REPRESENTATION THEORY OF THE DIAGRAM A_n

OVER THE RING $k[[x]]$

by

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(ABSTRACT)

Fix $R = K[[x]]$. Let Ω_n be the category whose objects are $((M_1,\ldots,M_n),(f_1,\ldots,f_{n-1}))$ where each M_i is a free R-module and $f_i : M_i \longrightarrow M_{i+1}$ for each $i = 1,\ldots,n-1$, and in which the morphisms are the obvious ones. Let $\mathcal{B}_n$ be the full subcategory of Ω_n in which each map f_i is a monomorphism whose cokernel is a torsion module. It is shown that there is a full dense functor $\Omega_n \longrightarrow \mathcal{B}_n$. If X is an object of $\mathcal{B}_n$, we say that X <u>diagonalizes</u> if X is isomorphic to a direct sum of objects $((M_1,\ldots,M_n),(f_1,\ldots,f_{n-1}))$ in which each M_i is of rank one. We establish an algorithm which diagonalizes any diagonalizable object X of $\mathcal{B}_n$, and which fails only in case X is not diagonalizable.

Let Λ be an artin algebra of finite type. We prove that for a fixed C in $\mathrm{mod}(\Lambda)$ there are only finitely many modules A in $\mathrm{mod}(\Lambda)$ (up to isomorphism) for which a short exact sequence of the form $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is indecomposable.

To my family

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I must also thank my wife, , for her love and support during this time. Without her help I would not have made it. Finally, I thank for putting up with a father who has been either absent or absorbed much of the time.

TABLE OF CONTENTS

Chapter I. Introduction.

Let R be a commutative ring with 1, and let A_n be the diagram $\underset{1}{\cdot}\longrightarrow\underset{2}{\cdot}\longrightarrow\cdots\longrightarrow\underset{n}{\cdot}$. An object in the category of representations of A_n by R-modules and R-module maps is an ordered n-tuple $(M_1,\ldots,M_n)$ of R-modules together with an ordered (n-1)-tuple $(f_1,\ldots,f_{n-1})$ of R-module maps such that $f_i:M_i\longrightarrow M_{i+1}$ for $i=1,2,\ldots,n-1$. Of course we think of putting the modules in at the vertices of A_n and the maps in for the arrows to obtain the picture $M_1\xrightarrow{f_1}M_2\xrightarrow{f_2}\cdots\xrightarrow{f_{n-1}}M_n$. If $X=M_1\xrightarrow{f_1}M_2\longrightarrow\cdots\xrightarrow{f_{n-1}}M_n$ and $X'=M_1'\xrightarrow{f_1'}M_2'\longrightarrow\cdots\xrightarrow{f_{n-1}'}M_n'$ are objects in this category, than a morphism η from X to X′ is an n-tuple $\eta=(\eta_1,\ldots,\eta_n)$ of R-module maps making the diagram

$$\begin{tikzcd}
M_1 \arrow[r, "f_2"] \arrow[d, "\eta_1"'] & M_2 \arrow[r, "f_2"] \arrow[d, "\eta_2"'] & \cdots \arrow[r, "f_{n-1}"] & M_n \arrow[d, "\eta_n"'] \\
M_1' \arrow[r, "f_1'"'] & M_2' \arrow[r, "f_2'"'] & \cdots \arrow[r, "f_{n-1}'"'] & M_n'
\end{tikzcd}$$

commute.

An object X <u>decomposes</u> in this category if $X\simeq X_1\oplus X_2$ where neither X_1 nor X_2 is zero. For now and for this entire thesis, R will denote the ring K[[x]] of formal power series in one variable over a field K, and Ω_n will denote the category of representations of A_n by finitely generated free R-modules and R-module maps; the problem of this thesis is to study certain questions about the category Ω_n. The impetus for this study comes mainly from two directions. The first is the following result, which is just a re-statement of the invariant factor theorem.

Proposition I.1. Let F_0 and F_1 be finitely generated free R-modules, and let $X=F_1\xrightarrow{f}F_0$ be in Ω_2. Then X decomposes as a direct sum of representations of the types $0\longrightarrow R$, $R\longrightarrow 0$, and $R\xrightarrow{f_i}R$, where $f_i\neq 0$.

If the cokernel of a map g between free R-modules is a torsion module, we will call g a full map. If the map f in proposition I.1 is a full monomorphism, then $\mathrm{rank}(F_1) = \mathrm{rank}(F_0)$, and only maps of the type $f_i:R\longrightarrow R$ can occur. Any map from R to R is just multiplication by an element of R, so in this case we can interpret the result as saying that there exist bases for F_1 and F_0 so that the matrix of f with respect to these bases is diagonal. Unfortunately even for A_3 not every representation $X = F_2\xrightarrow{f}F_1\xrightarrow{g}F_0$ in which f and g are full monomorphisms can be diagonalized, i.e. there do not always exist bases for F_2, F_1, and F_0 with respect to which the matrices of f and g are simultaneously diagonal. The main result of Chapter II is a partial remedy for this; it establishes an algorithm which determines whether a given representation of A_n by finitely generated free R-modules and full monomorphisms is diagonalizable, and if it is, diagonalizes it.

Our second reason to study representations of A_n by free R-modules is that it is a natural generalization of the representations of A_n by K-vector spaces and K-linear maps. Because of its intimate connection with categories of modules over artin rings, this latter class has been extensively studied and is well-understood; indeed the category of such objects is known to be of <u>finite representation type</u>, i.e. there are only finitely many non-isomorphic indecomposable representations (see Gabriel's paper [G72]; one consequence of this work is that if $X = K^n \xrightarrow{f_{n-1}} K^n \longrightarrow \cdots \xrightarrow{f_1} K^n$ is a representation of A_n over K then X diagonalizes). We see at once that the category Ω_n cannot be of finite type, as even Ω_2 is not; if we let μ_i denote multiplication by x^i, then $\{R \xrightarrow{\mu_i} R \mid i=0,1,\ldots\}$ is an infinite family of non-isomorphic indecomposable representations of A_2. Yet because R is so "close" to a field, it was hoped that at least the category Ω_n might be amenable to some "nice" description, say a description of all non-isomorphic indecomposable representations in terms of one-parameter families of these (such a category is of <u>tame</u> representation type; see [DR76]); this is in fact what proposition I.1 does for Ω_2. Unfortunately, this is not the case in general; as we shall show in Chapter III, Ω_3 is already of wild (i.e., not tame) representation type.

We mention one more area with which our problem makes contact, although we have not studied it from this point of view. Let K be the quotient field of R, and let S be a subring of $M_n(K)$, the ring of nxn matrices with entries from K. An <u>R-order in S</u> is a subring A of S such that $K\otimes A\simeq S$ (orders can be defined much more generally; for a definition and full discussion see [CR]). A (left) <u>R-lattice</u> over the R-order A is an A-module which is finitely generated and projective as an R-module. Orders and lattices arise naturally in the study of group representations (cf. [CR]); the orders A arising there are called <u>classical</u> because $K\otimes A$ is a semisimple ring.

If we let $S=T_n(K)$, the ring of lower-triangular nxn matrices with entries from K, then $T_n(R)$ is an R-order in S. The lattices over $T_n(R)$ are most easily pictured as columns

$L_0=\begin{pmatrix} F_{n-1} \\ \vdots \\ F_0 \end{pmatrix}$ of finitely generated free R-modules together with

maps $f_i:F_i\longrightarrow F_{i-1}$ for $i=1,2,\ldots,n-1$. The action of $T_n(R)$ on

$L=\left\{\begin{pmatrix} F_{n-1} \\ \vdots \\ F_0 \end{pmatrix}, (f_{n-1},\ldots,f_1)\right\}$ is as follows. Let $z = \begin{pmatrix} z_{n-1} \\ \vdots \\ z_0 \end{pmatrix} \in$

L_0 and $r = (r_{ij}) \in T_n(R)$; then the t^{th} entry of rz is

$r_{t1}(f_{t+1}\circ\cdots\circ f_{n-1})(z_{n-1}) + \cdots + r_{t,t-1}f_{t+1}(z_{t+1}) + r_{t,t}z_t$.

The functor which sends the object $X = F_{n-1}\xrightarrow{f_{n-1}}F_{n-2}\longrightarrow\cdots\xrightarrow{f_1}F_0$ in Ω_n to L is an equivalence of categories connecting the problem of representations of A_n with questions of R-lattices over $T_n(R)$. The order $T_n(R)$ is the natural next step away from classical, as $K\otimes T_n(R)$ is hereditary (i.e. every ideal is projective). Our results show that the category of R-lattices of $T_n(R)$ is of wild representation type for $n\geqslant 3$, and provide some information about the decomposition of certain special lattices over $T_n(R)$.

In Chapter II we confine our attention to the full subcategory $\mathcal{B}_n$ of Ω_n whose objects are those representations $F_{n-1}\xrightarrow{f_{n-1}}F_{n-2}\longrightarrow\cdots\xrightarrow{f_1}F_0$ of A_n in which each map f_i is a full monomorphism. If we choose bases for the modules F_i and write the maps as matrices, then $\mathcal{B}_n$ contains the representations in which each f_i is a square matrix with non-zero determinant. In this category we can discuss the problem of simultaneous diagonalization of the matrices, i.e. the problem of finding isomorphisms $\alpha_{n-1},\alpha_{n-2},\ldots,\alpha_0$ such that $\alpha_{i-1}f_i\alpha_i^{-1}$ is a diagonal matrix for each $i=1,2,\ldots,n-1$. The algorithm developed solves this problem whenever it can be solved.

Our strategy in proving that the algorithm works as claimed is as follows. We first show that any object of $\mathcal{B}_n$ can be written as a direct sum of an object of a special, easily diagonalized type and an object from a certain full subcategory $\mathcal{C}_n$. We then exhibit a functor λ from $\mathcal{C}_n$ to a category of (n-2)-tuples of short exact sequences of torsion R-modules (satisfying some conditions), and prove that λ is in fact a representation equivalence. This means that an object X in $\mathcal{C}_n$ decomposes iff the corresponding object λX decomposes. Of course any sequence of torsion R-modules is a sequence of $K[x]/(x^t)$-modules for some t, so we are able to use techniques from the theory of artin algebras to obtain information about the decomposition of λX. We then lift this information, via λ, to information about the decomposition of X. Although the devices of non-commutative ring theory enter into the proof in an apparently essential way, the algorithm itself makes no reference to them; it operates only on matrices in $M_n(R)$.

From the viewpoint of representation theory, an important property of $\mathcal{B}_n$ is that decompositions of objects in $\mathcal{B}_n$ are unique up to isomorphism. This is easily seen from the following considerations. The category $\mathcal{C}_n$ is, as we have said, representation equivalent to a category T_{n-2} of special (n-2)-tuples of short exact sequences of torsion

R-modules. This latter category is itself naturally equivalent, in a way to be explained in Chapter II, to the category U_{n-2} whose objects are sequences $Y = M_{n-2} \xrightarrow{\alpha_{n-2}} M_{n-3} \xrightarrow{\alpha_{n-3}} \cdots \xrightarrow{\alpha_1} M_0$ of torsion R-modules and monomorphisms. In the same way that an object of Ω_n is a module over $T_n(R)$, an object of U_{n-2} is a module over the ring $\Lambda_{n-1,t}$ of lower-triangular $(n-1)x(n-1)$ matrices with entries from $k[x]/(x^t)$ for some t. The ring $\Lambda_{n-1,t}$ is artinian, so $\mathrm{mod}(\Lambda_{n-1,t})$ is a Krull-Schmidt category; so decompositions of $\Lambda_{n-1,t}$-modules are unique up to isomorphism. The correspondence between decompositions in this category and decompositions in C_n allows us to conclude that the latter are also unique up to isomorphism.

