$ and $B$ with those on $E$ and $F$, yielding the assumed congruence.) Since $\Delta\mathcal{O}AB \cong t_1$ and $\Delta\mathcal{O}AB \ncong \Delta\mathcal{O}EF$, it is clearly follows that $\Delta\mathcal{O}EF \cong t_2$. However, this implies that $t_2$ has two edges of length $d_2$, a contradiction. Hence, Pair # 1 cannot occur.

**Pair # 2** Here we note that we can refine the situation of Figure 4.1 by noting that $\overline{AB} = d_2$ and $\overline{EF} = d_1$. Suppose that $\overline{\mathcal{O}C} = d_1$. If this is not the case, then we exchange the labels of $d_1$ and $d_2$, as well as $A$ and $B$ with $E$ and $F$.

Now note that $\Delta\mathcal{O}BC$ and $\Delta\mathcal{O}AC$ must be congruent to $t_1$ since the have 2 edges of length $d_1$. It follows then that $\overline{AC} = \overline{BC} = d_2$. But this implies that $\Delta ABC$ is an equilateral triangle with edge length $d_2$, a contradiction. So Pair # 2 cannot occur.

**Pair # 3** Here we refine the situation of Figure 4.1 by noting that $\overline{AB} = \overline{EF} = d_1$. If $\overline{\mathcal{O}C} = \overline{\mathcal{O}D} = d_1$ then we note that $\{\mathcal{O}, A, B, C, D\}$ are all related by a single distance and so would imply that the 3-simplex could contain 5 points, a contradiction. Hence, WLOG, $\overline{\mathcal{O}D} = d_2$. If $\overline{\mathcal{O}C} = d_2$ as well, then we note that the points $v = \{C, D, E, F\}$ are related by a single distance, vis. $d_1$, and so form the vertex set of a 3-simplex. Next note that each of the points $\mathcal{O}$, $A$, and $B$ relate to the points in $v$ solely by the distance $d_2$. Hence, each of these 3 points is equidistant from the points in $v$, but there can be only one such point, namely the circumcenter of the points in $v$. Hence, it must be that $\overline{\mathcal{O}C} = d_1$. But now the previous argument holds with $v = \{\mathcal{O}, A, B, C\}$. Thus, Pair # 3 cannot occur either.

Since all possible pairs of triangles determining exactly 2 distinct distances results in a contradiction, it follows that $S$ cannot determine 2 distinct distances. Further, since every possible number of distinct distances determined by $S$ results in a contradict, $S$ a 7 point set determining exactly 2 distinct triangles cannot exist in $\mathbb{R}^3$. Hence, $F_3(2) \leq 6$.

□

# Chapter 5

# Proofs of Lemmas:

We conclude by giving formal proofs of the lemmas upon which the proofs of our main results rest. They are proved in the order in which they were stated.

## 5.1 Definitions Needed for the Proofs of Lemmas

## 5.2 Proof of Lemma 3.1.5

It is well known in the literature that Lemma 3.1.5 holds and that in $\mathbb{R}^d, d \geq 2$ a set of points determining a single distinct distance has at most $d+1$ points. However, in the interest of completeness, we include here a proof by induction on the dimension, $d$, of the Lemma:

### 5.2.1 Proof of Lemma 3.1.5

*Proof.*

**Base Case:** $d = 3$**:** Let $S = \{A_1, A_2, A_3, A_4, A_5\} \subset \mathbb{R}^d = \mathbb{R}^3$ be a point set containing $d+2 = 5$ points, which defines a single distinct equilateral triangle, call it $T$.

Thus, the triangle $\Delta A_1 A_2 A_3$ must form the triangle $T$. Define $e$ to be the edge length of $T$, and let $P$ denote the plane defined by $\{A_1, A_2, A_3\}$.

Since $S$ defines an equilateral triangle, it follows that $A_4$ and $A_5$ must be equidistant from $\{A_1, A_2, A_3\}$ and lie upon a line normal to $P$, which goes through a point $p \in P$, where $p$ is equidistant to $\{A_1, A_2, A_3\}$, $p$ is called the circumcenter of the equilateral triangle $\Delta A_1 A_2 A_3$.

Since $p$ is the circumcenter of the equilateral triangle $\Delta A_1 A_2 A_3$, it follows that

$$d(A_1, p) = d(A_2, p) = d(A_3, p) = \frac{\sqrt{3}}{3} e$$

Since $S$ defines only the triangle $T$, to follows that $d(A_4, A_5) = e$. Since $A_4$, $A_5$, and $p$ lie upon the same line, it follows that:

$$d(A_4, p) + d(p, A_5) = d(A_4, A_5)$$

Since $A_4$ and $A_5$ are equidistant from $\{A_1, A_2, A_3\}$, they are also equidistant from the plane $P$, and hence, $d(A_4, p) = d(p, A_5) = \frac{e}{2}$. Applying the Pythagorean Theorem we obtain:

$$d(A_4, p)^2 + d(A_1, p)^2 = d(A_4, A_1)^2$$

Which yields:

$$\left(\frac{e}{2}\right)^2 + \left(\frac{\sqrt{3}e}{3}\right)^2 = (e)^2$$

Thus, we obtain:

$$\frac{1}{4}e^2 + \frac{1}{3}e^2 = e^2$$

Which implies that $\frac{7}{12} = 1$, a clear contradiction. We note that the vertices of a 3-simplex, the regular tetrahedron, gives a configuration in $\mathbb{R}^3$ which defines a single distinct equilateral triangle and has $3 + 1 = 4$ points. Therefore, a set $S$ defining a single distinct equilateral triangle in $\mathbb{R}^3$ can have at most $3 + 1 = 4$ points.

**Inductive Assumption:** Suppose that for dimensions $n < d$, a point set $S$ defining a single distinct equilateral triangle can have at most $n + 1$ points.

**Inductive Step:** Now let $S = \{A_1, \ldots, A_d, A_{d+1}, A_{d+2}\} \subset \mathbb{R}^d$ be a point set containing $d + 2$ points, which defines a single distinct equilateral triangle, call it $T$.
Thus, the points $\{A_1, \ldots, A_d\}$ must be such that any triplet forms the triangle $T$, and so they must form a $(d - 1)$-simplex (This by our Inductive Assumption). Call this $(d - 1)$-simplex $\Delta A_1 \cdots A_d$. Let $e$ be the edge length of $T$, and let $P$ be the $d - 1$ dimensional hyper-plane defined by $\{A_1, \ldots, A_d\}$.
Since $S$ defines an equilateral triangle, it follows that $A_{d+1}$ and $A_{d+2}$ must be equidistant to $\{A_1, \ldots, A_d\}$ and lie upon a line normal to $P$, which passes through a point $p \in P$ (vis. the circumcenter of $\Delta A_1 \cdots A_d$), where $p$ is equidistant from $\{A_1, \ldots, A_d\}$.
From this point, we can follow the same construction as in the Base Case, using the points $A_1$, $p$, $A_{d+1}$, and $A_{d+2}$ to arrive at a contradiction. We note here that the $d$-simplex gives a configuration in $\mathbb{R}^d$ which defines a single distinct equilateral triangle and has $d + 1$ points. Thus, a set $S$ in $\mathbb{R}^d$ defining a single distinct equilateral triangle can have at most $d + 1$ points, as desired.

□

## 5.3 Proof of Lemma 4.1.2

*Proof.* Let $S$ be a collection of points in $\mathbb{R}^3$ which determine exactly 2 distinct triangles and two distinct distances.
Since there are 2 distinct distances, the three choices for triangles in Table 4.2 are the cases to consider. For ease, we go through these pairs in reverse order.
Suppose that $\mathcal{O} \in S$ be such that a 5 point subset of $S$, $X = \{A, B, C, D, E\}$, is such that every member of $X$ is at a distance $d_1$ from $\mathcal{O}$.

**Pair # 3:** In this case, we note that must appear between 2 points in $X$, else the 5 points of $X$ would determine a single distance. Contradicting the fact that the 3-simplex has only 4 vertices.
WLOG, let $\overline{AB} = d_2$. But now the triangle $\Delta\mathcal{O}AB$ is such that only one edge is length $d_2$, and is therefore non-congruent to either of the assumed triangles.

**Pair # 2:** Here we note since every triangle with 2 vertices in $X$ and third vertex $\mathcal{O}$ has two edges of length $d_1$, all of the distances between the points in $X$ must be of distance $d_2$, but this means that the 5 points of $X$ are related by a single distance, a contradiction of the 3-simplex having 4 points. Hence, this pair is impossible.

**Pair # 1:** Here we note that two of the distances between points in $X$ mus be $d_2$, else we have 5 points related by $d_1$.
WLOG, suppose $\overline{AB} = \overline{CD} = d_2$. Then we note that given the allowable triangles, all remaining distances between points in $X$ must be $d_1$.
Consider the points $A, B, C, D$. Note first that they are all equidistant from both $\mathcal{O}$ and $E$, and so lie upon the circle defined by the intersection of two circles of radius $d_1$ centered at $\mathcal{O}$ and $E$. Further note that 4 of the distances between these 4 points are $d_1$ and the remaining 2, which do not share endpoints, are of $d_2$. Hence, these 4 points must be the vertex set of the square inscribed withing the circle of intersection.
If we rescale the point set such that $d_1 = 1$, then $d_2 = \sqrt{2}$ as it is the diagonal of the unit square. Let $p$ be the center point of this square, and note that $p$ is $\frac{1}{2}d_1 = \frac{1}{2}$ away from both $\mathcal{O}$ and $E$.
We can then use the Pythagorean theorem on the right triangle formed by the points $p, E, A$ to get:

$$\begin{aligned} d(A,E)^2 &= d(A,p)^2 + d(p,E)^2 \\ 1^2 &= \frac{\sqrt{2}}{2}^2 + \frac{1}{2}^2 \\ 1 &= \frac{2}{4} + \frac{1}{4} \end{aligned}$$

a clear contradiction. Hence, this pair is also impossible.