Chapter III is devoted to some examples of the correspondence which λ sets up between objects in C_3 and short exact sequences of torsion R-modules. We show for a few classes of sequences what kinds of representations correspond to them, and what kind of information about the representations can be obtained from knowledge of the sequences. As an example, we show that if $X = R^n \xrightarrow{f} R^n \xrightarrow{g} R^n$ is such that $gf \simeq x^t I$ for some t (where I is the nxn identity matrix), then X must diagonalize.

Chapter IV is entirely concerned with the construction of a full dense functor from Ω_n to $\mathcal{B}_n$.

Chapter V deals with a slightly different topic than do the earlier sections. Its main result is a theorem about the indecomposability of short exact sequences of Λ-modules, where Λ is an artin ring of finite representation type. This theorem says, in effect, that the exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ can be indecomposable only if A is not "too big" compared to C, where the sizes involved are related to the lengths of the socles of A and C. The section ends with an application of this theorem to objects in $\mathcal{C}_3$, an interesting consequence of which is the following. If $X = R^n \xrightarrow{f} R^n \xrightarrow{g} R^n$ is in $\mathcal{C}_3$ and the degree of the determinant of f is less than n, then X must decompose.

We mention that although we have, in this thesis, restricted our attention to the category mod(R), most of our techniques, and hence results, are good whenever R is a local PID.

Chapter II. Diagonalization.

As was stated in the introduction, the purpose of this chapter is to develop an algorithm which generalizes the invariant factor theorem. Thus the question of this chapter is: when does a representation in $\mathbb{B}_n$ decompose as a direct sum of representations $F_{n-1} \to F_{n-2} \to \cdots \to F_0$ in which each F_i has rank one? If the f_i are written as matrices, this is equivalent to asking when there are bases for $F_{n-1}, \cdots, F_0$ with respect to which each f_i is diagonal; therefore if a representation does decompose in this way we say that it diagonalizes. We are able to supply an answer to this question in the form of an algorithm for diagonalizing an object in $\mathbb{B}_n$ which fails only in case the object does not diagonalize.

§1. Statement of the problem. The functor λ.

The first step in the development of the algorithm is Lemma II.1.1; but first we need a few definitions.

Definition. Let C_n be the full subcategory of $\mathbb{B}_n$ consisting of those objects of $\mathbb{B}_n$ for which $(f_1 \circ \cdots \circ f_{n-1})(F_{n-1}) \subseteq xF_0$. (Because R is local, this condition ensures that $F_0 \to \operatorname{coker}(f_1 \circ \cdots \circ f_{n-1})$ is a projective cover.)

Definition. If in the diagram

$$\begin{array}{ccccccc} M_n & \xrightarrow{\alpha_n} & M_{n-1} & \xrightarrow{\alpha_{n-1}} & \cdots & \xrightarrow{\alpha_2} & M_1 \\ \tau_n \downarrow \uparrow \sigma_n & & \tau_{n-1} \downarrow \uparrow \sigma_{n-1} & & & & \tau_1 \downarrow \uparrow \sigma_1 \\ N_n & \xrightarrow[\beta_n]{} & N_{n-1} & \xrightarrow[\beta_{n-1}]{} & \cdots & \xrightarrow[\beta_2]{} & N_1 \end{array}$$

we have $\tau_i\sigma_i = 1_{N_i}$, $\alpha_i\sigma_i = \sigma_{i-1}\beta_i$, and $\beta_i\tau_i = \tau_{i-1}\alpha_i$ for $i=2,3,\ldots,n$, then we say that the σ_i are consistently split by the τ_i (or vice versa). If $(\alpha_n,\ldots,\alpha_1)$, $(\beta_n,\ldots,\beta_1)$ are morphisms in Ω_n then we say that $(\alpha_n,\ldots,\alpha_1)$ is consistently split by $(\beta_n\ldots,\beta_1)$ and that the bottom row is a summand of the top.

Lemma II.1.1. Let $X = F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_1} F_0$ in $\mathcal{B}_n$. If $f_1 \circ \cdots \circ f_{n-1}(F_{n-1}) \not\subseteq xF_0$, then X has a summand of the form $R = R = \cdots = R$ (n copies).

Proof. The hypothesis implies that there is an injection $R \xrightarrow{\alpha} F_{n-1}$ such that $f_1 \circ \cdots \circ f_{n-1} \circ \alpha(R) \cong R$. Because R is projective there is a splitting β of $f_1 \circ \cdots \circ f_{n-1} \circ \alpha$. Set $\sigma_{n-1} = \alpha$, $\sigma_j = f_{j+1} \circ \cdots \circ f_{n-1} \circ \alpha$ for $0 \leqslant j < n-1$, and set $\tau_0 = \beta$, $\tau_j = \beta \circ f_1 \circ \cdots \circ f_j$ for $1 \leqslant j \leqslant n-1$. Then in the diagram

$$\begin{array}{ccccccccc} F_{n-1} & \xrightarrow{f_{n-1}} & F_{n-\alpha} & \xrightarrow{f_{n-2}} & \cdots & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 \\ \sigma_{n-1} \uparrow \downarrow \tau_{n-1} & & \sigma_{n-2} \uparrow \downarrow \tau_{n-2} & & & & \sigma_1 \uparrow \downarrow \tau_1 & & \sigma_0 \uparrow \downarrow \tau_0 \\ R & = & R & = & \cdots & = & R & = & R \end{array}$$

$(\sigma_{n-1},\dots,\sigma_0)$ is a monomorphism consistently split by the epimorphism $(\tau_{n-1},\dots,\tau_0)$, so the bottom row is a summand of the top. ■

Lemma II.1.1 says that an object in $\mathcal{B}_n$ is a direct sum of an object in $\mathcal{C}_n$ and a representation of the form $R^d = R^d = \dots = R^d$, so we can concentrate on $\mathcal{C}_n$. Essential for the analysis of objects in $\mathcal{C}_n$ is the construction of a functor λ from $\mathcal{C}_n$ to a certain category of torsion R-modules and monomorphisms. This allows the introduction of artin algebra techniques into the problem.

Define the category T_n whose objects are ordered n-tuples $(\epsilon_1,\dots,\epsilon_n)$ of short exact sequences $\epsilon_i = 0 \to A_i \xrightarrow{a_i} B_i \xrightarrow{b_i} C_i \to 0$ of torsion R-modules with the property that $B_i = A_{i-1}$ for $2 \leqslant i \leqslant n$. A morphism in T_n from $(\epsilon_1,\dots,\epsilon_n)$ to $(\epsilon_1',\dots,\epsilon_n')$ is an ordered n-tuple of ordered triples of R-module maps $((\alpha_{11}, \alpha_{12}, \alpha_{13}),\dots, (\alpha_{n1}, \alpha_{n2}, \alpha_{n3}))$ satisfying: (i) $\alpha_{i2} = \alpha_{i-1,1}$ for $2 \leqslant i \leqslant n$; (ii) the diagram

$$\begin{array}{ccccccccc}
\epsilon_i = 0 & \longrightarrow & A_i & \xrightarrow{a_i} & B_i & \xrightarrow{b_i} & C_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_{i1}} & & \downarrow{\scriptstyle \alpha_{i2}} & & \downarrow{\scriptstyle \alpha_{i3}} & & \\
\epsilon_i' = 0 & \longrightarrow & A_i' & \longrightarrow & B_i' & \longrightarrow & C_i' & \longrightarrow & 0
\end{array}$$

commutes for $i=1,2, \dots, n$.

We now define the functor $\lambda: C_n \to T_{n-2}$ $(n>2)$. Let $X = F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \longrightarrow \cdots \xrightarrow{f_1} F_0$ be in C_n. In the commutative diagram

$$\begin{tikzcd}
 & F_{n-1} \arrow[r, equal] \arrow[d, "f_2\circ\cdots\circ f_{n-1}"'] & F_{n-1} \arrow[d, "f_1\circ\cdots\circ f_{n-1}"] & & \\
0 \arrow[r] & F_1 \arrow[r, "f_1"'] \arrow[d, "\rho_1"] & F_0 \arrow[r] \arrow[d, "\rho_0"] & C_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A_1 \arrow[r, "a_1"'] & B_1 \arrow[r, "b_1"'] & C_1 \arrow[r] & 0
\end{tikzcd}$$

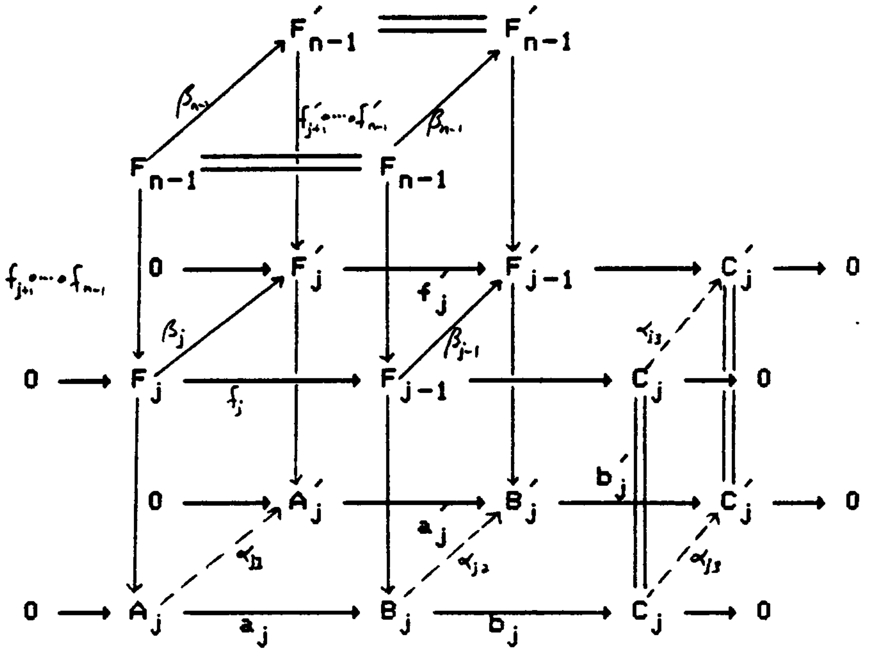
ρ_0 is a projective cover. The bottom row of this diagram is the short exact sequence ϵ_1. To construct ϵ_2, form the diagram

$$\begin{tikzcd}
 & F_{n-1} \arrow[r, equal] \arrow[d, "f_1\circ\cdots\circ f_{n-1}"'] & F_{n-1} \arrow[d, "f_2\circ\cdots\circ f_{n-1}"] & & \\
0 \arrow[r] & F_2 \arrow[r, "f_2"'] \arrow[d, "\rho_2"] & F_1 \arrow[r] \arrow[d, "\rho_1"] & C_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "a_2"'] & B_2 \arrow[r, "b_2"'] & C_2 \arrow[r] & 0;
\end{tikzcd}$$

then ϵ_2 is the bottom row. Clearly $B_2 = A_1$ as required. Continuing in this way we certainly get an object λX of T_{n-2}; the last step is the diagram

$$
\begin{tikzcd}
 & F_{n-1} \arrow[r, equal] \arrow[d, "f_{n-1}"'] & F_{n-1} \arrow[d, "f_{n-2}\circ f_{n-1}"] & & \\
0 \arrow[r] & F_{n-2} \arrow[r, "f_{n-2}"'] \arrow[d, "\rho_{n-2}"] & F_{n-3} \arrow[r] \arrow[d, "\rho_{n-3}"] & C_{n-2} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A_{n-2} \arrow[r, "a_{n-2}"'] & B_{n-2} \arrow[r, "b_{n-2}"'] & C_{n-2} \arrow[r] & 0.
\end{tikzcd}
$$

Suppose that $X' = F'_{n-1} \xrightarrow{f'_{n-1}} F'_{n-2} \xrightarrow{f'_{n-2}} \cdots \xrightarrow{f'_1} F'_0$, and that $\beta=(\beta_{n-1},\ldots,\beta_0):X \longrightarrow X'$ in C_n. We get a morphism $\lambda\beta:\lambda X \longrightarrow \lambda X'$ in T_{n-2} in the following way. We need to fill in the diagram



with α_{j1}, α_{j2}, α_{j3}. But the left-hand "wall" induces α_{j1}, as its columns are exact; the middle wall induces α_{j2}; and

α_{j3} can be gotten from either the middle or the bottom "floor". We will prove that the square

$$\begin{tikzcd} A_j \arrow[r, "a_j"] \arrow[d, "\alpha_{j1}"'] & B_j \arrow[d, "\alpha_{j2}"] \\ A_j' \arrow[r, "a_j'"'] & B_j' \end{tikzcd}$$

commutes; the proof that the right-hand half of the bottom floor commutes is the same. From the diagram we have that $a_j p_1 = p_2 f_j$, so $\alpha_{j2} a_j p_1 = \alpha_{j2} p_2 f_j$. Then $\alpha_{j2} a_j p_1 = \alpha_{j2} p_2 f_j = p_2' \beta_{j-1} f_j = p_2' f_j' \beta_j = a_j' \alpha_{j1} p_1$; as p_1 is an epimorphism, this gives $\alpha_{j2} a_j = a_j' \alpha_{j1}$ as was wanted.

Because of the way in which the maps $(\alpha_{j1}, \alpha_{j2}, \alpha_{j3})$ are induced, it is clear that $((\alpha_{11}, \alpha_{12}, \alpha_{13}), \ldots, (\alpha_{n-2}, \alpha_{n-2}, \alpha_{n-2,3}))$ is a morphism from λX to $\lambda X'$ in T_{n-2}.

That λ is an additive functor is not hard to verify. We want more than this, though; we want to show that λ is a representation equivalence, i.e. that λ is full and dense, and that it reflects isomorphisms and preserves indecomposables. To prove that λ has these properties we need the following (non-functorial) construction, which provides a sort of inverse for λ.

Let $Y=(\epsilon_1, \ldots, \epsilon_{n-2})$ be in T_{n-2}, and let $F_0 \xrightarrow{\rho_0} B_1$ be the projective cover of B_1 in mod(R) (so F_0 is a finitely generated free R-module). Construct an object $F_{n-1} \xrightarrow{\eta} F_1 \xrightarrow{f_1} F_0$ as follows: let $0 \longrightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{b_1 \rho_0} C_1 \longrightarrow 0$ be exact (where $\epsilon_1 = 0 \longrightarrow A_1 \xrightarrow{a_1} B_1 \xrightarrow{b_1} C_1 \longrightarrow 0$). Then the diagram

$$\begin{tikzcd}
0 \arrow[r] & F_1 \arrow[r, "f_1"] \arrow[d, "\rho_1"] & F_0 \arrow[r, "b_1\rho_0"] \arrow[d, "\rho_0"] & C_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A_1 \arrow[r, "a_1"'] & B_1 \arrow[r, "b_1"'] & C_1 \arrow[r] & 0
\end{tikzcd}$$

commutes, where ρ_1 is just the induced map. By the Snake Lemma ρ_1 is an epimorphism and $\mathrm{Ker}\rho_1 \simeq \mathrm{Ker}\rho_0$; let $\eta_1 : F_{n-1} \longrightarrow F_1$ be the kernel of ρ_1. Then the diagram

$$\begin{tikzcd}
 & F_{n-1} \arrow[r, equal] \arrow[d, "\eta_1"] & F_{n-1} \arrow[d, "f_1\eta_1"] & & \\
0 \arrow[r] & F_1 \arrow[r, "f_1"'] \arrow[d, "\rho_1"'] & F_0 \arrow[r] \arrow[d, "\rho_0"] & C_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A_1 \arrow[r, "a_1"'] & B_1 \arrow[r, "b_1"'] & C_1 \arrow[r] & 0
\end{tikzcd}$$

commutes. The next step is to repeat this construction with ρ_1 in place of ρ_0; we get a commutative diagram

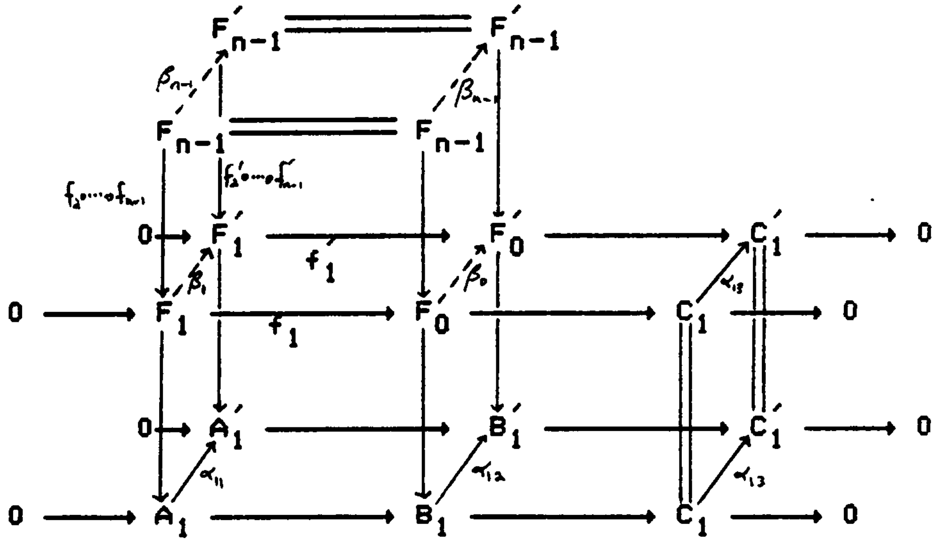
$$\begin{tikzcd}
 & F_{n-1} \arrow[r, equal] \arrow[d, "\eta_2"] & F_{n-1} \arrow[d, "f_2\eta_2"] & & \\
0 \arrow[r] & F_2 \arrow[r, "f_2"'] \arrow[d, "\rho_2"'] & F_0 \arrow[r] \arrow[d, "\rho_0"] & C_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "a_2"'] & B_2 \arrow[r, "b_2"'] & C_2 \arrow[r] & 0
\end{tikzcd}$$

After n-2 steps we get an object $F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_1} F_0$ in T_n, which we will call θY.

We will first use θ to show that λ is full. Suppose that $Y = (\epsilon_1, \ldots, \epsilon_{n-2})$, $Y' = (\epsilon_1', \ldots, \epsilon_{n-2}')$ in T_{n-2}, $\alpha : Y \longrightarrow Y'$ in T_{n-2}, and that

$\theta Y = F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_1} F_1$ and $\theta Y' = F_{n-1}' \xrightarrow{f_{n-1}'} \cdots \xrightarrow{f_1'} F_0'$.

Then we claim that there must be maps β_{n-1}, β_1, β_0 to make



commute. Both $F_0 \longrightarrow B_1$ and $F_0' \longrightarrow B_1'$ are projective covers, so we can lift α_{12} to a map $\beta_0 : F_0 \longrightarrow F_0'$; then β_{n-1} is also induced. Similarly because the rows of the middle floor are exact, β_1 is defined. It is easy to check that the whole diagram commutes. Note that because $F_0 \longrightarrow B$ and $F_0' \longrightarrow B'$ are projective covers, β_0 is an isomorphism if α_{12} is; and so if the other α_{1j} are isomorphisms, so are β_{n-1} and β_1.

The next step is to find β_2 to make the diagram

$$\begin{tikzcd}
 & & & F_{n-1}' \arrow[rr, equal] \arrow[dd, "f_3'\circ\cdots\circ f_{n-1}'"] & & F_{n-1}' \arrow[dd] & & & & \\
 & & F_{n-1} \arrow[ur, "\beta_{n-1}"] \arrow[rr, equal] \arrow[dd, "f_3\circ\cdots\circ f_{n-1}"'] & & F_{n-1} \arrow[ur, "\beta_{n-1}"] \arrow[dd] & & & & & \\
 & 0 \arrow[rr] & & F_2' \arrow[rr, "f_2'"] \arrow[dd] & & F_0' \arrow[rr] \arrow[dd] & & C_2' \arrow[rr] \arrow[dd, equal] & & 0 \\
0 \arrow[rr] & & F_2 \arrow[ur, "\beta_2"] \arrow[rr, "f_2"] \arrow[dd] & & F_0 \arrow[ur, "\beta_0"] \arrow[rr] \arrow[dd] & & C_2 \arrow[ur, "\alpha_{23}"] \arrow[rr] \arrow[dd, equal] & & 0 & \\
 & 0 \arrow[rr] & & A_2' \arrow[rr] & & B_2' \arrow[rr] & & C_2' \arrow[rr] & & 0 \\
0 \arrow[rr] & & A_2 \arrow[ur, "\alpha_{21}"] \arrow[rr] & & B_2 \arrow[ur, "\alpha_{22}"] \arrow[rr] & & C_2 \arrow[ur, "\alpha_{23}"] \arrow[rr] & & 0 &
\end{tikzcd}$$

commute; but β_2 is induced in this diagram exactly as β_1 was induced in the last, and again the whole diagram commutes. Again, note that if α is an isomorphism then because β_1 must be an isomorphism from the first diagram, β_2 must also be an isomorphism. Continuing this process gives a morphism $\beta=(\beta_{n-1},\ldots,\beta_0):\theta Y \longrightarrow \theta Y'$ in $\mathcal{C}_n$. β is certainly not unique, but it is clear from the construction that $\lambda\beta=\alpha$, so that λ is full. Because β must be an isomorphism when α is one, we get $\theta\lambda X \simeq X$ for X in $\mathcal{C}_n$ (take α to be the identity on λX). Thus if $\lambda X \simeq \lambda X'$, then $\theta\lambda X \simeq \theta\lambda X'$, so $X \simeq X'$; so λ reflects isomorphisms. Finally if Y is in T_n then clearly $\lambda\theta Y \simeq Y$, so that λ is dense.