Since all pairs of allowable triangles lead to a contradiction, $S$ cannot be such that one point has 5 others which are equidistant from it. □

## 5.4 Proof of Lemma 4.1.4

*Proof.* Let $S$ be a collection of $n$ points determining $t$ distinct triangles. Let $t_1$ and $t_2$ be two of those $t$ triangles. Note that there exist $A, B, C \in S$, such that $\Delta ABC \cong t_1$. Further, there exist points $D.E.F \in S$ such that $\{A, B, C\} \neq \{D, E, F\}$ (although some of the points may be common to both sets) and $\Delta DEF \cong t_2$. Now consider the process of exchanging a single point in $\{A, B, C\}$ with another point in $S$, with the sole restriction that the newly chosen point cannot be co-linear with the two points which remain from the previous triplet. Note that this new triplet of points is, by definition, one of the $t$ triangles determined by $S$. Label this triangle $T_1$. Note that $t_1$ and $T_1$ associate since they share an edge (vis. that formed by the two points in common between the two triplets). We can continue this process, labeling the resulting triangles by $T_i$, where $i$ is the total number of exchanges that have been made, until we obtain the triplet $\{D, E, F\}$. Let $k$ be the index of the final $T_i$, and let $T_0 = t_1$. Then we have constructed a sequence of triangles $T_1, \ldots, T_k$, such that every pair $(T_{j-1}, T_j)$ share an edge in common, $t_1 = T_0$, and $t_2 = T_k$. So, by definition, we have that $t_1$ and $t_2$ associate. □

## 5.5 Proof of Lemma 4.1.5

*Proof.* We proceed by induction upon, $t$, the number of distinct triangles determined by our point set, $S$.

**Base Case:** Suppose that $S$ is a point set in $\mathbb{R}^d$, $d \geq 2$, which determine exactly 1 triangle. Since there is only 1 triangle, the number of distinct edges is at most 3. Since $3 = 2(1) + 1 = 2t + 1$, the base case holds.

**Inductive Assumption:** Suppose that given a point set, $X$, determining $t - 1$ distinct triangles, the maximum number of distinct distances determined by $X$ is $2(t-1)+1 = 2t - 1$.

**Inductive Step:** Suppose that $S$ is a point set determining exactly $t$ distinct triangles. Label these $t$ triangles by $T_1, \ldots, T_t$. Let $X \subset S$ be the subset of points which determine only the triangles $T_1, \ldots, T_{t-1}$. Then by our assumption, we note that $X$ determines at most $2t-1$ distinct points. We further note that by Lemma 4.1.4 $T_t$ must associate with some triangle $T_i$ with $1 \leq i \leq t-1$. Since these 2 triangles associate, $T_t$ must have at least 1 edge in common with $T_i$, and so it can introduce at most 2 new edge lengths. So, the maximum number of distinct edges determined by $S$ is the maximum number of distinct edges determined by the first $t-1$ triangles plus at most 2 new edges from the $t^{\text{th}}$ triangle. Hence, the maximum number of distinct distances determined by $S$ is $2t - 1 + 2 = 2t + 1$, as desired.

□

# Bibliography


[BMP] P. Brass, W. Moser, and J. Pach, *Research Problems in Discrete Geometry*, Springer Verlag, 2005, 200.

[Er46] P. Erdős, On sets of distances of $n$ points, *Amer. Math. Monthly* **52:5** (1946), 248-250.

[Er75] P. Erdős, "On some problems of elementary and combinatorial geometry." *Annali di Matematica Pura ed Applicata,* vol. 103, 1975, pp. 99-108.

[EF] P. Erdős and P. Fishburn, Maximum planar sets that determine $k$ distances, *Discrete Math.* **160:1-3** (1996), 115-125.

[Ep] A. Epstein, A. Lott, S. J. Miller, and E. A. Palsson, *Optimal point sets determining few distinct triangles*, Integers, **18** (2018), #A16.

[GK] L. Guth and N. H. Katz, On the Erdős distinct distances problem in the plane, *Ann. of Math.* **181:1** (2015), 155-190.

[Lis] P. Lisonek, New maximal two-dimensional sets, *J. Combin. Theory, Series A, 77* (1997), 318-338

[Sh] M. Shinohara, Uniqueness of maximum planar five-distance sets, *Discrete Math.* **308:14** (2008), 3048-3055.

[We] X. Wei, A proof of Erdős-Fishburn's conjecture for $g(6) = 13$, *Electron. J. Combin.* **19:4** (2012).