Because λ is additive, λX decomposes if X does; we want to show that the converse is also true. Suppose λX decomposes as $Y\oplus Y'$, where $Y=(\epsilon_1,\ldots,\epsilon_{n-2})$ and $Y'=(\epsilon_1',\ldots,\epsilon_{n-2}')$. We will be done if we can show that $\theta\lambda X$ decomposes, as $\theta\lambda X\simeq X$. We think of $Y\oplus Y'$ as the object $(\epsilon_1'',\ldots,\epsilon_{n-2}'')$, where

$$\epsilon_i''=0\longrightarrow A_i\oplus A_i' \xrightarrow{\begin{bmatrix} a_i & 0 \\ 0 & a_i' \end{bmatrix}} B_i\oplus B_i' \xrightarrow{\begin{bmatrix} b_i & 0 \\ 0 & b_i' \end{bmatrix}} C_i\oplus C_i' \longrightarrow 0.$$

The projective cover of $B_1\oplus B_1'$ is the sum $F_0\oplus F_0' \xrightarrow{\rho=\begin{bmatrix} \rho_0 & 0 \\ 0 & \rho_0' \end{bmatrix}} B_1\oplus B_1'$ of the projective covers. The kernel of $\begin{bmatrix} b_1\rho_0 & 0 \\ 0 & b_1'\rho_0' \end{bmatrix}: F_0\oplus F_0' \longrightarrow C_1\oplus C_1'$ is just the sum of the kernels, namely $\begin{bmatrix} f_1 & 0 \\ 0 & f_1' \end{bmatrix}: F_1\oplus F_1' \longrightarrow F_0\oplus F_0'$. In the diagram

$$\begin{tikzcd}[column sep=large]
0 \arrow[r] & F_1 \oplus F_1' \arrow[r, "\begin{bmatrix} f_1 & 0 \\ 0 & f_1' \end{bmatrix}"] \arrow[d, "\sigma"] & F_0 \oplus F_0' \arrow[r, "\begin{bmatrix} b_1\rho_0 & 0 \\ 0 & b_1'\rho_0' \end{bmatrix}"] \arrow[d, "\begin{bmatrix} \rho_0 & 0 \\ 0 & \rho_0' \end{bmatrix} = \rho"] & C_1 \oplus C_1' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A_1 \oplus A_1' \arrow[r, "\begin{bmatrix} a_1 & 0 \\ 0 & a_1' \end{bmatrix}"'] & B_1 \oplus B_1' \arrow[r, "\begin{bmatrix} b_1 & 0 \\ 0 & b_1' \end{bmatrix}"'] & C_1 \oplus C_1' \arrow[r] & 0
\end{tikzcd}$$

we can take σ to be diagonal, $\sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$; then its kernel is also diagonal, i.e. a map $\zeta = \begin{bmatrix} f_2\circ\cdots\circ f_{n-1} & 0 \\ 0 & f_2'\circ\cdots\circ f_{n-1}' \end{bmatrix} : F_{n-1} \oplus F_{n-1}' \longrightarrow F_1 \oplus F_1'$. Then the diagram

$$\begin{tikzcd}[column sep=large]
 & F_{n-1} \oplus F_{n-1}' \arrow[r, equal] \arrow[d, "\zeta"] & F_{n-1} \oplus F_{n-1}' \arrow[d] & & \\
0 \arrow[r] & F_1 \oplus F_1' \arrow[r, "\begin{bmatrix} f_1 & 0 \\ 0 & f_1' \end{bmatrix}"'] \arrow[d, "\sigma"] & F_0 \oplus F_0' \arrow[r] \arrow[d, "\rho"] & C_1 \oplus C_1' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A_1 \oplus A_1' \arrow[r, "\begin{bmatrix} a_1 & 0 \\ 0 & a_1' \end{bmatrix}"'] & B_1 \oplus B_1' \arrow[r, "\begin{bmatrix} b_1 & 0 \\ 0 & b_1' \end{bmatrix}"'] & C_1 \oplus C_1' \arrow[r] & 0
\end{tikzcd}$$

commutes. The only crucial part of this construction was that the map ρ could be taken to be diagonal; as the map σ takes the role of ρ in the diagram for ϵ_2, this process can continue. Thus $\theta\lambda X$ is decomposable.

We need to mention one more property of λ, and that is that if $X = F_{n-1} \longrightarrow F_{n-2} \longrightarrow \cdots \longrightarrow F_0$ with each F_i of rank 1, and $\lambda X = (\epsilon_1, \ldots, \epsilon_{n-2})$, then each ϵ_i is of the form $0 \longrightarrow V_i \longrightarrow V_{i+j} \longrightarrow V_j \longrightarrow 0$, where V_i is the indecomposable R-module of length i and the maps are the obvious inclusion and surjection. (Henceforth we will write sequences of this sort without referring to the maps; we will always mean the obvious inclusion and surjection. Also, we will let V_0 be the zero module.) If an object Y in T_n decomposes as a direct sum of objects of this type, we call Y <u>diagonalizable</u>; by the foregoing, an object X in C_n is diagonalizable in C_n iff λX is diagonalizable in T_{n-2}.

If $(\epsilon_1, \ldots, \epsilon_n)$ is in T_n, we get an object in U_n, the category of sequences of torsion R-modules and monomorphisms, by taking the sequence $A_n \xrightarrow{a_n} A_{n-1} \xrightarrow{a_{n-1}} \cdots \xrightarrow{a_2} A_1 \xrightarrow{a_1} B_1$. By doing the obvious thing on morphisms we get a functor which is clearly an equivalence of categories. We will denote the composition of λ with this equivalence by λ'; thus if $X \in C_n$ then λX is in T_{n-2} and $\lambda' X$ is in U_{n-2}.

§2. The case of A_3.

Our approach to developing the promised algorithm takes us through the cases of A_3 and A_4 before getting to the general case. At each step we will need the results obtained at the previous step. We first establish some preliminary lemmas.

Lemma II.2.1. There is an algorithm to find a basis for the kernel of an epimorphism $\tau: R^n \longrightarrow R^m$.

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis for R^n. Abstractly we know that there must be a subset of m 'vectors' of the set $\{\tau e_1, \ldots, \tau e_n\}$ which forms a basis for R^m, as suppose $\{\tau e_{i_1}, \ldots, \tau e_{i_s}\}$ is a subset of minimum cardinality which still spans R^m. Then if $s>m$, this set is linearly dependent as a set of vectors in K^m, where K is the quotient field of R; say $\sum_{i=1}^{m} \alpha_i \tau e_i = 0$. Clearing denominators, we get a dependence relation with coefficients in R, so $s=m$. Re-order so that $\tau e_1, \ldots, \tau e_m$ are the basis elements; then if $j>m$, $\tau e_j = \sum_{i=1}^{m} r_{ji} \tau(e_i)$ for some $r_{ji} \in R$, so $e_j - \sum_{i=1}^{m} r_{ji} e_i \in \ker\tau$. There are exactly n-m of these, and they span a summand of R^n of rank n-m ($\{e_1, \ldots, e_m\}$ spans a free complement to it) which is inside $\ker\tau$. As $\ker\tau$ is itself a summand of R^n of rank n-m, $\{e_j - \sum_{i=1}^{m} r_{ji} e_i \mid j=m+1, \ldots, n\}$ is a basis for $\ker\tau$. ∎

<u>Proposition</u> <u>II.2.2</u>. Suppose $F_2 \xrightarrow{f} F_1 \xrightarrow{g} F_0$ is in $\mathcal{B}_3$. If for some fixed bases for F_2, F_1, and F_0 the matrix of f is $\begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}$ and the matrix of gf is $\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$, where the blocks are the same size, then the matrix of g with respect to these bases is also block diagonal with blocks of the same size.

<u>Proof</u>. The map f is a monomorphism and hence has a non-zero determinant. Thus in $M_n(K)$, f has an inverse, which is $f^{-1}=\begin{pmatrix} f_1^{-1} & 0 \\ 0 & f_2^{-1} \end{pmatrix}$. Then $g=\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}\begin{pmatrix} f_1^{-1} & 0 \\ 0 & f_2^{-1} \end{pmatrix}$, which is block-diagonal. ∎

For the rest of this section we will let $X=F_2 \xrightarrow{f} F_1 \xrightarrow{g} F_0$, and $\lambda X=0 \longrightarrow A \xrightarrow{a} B \longrightarrow C \longrightarrow 0$. When we write $V_r \longrightarrow V_s$ we always mean that the map between these indecomposable modules is the inclusion if $r \leq s$, and the surjection if $r>s$.

Suppose the diagram
$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ \tau_1 \downarrow \uparrow \sigma_1 & & \tau_2 \downarrow \uparrow \sigma_2 \\ V_r & \longrightarrow & V_s \end{array}$$
commutes, where σ_1, σ_2 are monomorphisms which are consistently split by the epimorphisms τ_1, τ_2. Then we can lift σ_i, τ_i to maps $\hat{\sigma}_i$,

$\hat{\tau}_i$, σ_i', τ_i' to make the diagram

$$\begin{tikzcd}
 & F_2 \arrow[rr, equal] \arrow[dl, shift right, "\tau_1'"'] \arrow[dd] & & F_2 \arrow[dl, shift right, "\tau_2'"'] \arrow[dd] \\
R \arrow[ur, shift right, "\sigma_1'"'] \arrow[rr, equal] \arrow[dd] & & R \arrow[ur, shift right, "\sigma_2'"'] \arrow[dd] & \\
 & F_1 \arrow[rr] \arrow[dl, shift right, "\hat{\tau}_1"'] \arrow[dd] & & F_0 \arrow[dl, shift right, "\hat{\tau}_2"'] \arrow[dd] \\
R \arrow[ur, shift right, "\hat{\sigma}_1"'] \arrow[rr] \arrow[dd] & & R \arrow[ur, shift right, "\hat{\sigma}_2"'] \arrow[dd] & \\
 & A \arrow[rr] \arrow[dl, shift right, "\tau_1"'] & & B \arrow[dl, shift right, "\tau_2"'] \\
V_r \arrow[ur, shift right, "\sigma_1"'] \arrow[rr] & & V_s \arrow[ur, shift right, "\sigma_2"'] &
\end{tikzcd}$$

commute; of course $\sigma_1'=\sigma_2'$ and $\tau_1'=\tau_2'$. Take bases as follows: for F_2, take $\sigma_1'(1)$ together with a basis for $\mathrm{Ker}\,\tau_1'$; for F_1, take $\hat{\sigma}_1(1)$ together with a basis for $\mathrm{Ker}\,\hat{\tau}_1$; for F_0, take $\hat{\sigma}_2(1)$ together with a basis for $\mathrm{Ker}\,\hat{\tau}_2$. Then f clearly has the form $\left(\begin{array}{c|ccc} x^r\mu & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & * & \\ 0 & & & \end{array}\right)$ and gf the form $\left(\begin{array}{c|ccc} x^s\mu' & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & * & \end{array}\right)$, where μ, μ' are units; by the proposition just proved, g must have the form $\left(\begin{array}{c|ccc} x^{s-r}\mu'' & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & * & \end{array}\right)$ for some unit μ''.

This together with Lemma II.2.1 implies that if we specify a set of monomorphisms $\{\sigma_i\}$ with consistent splittings $\{\tau_i\}$ to make an object Y a summand of an object λX in T_n (or $\lambda' X$ in U_n), then we have in principle specified an algorithm for splitting off the corresponding summand of X in C_n.

In what follows we will often write a representation $F_{n-1}\xrightarrow{f_{n-1}}F_{n-1}\xrightarrow{f_{n-2}}\cdots\xrightarrow{f_1}F_0$ as $\xrightarrow{f_{n-1}}\xrightarrow{f_{n-2}}\cdots\xrightarrow{f_1}$, or an object $A_{n-1}\xrightarrow{\alpha_{n-1}}A_{n-2}\xrightarrow{\alpha_{n-2}}\cdots\xrightarrow{\alpha_1}A_0$ of U_n as $\xrightarrow{\alpha_{n-1}}\xrightarrow{\alpha_{n-2}}\cdots\xrightarrow{\alpha_1}$. Also, maps between free modules will be written and treated as matrices without explicit mention of the bases with respect to which they are so written. In those cases in which the matrix takes a special form, it will usually be obvious that such bases exist; if there is no indication to the contrary, the standard basis for R^n will be assumed.

<u>Proposition II.2.3</u>. Fix i, $0\leqslant i\leqslant n-1$. The diagram

$$\begin{tikzcd}
\cdot \arrow[r, "f_{n-1}"] \arrow[d, "\alpha_{n-1}"'] & \cdot \arrow[r, "f_{n-2}"] \arrow[d, "\alpha_{n-2}"] & \cdots & \cdot \arrow[r, "f_1"] \arrow[d, "\alpha_1"] & \cdot \arrow[d, "\alpha_0"] \\
\cdot \arrow[r, "g_{n-1}"'] & \cdot \arrow[r, "g_{n-2}"'] & \cdots & \cdot \arrow[r, "g_1"'] & \cdot
\end{tikzcd}$$

of free modules and matrices commutes iff the diagram obtained from it by replacing f_i, g_i by $x^t f_i, x^t g_i$, respectively, commutes.

<u>Proof</u>. This is obvious. ■

The consequence of this proposition that we need is that a representation $X=\xrightarrow{f_{n-1}}\xrightarrow{f_{n-2}}\cdots\xrightarrow{f_1}$ diagonalizes iff the representation obtained from X by factoring powers of x out of each f_i diagonalizes.

For an object X in C_n, our approach to finding a diagonalization of X is to detect summands of X which are of the form $F_{n-1}\longrightarrow F_{n-2}\longrightarrow\cdots\longrightarrow F_0$ where each F_i has rank 1. In C_3, if $X=F_2\xrightarrow{f}F_1\xrightarrow{g}F_0$, we factor powers of x out of f and g until a unit entry appears in each; if the result is $X'=F_2\xrightarrow{f'}F_1\xrightarrow{g'}F_0$, we check to see whether $g'f'(F_2)\subseteq xF_0$. If not, we can split off a summand of the form $R = R = R$ by Lemma II.1.1. We would obtain an isomorphism

$$\begin{array}{ccccc} F_2 & \xrightarrow{f'} & F_1 & \xrightarrow{g'} & F_0 \\ \alpha\downarrow\wr & & \downarrow\wr\beta & & \downarrow\wr\gamma \\ F_2 & \xrightarrow[f"]{} & F_1 & \xrightarrow[g"]{} & F_0 \end{array}$$

where both f" and g" are block-diagonal with a 1x1 block in the upper left-hand corner. If $f'=x^{-s}f$ and $g'=x^{-t}g$, then the diagram

$$\begin{array}{ccccc} F_2 & \xrightarrow{f} & F_1 & \xrightarrow{g} & F_0 \\ \alpha\downarrow\wr & & \downarrow\wr\beta & & \downarrow\wr\gamma \\ F_2 & \xrightarrow[x^s f"]{} & F_1 & \xrightarrow[x^t g"]{} & F_0 \end{array}$$

commutes by the proposition just established, so we have split off a summand of X of the form $\xrightarrow{x^s}\ \xrightarrow{x^t\mu}$ for some unit μ. What can we do if $g'f'(F_2) \subseteq xF_0$? We will want the notation $s(M)$ for the length of the socle of a module M in $\mathrm{mod}(R)$.

Let $\lambda X' = 0 \longrightarrow A' \longrightarrow B' \longrightarrow C' \longrightarrow 0$. Because g' has a unit entry while $g'f'(F_2) \subseteq xF_0$, $s(\mathrm{coker}(g')) < s(\mathrm{coker}(g'f'))$. This means that if $\lambda X'$ diagonalizes, it must have a summand of the form $0 \longrightarrow V_r = V_r \longrightarrow 0 \longrightarrow 0$ for some indecomposable module V_r, which corresponds to a summand of X of the form $\xrightarrow{x^r} \xrightarrow{\nu}$ for some unit ν. Our next few lemmas have to do with detecting such a summand.

Suppose that $\epsilon = 0 \longrightarrow A \xrightarrow{a} B \longrightarrow C \longrightarrow 0$ is exact and that for some indecomposable module A' the sequence $0 \longrightarrow A' = A' \longrightarrow 0 \longrightarrow 0$ is a summand of ϵ. Let $A = \coprod V_{a_i} \oplus A''$ be any decomposition of A in which each V_{a_i} is isomorphic to A' and A'' has no summand isomorphic to A', and for each i let

$\sigma_i : V_{a_i} \longrightarrow A$ be the induced inclusion. Then the diagram

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ \uparrow{\scriptstyle \sigma_i} & & \uparrow{\scriptstyle a\sigma_i} \\ V_{a_i} & = & V_{a_i} \end{array}$$

commutes.

<u>Lemma II.2.4</u>. With the notation of the last paragraph, for some i there exist epimorphisms $\rho : A \longrightarrow V_{a_i}$ and $\tau : B \longrightarrow V_{a_i}$

which consistently split σ_i, $a\sigma_i$.

<u>Proof</u>. Let $\pi_A: A \longrightarrow A'$ and $\pi_B: B \longrightarrow A'$ be the consistently splittable epimorphisms we know exist. By the proof of the Krull-Schmidt theorem ([R73], p.80), one of the compositions $\pi_A\sigma_i$ must be an isomorphism; then $\tau = (\pi_A\sigma_i)^{-1}\pi_B$ and $\rho = \tau a$ consistently split $a\sigma_i$, σ_i. ■

The advantage that Lemma II.2.4 gives us is that it allows us to test conditions for the existence of a summand of ϵ for any fixed decomposition of A_i; if we fail to find a summand using the given decomposition, then there is no summand. Of course there is one decomposition of A which is always easily found, namely the one induced by the diagonalization of f.

For our next step we need the special short exact sequences called the almost split or Auslander-Reiten sequences ([AR75]). One defining property of these is this: $0 \longrightarrow A \xrightarrow{\alpha} E \longrightarrow C \longrightarrow 0$ is an almost split sequence if A and C are indecomposable and, given a map $\sigma: A \longrightarrow M$, then there is a map φ to make the diagram

$$(*) \quad \begin{tikzcd} M & \\ A \arrow[u, "\sigma"] \arrow[r, "\alpha"'] & E \arrow[ul, dashed, "\varphi"'] \end{tikzcd}$$

commute iff σ is not a splittable monomorphism.

<u>Proposition II.2.5</u>. In the category of torsion R-modules the almost split sequences are the sequences

$$0 \longrightarrow V_r \xrightarrow{\alpha} V_{r-1} \oplus V_{r+1} \xrightarrow{\beta} V_r \longrightarrow 0,$$

where $\alpha = \begin{bmatrix} \text{surj} \\ \text{incl} \end{bmatrix}$ and $\beta = [-\text{incl} \ \ \text{surj}]$.

<u>Proof</u>. Certainly these sequences have both ends indecomposable. Suppose that $\sigma : V_r \longrightarrow M$; clearly it suffices to consider the case in which M is an indecomposable module V_s. If σ is a splittable monomorphism in this case, then it is an isomorphism, and so if a map $\varphi : E \longrightarrow V_s$ existed to make the diagram (*) commute, then α would be splittable. This is obviously impossible, so if σ is a splittable monomorphism then no such map φ exists. Suppose that σ is not a splittable monomorphism. Then writing elements of V_j as $a_0 + \cdots + a_{j-1}x^{j-1}$ we see that σ must have the form $\sigma(a_0 + \cdots + a_{r-1}x^{r-1}) = a_0 x^t + \cdots + a_{t+s-1}x^{s-1}$ for some $t \geqslant 0$. There are two cases (the case $s=r$, $t=0$ is the case of σ a splittable monomorphism so we leave it out).

<u>Case 1</u>. If $s<r$, we define φ by

$\varphi(a_0 + \cdots + a_{r-2}x^{r-2}, b_0 + \cdots + b_r x^r) = a_0 x^t + \cdots + a_{t+s-1}x^{s-1}$.

<u>Case 2</u>. If $s \geqslant r$ then $t>0$, and we set

$\varphi(a_0 + \cdots + a_{r-2}x^{r-2}, b_0 + \cdots + b_r x^r) = b_0 x^{t-1} + \cdots + b_{t+s}x^{s-1}$. In either case φ makes the diagram (*) commute, and so the sequences given are almost split sequences. The work of Auslander and Reiten cited implies that almost split sequences are unique up to isomorphism, so the sequences given are, up to isomorphism, all of the almost split sequences. ■

Henceforth we will write $0 \longrightarrow V_r \longrightarrow E_r \longrightarrow V_r \longrightarrow 0$ for the unique almost split sequence corresponding to V_r. The property used in defining the almost split sequence has as a consequence the fact that there is a map φ to make

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{a} & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \uparrow\sigma & & \uparrow\varphi & & & & \\ 0 & \longrightarrow & V_r & \longrightarrow & E_r & \longrightarrow & V_r & \longrightarrow & 0 \end{array} \qquad \text{(II.2.6)}$$

commute iff $a\sigma$ is not a splittable monomorphism, i.e. if the top row has no summands of the form $0 \longrightarrow V_r = V_r \longrightarrow 0 \longrightarrow 0$.

If $\lambda X=Y$, or if $\lambda' X=Y$, let us say that X <u>lies</u> <u>over</u> Y. It is an easy calculation that (*1) $R \xrightarrow{x} R \xrightarrow{x} R$ lies over the almost split sequence for V_1, and that (*2) $R^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & x^{a_r} \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x^{a_r-1} & -1 \\ 0 & x \end{pmatrix}} R^2$ lies over the almost split sequence for V_{a_r}, $a_r>1$.

Let $X=F_2 \xrightarrow{f} F_1 \xrightarrow{g} F_0$. We can without loss of generality assume that f is diagonal, say $f = \begin{pmatrix} x^{a_1}\mu_1 & & 0 \\ & \ddots & \\ 0 & & x^{a_n}\mu_n \end{pmatrix}$ with each

μ_i a unit and $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$. We write $\coprod_{i=1}^{n} V_{a_i}$ for the induced decomposition of A, and $\sigma_r : V_{a_r} \longrightarrow A$ for the specific inclusions associated with this decomposition. Then if α and β are maps such that

$$\begin{array}{ccccccc}
F_2 & \xrightarrow{\ f\ } & F_1 & \longrightarrow & A & \longrightarrow & 0 \\
\uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\beta} & & \uparrow{\scriptstyle\sigma_r} & & \\
R^2 & \xrightarrow[\begin{pmatrix} 1 & 0 \\ 0 & x^{a_r} \end{pmatrix}]{} & R^2 & \longrightarrow & V_r & \longrightarrow & 0
\end{array}$$

commutes, i.e. α and β induce the inclusion σ_r, then we want to find conditions on g which imply that there is no map γ such that (α,β,γ) is a map from (*2) to X in C_3. (We also want to do this for maps of (*1) into X.) If no such map (α,β,γ) exists, then there is no map φ to make the diagram II.2.6 commute, so

$0 \longrightarrow V_{a_r} = V_{a_r} \longrightarrow 0 \longrightarrow 0$ must be a summand of λX, which means that $R \xrightarrow{x^{a_r}} R \xrightarrow{\nu} R$ is a summand of X for some unit ν. A simple condition which implies that a map γ does exist is that no entry in the r^{th} column of g be a unit, as then the diagram

$$\begin{tikzcd} {} \arrow[r, "f"] & {} \arrow[r, "g"] & {} \\ {} \arrow[u, "\alpha"] \arrow[r, "\begin{pmatrix}1&0\\0&x^{a_r}\end{pmatrix}"'] & {} \arrow[u, "\beta"'] \arrow[r, "\begin{pmatrix}x^{a_r-1}&-1\\0&x\end{pmatrix}"'] & {} \arrow[u, "\gamma"'] \end{tikzcd}$$

commutes with $\alpha=\begin{pmatrix}0&0\\ \vdots&\vdots\\ \vdots&1\\ \vdots&\vdots\\ 0&0\end{pmatrix}$, $\beta=\begin{pmatrix}0&0\\ \vdots&\vdots\\ \vdots&\mu_r\\ \vdots&\vdots\\ 0&0\end{pmatrix}$ (the 1 and μ_r appear in the r^{th} row) and $\gamma=\begin{pmatrix}0&x^{-1}g_{1r}\mu_r\\ \vdots&\vdots\\ \vdots&\vdots\\ 0&x^{-1}g_{nr}\mu_r\end{pmatrix}$ for $a_r>1$; for $a_r=1$ the diagram

$$\begin{tikzcd} {} \arrow[r, "f"] & {} \arrow[r, "g"] & {} \\ {} \arrow[u, "\alpha'"'] \arrow[r, "x"'] & {} \arrow[u, "\beta'"'] \arrow[r, "x"'] & {} \arrow[u, "\gamma'"'] \end{tikzcd}$$

commutes with α', β', and γ' the second columns of the α, β, and γ just given.

We may suppose that either $a_r<a_{r+1}$ or that $r=n$; if not, just shift rows and columns to get it that way. Suppose α, β are maps making

$$\begin{tikzcd} F_2 \arrow[r, "f"] & F_1 \\ R^2 \arrow[u, "\alpha"'] \arrow[r, "\begin{pmatrix}1&0\\0&x^{a_r}\end{pmatrix}"'] & R^2 \arrow[u, "\beta"'] \end{tikzcd}$$

commute and inducing the inclusion $V_{a_r}\xrightarrow{\sigma_r}A$ as before. If (α,β,γ) is to be a map of (*2) into X, then the following diagram must commute:

$$\begin{tikzcd}
{} \arrow[r, "\begin{pmatrix} x^{a_1}\mu_1 & & 0 \\ & \ddots & \\ 0 & & x^{a_n}\mu_n \end{pmatrix}"] & {} \arrow[r] & V_{a_r} \oplus M \arrow[r] & 0 \\
R^2 \arrow[u, "\alpha"] \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & x^{a_r} \end{pmatrix}"'] & R^2 \arrow[u, "\beta"] \arrow[r] & V_{a_r} \arrow[u, "\sigma_r"] \arrow[r] & 0
\end{tikzcd}$$

If e_1, e_2 are the usual basis elements for R^2, then β must send e_1 into $\mathrm{im}(f)$ and e_2 to something that goes to $\bar{1}$; so if $\beta = \begin{pmatrix} c_1 & d_1 \\ \vdots & \vdots \\ c_n & d_n \end{pmatrix}$, then d_r must be a unit, x^{a_i} must divide c_i for all i, and x^{a_i} must divide d_i for $i \neq r$.

A further condition on (α, β, γ) is that the diagram

$$\begin{tikzcd}
{} \arrow[r, "g"] & {} \\
{} \arrow[u, "\beta"] \arrow[r, "\begin{pmatrix} x^{a_r-1} & -1 \\ 0 & x \end{pmatrix}"'] & {} \arrow[u, "\gamma"']
\end{tikzcd}$$

commute. Let $\gamma = \begin{pmatrix} s_1 & t_1 \\ \vdots & \vdots \\ s_n & t_n \end{pmatrix}$; then

$$g \cdot \begin{pmatrix} c_1 & d_1 \\ \vdots & \vdots \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} s_1 & t_1 \\ \vdots & \vdots \\ s_n & t_n \end{pmatrix} \begin{pmatrix} x^{a_r-1} & -1 \\ 0 & x \end{pmatrix} = \begin{pmatrix} s_1 x^{a_r-1} & -s_1 + x t_1 \\ \vdots & \vdots \\ s_n x^{a_r-1} & -s_n + x t_n \end{pmatrix}.$$

The equation for the first column is $g \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} s_1 x^{a_r-1} \\ \vdots \\ s_n x^{a_r-1} \end{pmatrix}$, so

$$\begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix} = x^{1-a_r} \cdot g \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}; \text{ so we get } x \cdot \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} = g \cdot \begin{pmatrix} d_1 + c_1 x^{1-a_r} \\ \vdots \\ d_n + c_n x^{1-a_r} \end{pmatrix}$$

from the equation for the second column. Expanding the right-hand side of this gives

$$x \cdot \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} = \begin{pmatrix} g_{11}(d_1 + c_1 x^{1-a_r}) + \dots + g_{1r}(d_r + c_r x^{1-a_r}) \\ \vdots \\ g_{n1}(d_1 + c_1 x^{1-a_r}) + \dots + g_{nr}(d_r + c_r x^{1-a_r}) \end{pmatrix} \tag{II.2.7}$$

$$+ \begin{pmatrix} g_{1,r+1}(d_{r+1} + c_{r+1} x^{1-a_r}) + \dots + g_{1n}(d_n + c_n x^{1-a_r}) \\ \vdots \\ g_{n,r+1}(d_{r+1} + c_{r+1} x^{1-a_r}) + \dots + g_{nn}(d_n + c_n x^{1-a_r}) \end{pmatrix}.$$

The second term contains no units, as d_{r+j} is divisible by $x^{a_{r+j}-a_r}$ and c_{r+j} is divisible by $x^{a_{r+j}}$ for each $j>r$; thus there is a solution to the mapping problem iff there are no units in the first term on the right-hand side.

For $a_r=1$ we get a similar condition. Suppose α' and $\beta' = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$ make the diagram

$$
\begin{tikzcd}[column sep=large]
{} \arrow[r, "\begin{pmatrix} x^{a_1}\mu_1 & & 0 \\ & \ddots & \\ 0 & & x^{a_n}\mu_n \end{pmatrix}"] & {} \arrow[r, "proj"] & V_1 \oplus M \arrow[r] & 0 \\
{} \arrow[u, "\alpha"] \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & x^{a_r} \end{pmatrix}"'] & {} \arrow[u, "\beta"] \arrow[r, "proj"'] & V_1 \arrow[u, "incl"] \arrow[r] & 0
\end{tikzcd}
$$

commute; then as before we must have x^{a_i} dividing d_i if $i \neq r$ and d_r a unit. Thus if
$\begin{tikzcd} {} \arrow[r, "\alpha"] & {} \\ {} \arrow[u, "\beta"] \arrow[r, "x"'] & {} \arrow[u, "\gamma"'] \end{tikzcd}$
commutes, where $\gamma = \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}$,

we get (II.2.8) $x \cdot \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} = g \cdot \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$. Again there are no solutions to this iff there is a unit on the right-hand side; again these could occur only in the first r terms of each entry.

<u>Definition</u>. If $r \in R$, r can be written $r = x^{\beta}\mu$ for some unit μ; we write $\|r\| = \beta$.

<u>Lemma</u> <u>II.2.9</u>. (i) For $a_r = 1$, there is no solution to the mapping problem above iff (a) some entry of the r^{th} column of g is a unit; (b) for some unit g_{tr} in the r^{th} column of g, either $g_{ts} = 0$ or $\|g_{ts}\| + a_s > 0$ (i.e. x divides $g_{ts}d_s$) for each s with $1 \leqslant s \leqslant r-1$.

(ii) For $a_r > 1$, there is no solution to the mapping problem above iff (a′) some entry of the r^{th} column of g is a unit; (b′) for some unit g_{tr} in the r^{th} column of g, either $g_{ts}=0$ or $\|g_{ts}\| \geqslant \max(1, a_r - a_s)$ for each s with $1 \leqslant s \leqslant r-1$.

<u>Proof</u>. We have already seen the necessity of (a) and (a′). Suppose that (a) holds but that (b) fails for some s. Then g_{ts} is a unit, and the following is a solution: set $d_i=0$ for $i \neq s,r$; set $d_r=1$, and set $d_s=-g_{tr}g_{ts}^{-1}$. Now suppose that (a′) holds but that (b′) fails for some s. Then the following is a solution: set $c_i=0$ for $i \neq s$ and $d_i=0$ for $i \neq r$; set $d_r=1$, and $c_s=-x^{a_r-\beta-1}\mu^{-1}g_{tr}$, where $g_{ts}=x^{\beta}\mu$ with μ a unit. Thus if no map φ exists, (a) and (b) or (a′) and (b′) must hold. On the other hand if (a) and (b) or (a′) and (b′) hold then it is easy to see that the t^{th} entry on the right-hand side of II.2.8, or II.2.7 respectively, is a unit while this cannot be so on the left; so there is no solution, hence no map φ. ∎

Lemma II.2.9 shows how to detect a summand of X of the form $\xrightarrow{x^{a_r}}\xrightarrow{\nu}$; what we need now is a way to split off these summands once their presence has been detected, i.e. to find bases for F_2, F_1, and F_0 so that f and g become block-diagonal with 1×1 blocks in their upper left-hand corners. The conditions just obtained for X to have a summand of this type give a way to do this; we illustrate for a summand $\xrightarrow{x^{a_r}}\xrightarrow{1}$. We assume that f is already diagonal. We can use elementary row operations on g to get $g_{tr}=1$, $g_{jr}=0$ for $j\neq t$ without affecting f. If $g_{t,r-1}=0$, there is no need to do anything here. If $g_{t,r-1}\neq 0$, the next step is to multiply the r^{th} column of g by $-g_{t,r-1}$ and add the result to the $(r-1)^{st}$ column of g. In order to end up with a representation isomorphic to X, we are then forced to multiply the $(r-1)^{st}$ row of f by $g_{t,r-1}$ and add the result to the r^{th} row of f; then the new $(r,r-1)$ entry of f is $g_{t,r-1}x^{a_{r-1}}\mu_{r-1}$. But the conditions given in Lemma II.2.9 imply that $\|g_{t,r-1}x^{a_{r-1}}\mu_{r-1}\|\geqslant a_r$ (whether $a_r=1$ or $a_r>1$), and so we can use an elementary column operation on f (which doesn't affect g) to get f back to diagonal form. Continue this with $g_{t,r-2}$, etc.

Thus this algorithm actually says that $\xrightarrow{x^{a_r}}\xrightarrow{\nu}$ is a summand of X iff we can find bases in which it appears as 1×1 blocks in the most naive possible way. We now have the algorithm complete. Explicitly:

<u>Theorem</u> <u>II.2.10</u>. Let $X = F_2\xrightarrow{f}F_1\xrightarrow{g}F_0$ be in $\mathcal{B}_3$. The following algorithm diagonalizes X if X can be diagonalized.

Step 1. Use Lemma II.1.1 to write X as a direct sum of a diagonalized representation Y and a representation $X_0=F_2' \xrightarrow{f_0} F_1' \xrightarrow{g_0} F_0'$ in C_3.

Step 2. Diagonalize f_0. That is, obtain a representation $\hat{X}_0=F_2' \xrightarrow{\hat{f}_0} F_1' \xrightarrow{\hat{g}_0} F_0'$, isomorphic to X_0, in which $\hat{f}_0$ is a diagonal matrix, say $\hat{f}_0=\begin{pmatrix} x^{a_1}\mu_1 & & 0 \\ & \ddots & \\ 0 & & x^{a_n}\mu_n \end{pmatrix}$.

Step 3. Factor powers of x out of $\hat{f}_0$ and $\hat{g}_0$ until a unit entry appears in each; call the result $X'=F_2' \xrightarrow{f'} F_1' \xrightarrow{g'} F_0'$. Check to see whether $g'f'(F_2') \subseteq xF_0'$. If so, proceed to step 4. If not, use Lemma II.1.1 to split off any summands of the form $R = R = R$ that may occur. Multiply the appropriate powers of x back into these to get the actual summands; the result is a sum $X'' \oplus Y''$ of representations in which Y'' is diagonal. If $X''=0$, we're done; if not, set $\hat{X}_0=X''$ and repeat step 3.

Step 4. Using Lemma II.2.9 check for summands of the type $\xrightarrow{x^{a_r}}\xrightarrow{\nu}$, ν a unit. X' must have at least one of these if it diagonalizes, so if it hasn't, stop; X does not diagonalize. (This may necessitate shifting rows and columns to get the condition: $a_r < a_{r+1}$ or $r=n$, needed for Lemma II.2.9.) Again multiply the appropriate powers of x back in to obtain the actual summands; again the result will be a sum $X'' \oplus Y''$ with Y'' diagonal; again if $X'' \neq 0$ set $\hat{X}_0 = X''$ and repeat step 3. ∎

This algorithm says to diagonalize $\xrightarrow{f}\xrightarrow{g}$ as follows. Diagonalize f, and split off any summands of the form $R = R = R$. Then, after factoring powers of x out of g, use the criteria of Lemma II.2.9 (and then elementary row and column operations) to get a 1×1 block in g; repeat. If at any stage this cannot be done, then $\xrightarrow{f}\xrightarrow{g}$ does not diagonalize. We note that, as we mentioned in the introduction, this algorithm makes no reference to the apparatus of its proof, but operates only on matrices.

§3. The general case.

Before we can settle the general case we must first manage the case of A_4. To this end set $X = F_3 \xrightarrow{f} F_2 \xrightarrow{g} F_1 \xrightarrow{h} F_0$ in C_4. As in the previous section we can factor powers of x out of each of f, g, and h until each contains a unit entry, so we will assume that this has been done. We will use the notation ((x)) for the module coker(x) and work in the category U_2 unless otherwise stated. In U_2 the image of X is $\lambda' X = ((f)) \xrightarrow{\alpha} ((gf)) \xrightarrow{\beta} ((hgf))$.

Now assume that X diagonalizes. We first want to show that $\lambda'X$ must have a summand of the form $V_{r-t} \longrightarrow V_r = V_r$. Consider the short exact sequence $0 \longrightarrow ((gf)) \longrightarrow ((hgf)) \longrightarrow ((h)) \longrightarrow 0$, which corresponds to $\xrightarrow{gf}\xrightarrow{h}$ under λ. As X diagonalizes, this must diagonalize; because h has a unit entry, $s(((h))) < s(((hgf)))$, so this sequence must have a summand of the form $0 \longrightarrow V_r = V_r \longrightarrow 0 \longrightarrow 0$. This implies that $\lambda'X$ has a summand of the form $V_{r-t} \longrightarrow V_r = V_r$ as claimed.

Examine the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & 0 \longrightarrow & ((f)) & \xrightarrow{\alpha} & ((gf)) & \longrightarrow & ((g)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\beta} & & & & \\
 & & ((hgf)) & = & ((hgf)) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 \longrightarrow ((g)) \longrightarrow & & ((hg)) & \longrightarrow & ((h)) & \longrightarrow 0 & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \qquad \text{(II.3.1)}$$

This diagram is completely determined by α and β, so it is not hard to see that if $\xrightarrow{\alpha}\xrightarrow{\beta}$ has a summand of the form $V_{r-t} \longrightarrow V_r = V_r$, this diagram is forced to have a summand of the form

$$\begin{tikzcd} V_{r-t} \arrow[r] \arrow[d] & V_r \arrow[r] \arrow[d, equal] & V_t \\ V_r \arrow[r, equal] \arrow[d] & V_r \arrow[d] & \\ V_t \arrow[r, equal] & V_t \arrow[r] & 0 \end{tikzcd} \tag{II.3.2}$$

Our strategy is to use the techniques developed for A_3 to find summands of the top row of II.3.1 which "extend" to summands of the type II.3.2. If none of this sort exist, then by the observations just made X cannot diagonalize. If they do exist, we must exhibit an algorithmic way to split them off, i.e. a consistently split inclusion of $V_{r-t}\longrightarrow V_r$ V_r into $\lambda' X$.

Suppose V is an indecomposable summand of ((gf)) for which $0\longrightarrow V = V\longrightarrow 0\longrightarrow 0$ is a summand of the right-hand column of II.3.1, i.e. of the sequence $0\longrightarrow((gf))\longrightarrow((hgf))\longrightarrow((h))\longrightarrow 0$. Then by Lemma II.2.4 there must be, in the specific decomposition of ((gf)) obtained in diagonalizing the top row of II.3.1, a summand $V_r \cong V$ for which $0\longrightarrow V_r = V_r\longrightarrow 0\longrightarrow 0$ is a summand of the right-hand column of II.3.1. That is if

$$\begin{tikzcd} ((f)) \arrow[r, "\alpha"] \arrow[d, shift left, "\tau_1"] & ((gf)) \arrow[d, shift left, "\tau_2^*"] \\ V_{r-t} \arrow[r] \arrow[u, shift left, "\sigma_1"] & V_r \arrow[u, shift left, "\sigma_2"] \end{tikzcd}$$

commutes where (σ_1,σ_2) is a monomorphism consistently split by (τ_1,τ_2^*), then there must exist a monomorphism σ_3 and a consistent splitting (τ_2', τ_3) of (σ_2, σ_3) to make the diagram

$$\begin{tikzcd} ((gf)) \arrow[r, "\beta"] \arrow[d, shift left, "\tau_2'"] & ((hgf)) \arrow[d, shift left, "\tau_3"] \\ V_r \arrow[u, shift left, "\sigma_2"] \arrow[r, equal] & V_r \arrow[u, shift left, "\sigma_3"] \end{tikzcd}$$

commute. Then if in the diagram

$$\begin{tikzcd} ((f)) \arrow[r, "\alpha"] \arrow[d, shift left, "\tau_1"] & ((gf)) \arrow[r, "\beta"] \arrow[d, shift left, "\tau_2"] & ((hgf)) \arrow[d, shift left, "\tau_3"] \\ V_{r-t} \arrow[u, shift left, "\sigma_1"] \arrow[r] & V_r \arrow[u, shift left, "\sigma_2"] \arrow[r, equal] & V_r \arrow[u, shift left, "\sigma_3"] \end{tikzcd}$$

we define $\tau_2 = \tau_3\beta$, we get a consistent splitting (τ_1, τ_2, τ_3) of the monomorphism $(\sigma_1, \sigma_2, \sigma_3)$. Thus if a summand of the right sort can be found, it can be algorithmically split off.

We have proved the following theorem, which really states that the algorithm of theorem II.2.10 can continue to the case of A_4.

<u>Theorem</u> <u>II.3.3</u>. Let $X = F_3 \xrightarrow{f} F_2 \xrightarrow{g} F_1 \xrightarrow{h} F_0$ be in $\mathcal{B}_4$. If X can be diagonalized, the following algorithm diagonalizes X.

<u>Step</u> <u>1</u>. Use Lemma II.1.1 to write X as a direct sum of a diagonalized representation Y and a representation $X_0 = F_3' \xrightarrow{f_0} F_2' \xrightarrow{g_0} F_1' \xrightarrow{h_0} F_0'$ in $\mathcal{C}_4$.

<u>Step</u> <u>2</u>. Factor powers of x out of f_0, g_0, and h_0 until a unit entry appears in each; call the result $X' = F_3' \xrightarrow{f'} F_2' \xrightarrow{g'} F_1' \xrightarrow{h'} F_0'$. Check to see whether $h'g'f'(F_3') \subseteq xF_0'$. If so, proceed to step 3. If not, split off such summands

of the form $R = R = R = R$ as may occur. Multiply the appropriate powers of x back into these to get the actual summands; the result is a sum $X'' \oplus Y''$ with Y'' diagonal. If $X''=0$, stop; if not set $X_0=X''$ and repeat step 2.

<u>Step</u> <u>3</u>. Diagonalize $\xrightarrow{f'}\xrightarrow{g'}$. If this is not possible, stop; X does not diagonalize. If it is possible, use the techniques developed for A_3 to determine whether there is any summand V_r of $((g'f'))$ for which $0 \longrightarrow V_r = V_r \longrightarrow 0 \longrightarrow 0$ is a summand of the right-hand column of II.3.1. If there are none, then by the observations above, X cannot diagonalize. If there is one, we have found a summand $V_{r-t} \longrightarrow V_r = V_r$ of $\lambda'X$.

<u>Step</u> <u>4</u>. Split off the summand found in Step 3 in the manner indicated above. Multiply the appropriate powers of x back in as before; the result will be, as before, a sum $X'' \oplus Y''$ with Y'' diagonal. If $X'' = 0$, we're done; if not, set $X_0=X''$ and go back to Step 2. ∎

For the general case we prove, by induction on n, that this algorithm can continue. Let $X=F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \longrightarrow \cdots \xrightarrow{f_1} F_0$ and assume that each f_i has a unit entry but that $f_1 \circ \cdots \circ f_{n-1}(F_{n-1}) \subseteq xF_0$; as before we can easily reduce to this case. For each i let C_i denote the module $\mathrm{coker}(f_i)$, and let $C_{i,j}$ denote the module

$\operatorname{coker}(f_i \circ \cdots \circ f_j)$. Then $\lambda' X = C_{n-1} \xrightarrow{\alpha_{n-1}} C_{n-2,n-1} \xrightarrow{\alpha_{n-2}} \cdots \xrightarrow{\alpha_2} C_{1,n-1}$. Arguing in the same way as for A_4 we can show that $\lambda' X$ must have a summand of the form $V_{r_{n-1}} \to V_{r_{n-2}} \to \cdots \to V_{r_2} = V_{r_2}$ if X is to diagonalize. Just as in the previous case, this forces the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & C_{n-1} \arrow[r, "\alpha_3 \circ \cdots \circ \alpha_{n-1}"] \arrow[d] & C_{2,n-1} \arrow[r] \arrow[d, "\alpha_2"] & C_{2,n-2} \arrow[r] & 0 \\
 & C_{1,n-1} \arrow[r, equal] \arrow[d] & C_{1,n-1} \arrow[d] & & \\
0 \arrow[r] & C_{2,n-2} \arrow[r] & C_{1,n-2} \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd} \qquad (II.3.4)$$

to have a summand of the form

$$\begin{tikzcd}
V_{r_{n-1}} \arrow[r] \arrow[d] & V_{r_2} \arrow[r] \arrow[d, equal] & V_t \\
V_{r_2} \arrow[r, equal] \arrow[d] & V_{r_2} \arrow[d] & \\
V_t \arrow[r, equal] & V_t \arrow[r] & 0.
\end{tikzcd}$$

Now suppose X diagonalizes. Our induction hypothesis is that we can diagonalize, algorithmically, any diagonalizable object in C_{n-1}, so by this hypothesis we can diagonalize $F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_2} F_1$. This gives a diagonalization of the top row of II.3.4; just as in the case of A_4, there must be an indecomposable summand V of $C_{2,n-1}$ such that $0 \longrightarrow V = V \longrightarrow 0 \longrightarrow 0$ is a summand of the right-hand column of II.3.4. Again by Lemma II.2.4, there must then be, in the specific decomposition of $C_{2,n-1}$ obtained in diagonalizing $F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \longrightarrow \cdots \xrightarrow{f_2} F_1$, a summand $V_{r_2} \cong V$ for which $0 \longrightarrow V_{r_2} = V_{r_2} \longrightarrow 0 \longrightarrow 0$ is a summand of the right-hand column of II.3.4. Thus if $(\sigma_{n-1} \ldots, \sigma_2)$ is the monomorphism with consistent splitting $(\tau_{n-1}, \ldots, \tau_3, \tau_2^*)$, found by diagonalizing $F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \longrightarrow \cdots \xrightarrow{f_2} F_1$, which makes the diagram

$$\begin{tikzcd}
C_{n-1} \arrow[r, "\alpha_{n-1}"] & C_{n-2,n-1} \arrow[r, "\alpha_{n-2}"] & \cdots \arrow[r, "\alpha_3"] & C_{2,n-1} \arrow[d, shift left, "\tau_2^*"] \\
V_{r_{n-1}} \arrow[u, shift left, "\sigma_{n-1}"] \arrow[r] & V_{r_{n-2}} \arrow[u, shift left, "\sigma_{n-2}"] \arrow[r] & \cdots \arrow[r] & V_{r_2} \arrow[u, shift left, "\sigma_2"]
\end{tikzcd}$$

commute, then there must exist a monomorphism σ_1 and a consistent splitting (τ_2', τ_1) of (σ_2, σ_1) to make the diagram

$$\begin{array}{ccc} C_{2,n-1} & \xrightarrow{\alpha} & C_{1,n-1} \\ \sigma_2 \uparrow\downarrow \tau_2' & & \sigma_1 \uparrow\downarrow \tau_1 \\ V_{r_2} & = & V_{r_2} \end{array}$$

commute. Then if in the diagram

$$\begin{array}{ccccccccc} C_{n-1} & \xrightarrow{\alpha_{n-1}} & C_{n-2,n-1} & \xrightarrow{\alpha_{n-2}} \cdots \xrightarrow{\alpha_3} & C_{2,n-1} & \xrightarrow{\alpha_2} & C_{1,n-1} \\ \sigma_{n-1} \uparrow\downarrow \tau_{n-1} & & \sigma_{n-2} \uparrow\downarrow \tau_{n-2} & & \sigma_2 \uparrow\downarrow \tau_2 & & \sigma_1 \uparrow\downarrow \tau_1 \\ V_{r_{n-1}} & \longrightarrow & V_{r_{n-2}} & \longrightarrow \cdots \longrightarrow & V_{r_2} & = & V_{r_2} \end{array}$$

we define $\tau_2 = \tau_1 \alpha_2$, we get a consistent splitting $(\tau_{n-1}, \ldots, \tau_1)$ of the monomorphism $(\sigma_{n-1}, \ldots, \sigma_1)$.

We have proved the following theorem.

<u>Theorem</u> <u>II.3.5</u>. If the object $X = F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_1} F_0$ in $\mathbb{B}_n$ diagonalizes, then the following algorithm diagonalizes X.

<u>Step</u> <u>1</u>. Use Lemma II.1.1 to write X as a direct summand of a diagonal representation Y and a representation $X_0 = F_{n-1}' \xrightarrow{f_{n-1,0}} F_{n-2}' \xrightarrow{f_{n-2,0}} \cdots \xrightarrow{f_{1,0}} F_0'$ in $\mathcal{C}_n$.

<u>Step</u> <u>2</u>. Factor powers of x out of each $f_{i,0}$ until each contains a unit entry; call the result $X' = F_{n-1}' \xrightarrow{f_{n-1}'} F_{n-2}' \xrightarrow{f_{n-2}'} \cdots \xrightarrow{f_1'} F_0'$. Check to see whether $f_1' \circ \cdots \circ f_{n-1}'(F_{n-1}') \subseteq xF_0'$. If so, proceed to step 3. If not, split off any summands of the form $R = R = \cdots = R$

that may occur. Multiply the appropriate powers of x back into these to get the actual summands; the result is a sum $X'' \oplus Y''$ with Y'' diagonal. If $X''=0$, stop; if not, set $X_0=X''$ and repeat step 2.

__Step__ __3__. Diagonalize $F'_{n-1}\xrightarrow{f'_{n-1}}F'_{n-2}\longrightarrow\cdots\xrightarrow{f'_2}F'_1$. If this not possible, then X does not diagonalize. If it is possible, use the techniques described above to determine whether there is any summand V_{r_2} of $C'_{2,n-1} = ((f'_2\circ\cdots\circ f'_{n-1}))$ for which $0\longrightarrow V_{r_2} = V_{r_2}\longrightarrow 0\longrightarrow 0$ is a summand of the right-hand column of II.3.4. If there are none, then X does not diagonalize. If there is one, we have found a summand

$V_{r_{n-1}}\longrightarrow V_{r_{n-2}}\longrightarrow\cdots\longrightarrow V_{r_2} = V_{r_2}$ of $\lambda'X$.

__Step__ __4__. Split off the summand found in step 3 in the manner indicated above. Multiply the appropriate powers of x back in as before; the result will again be a sum $X'' \oplus Y''$ with Y'' diagonal. If $X''=0$, we're done; if not, set $X_0=X''$ and go back to step 2. ∎

Theorem II.3.5 says that the method of diagonalization established in theorems II.2.10 and II.3.3 for the cases of A_3 and A_4, respectively, can continue indefinitely. We remarked after II.2.10 that the method established there operated only on matrices; as the general algorithm is just a repetition of that method, it too operates only on matrices.

Chapter III. Examples in C_3.

In this chapter we exhibit some examples of the correspondence between objects in C_3 and short exact sequences, and of how considerations of the latter can be made to yield information about the former. We end the chapter by sketching a proof that C_3 (and hence Ω_3) is of wild representation type. For this chapter we will let ϵ be the short exact sequence

$0 \longrightarrow A \xrightarrow{a} B \xrightarrow{b} C \longrightarrow 0.$

<u>Example 1.</u> One term zero and split sequences.

(a) $A=0$. Then ϵ is the sequence $0 \longrightarrow 0 \longrightarrow B \xrightarrow{\sim} C \longrightarrow 0$, and it is easy to see that if $\theta\epsilon$ is $\xrightarrow{f} \xrightarrow{g}$ then f must be an isomorphism. Conversely if $X = \xrightarrow{f} \xrightarrow{g}$ in C_3 with f an isomorphism then λX plainly has the form $0 \longrightarrow 0 \longrightarrow B \xrightarrow{\sim} C \longrightarrow 0$. Clearly such an object diagonalizes.

(b) $C=0$. This time ϵ takes the form $0 \longrightarrow A \xrightarrow{\sim} B \longrightarrow 0 \longrightarrow 0$, and so if $\theta\epsilon = \xrightarrow{f} \xrightarrow{g}$ then g must be an isomorphism. Once again the converse holds, and once again every such object diagonalizes.

(c) ϵ split. Then ϵ is isomorphic to a direct sum of sequences of the types found in (a) and (b), and so corresponds to an object of the form $\xrightarrow{\begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}} \xrightarrow{\begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}}$ in C_3, where either (i) f_1 and g_2 are isomorphisms or (ii) f_2 and g_1 are isomorphisms. Again it is clear that any object of this sort diagonalizes.

__Example 2__. One term projective.

By 'one term projective' we mean that there is an integer t such that ϵ is a short exact sequence of $k[x]/(x^t)$-modules and that one of A, B, or C is projective in mod $(k[x]/(x^t))$. If A is projective in this category then it is also injective, so ϵ splits. If C is projective then of course ϵ also splits, so the only interesting case is when B alone is projective. Suppose $B \simeq \coprod^t V_p$. Then if $\theta\epsilon = \xrightarrow{f}\xrightarrow{g}$ then it must be that gf is isomorphic to x^pI, where I is the $t \times t$ identity matrix; conversely if $X = \xrightarrow{f}\xrightarrow{g}$ with $gf \simeq x^pI$, then λX has its middle term projective. The surprise is that these always decompose.

__Proposition III.1__. If $\epsilon = 0 \longrightarrow A \xrightarrow{a} B \xrightarrow{b} C \longrightarrow 0$ is exact with $B \simeq \coprod^t V_p$, then ϵ diagonalizes.

__Proof__. Suppose $s(C)=m$, and let $\coprod^m V_p \xrightarrow{\psi} C$ be the projective cover. By projectivity of B we get the map β in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{a} & B & \xrightarrow{b} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \| & & \\ 0 & \longrightarrow & K & \xrightarrow[q]{} & \coprod^{m} V_p & \xrightarrow[\psi]{} & C & \longrightarrow & 0 \end{array}$$

in which α is just the induced map. β is an epimorphism ($\psi\beta=b$, and ψ is essential), so there is a map $\tau\colon \coprod^{m} V_p \longrightarrow B$ to split it. Because β is an epimorphism the Snake Lemma tells us that α is also an epimorphism; a diagram chase says that we can define $\sigma=a^{-1}\tau a$ to get a splitting of α for which $a\sigma=\tau q$. Thus the bottom row is a summand of the top, and so ϵ decomposes into the direct sum of the (already diagonal) sequence $0\longrightarrow K\longrightarrow \coprod^{m} V_p\longrightarrow C\longrightarrow 0$ and a sequence of the form $0\longrightarrow A'\xrightarrow{\sim}B'\longrightarrow 0\longrightarrow 0$; so ϵ must diagonalize. ∎

<u>Example 3</u>. Both ends indecomposable.

Let $\delta_t = 0\longrightarrow V_i\xrightarrow{a_t}V_{i+j-t} \oplus V_t \xrightarrow{b_t}V_j\longrightarrow 0$,

where $0\leqslant t\leqslant \min(i,j)$, $a_t(a_0+\cdots+a_{i-1}x^{i-1}) =$

$$(a_0x^{j-1}+\cdots+a_{i-1}x^{i+j-t-1},a_0+\cdots+a_{t-1}x^{t-1}),$$

and $b_t(a_0+\cdots+a_{i+j-t-1}x^{i+j-t-1},\ b_0+\cdots+b_{t-1}x^{t-1}) =$

$$a_0+\cdots+a_{j-t-1}x^{j-t-1}+(a_{j-t}-b_0)x^{j-t}+\cdots+(a_{j-1}-b_{t-1})x^{j-1};$$

then $\theta\delta_t = \xrightarrow{\begin{bmatrix} x^{i-t} & -1 \\ 0 & x^t \end{bmatrix}} \xrightarrow{\begin{bmatrix} x^{j} & x^{j-t} \\ 0 & 1 \end{bmatrix}}$. (We used the special case of the almost-split sequence ($i=j, t=i-1$) in Chapter II.) We want to sketch a proof that these are, up to isomorphism, all the short exact sequences with left-hand end V_i and right-hand end V_j.

First, if $\epsilon = 0 \longrightarrow V_i \xrightarrow{a} B \xrightarrow{b} V_j \longrightarrow 0$ is exact, then $B \simeq V_{i+j-t} \oplus V_t$, as $s(B) \leqslant 2$. Note that $t \leqslant \min(i,j)$, as otherwise both indecomposable summands of B have length less than j. If t=i or t=j then ϵ splits, as set $r = \max(i,j)$; then over ${}^{k[x]}/_{(x^r)}$, either V_i is injective or V_j is projective. Suppose then that $t < \min(i,j)$. Then a must map V_i monomorphically into V_{i+j-t}, so the isomorphism $B \simeq V_{i+j-t} \oplus V_t$ extends to an isomorphism of sequences, i.e. we may work with the sequence $0 \longrightarrow V_i \xrightarrow{a} V_{i+j-t} \oplus V_t \xrightarrow{b} V_j \longrightarrow 0$. Then up to multiplication by units,

$a(a_0 + \cdots + a_{i-1}x^{i-1}) = (a_0 x^{j-t} + \cdots + a_{i-1}x^{i+j-1},$

$a_0 x^p + \cdots + a_{t-p-1}x^{t-1})$. The cokernel map is forced to be

$b(a_0 + \cdots + a_{i+j-t-1}x^{i+j-t-1},\ b_0 + \cdots + b_{t-1}x^{t-1}) =$

$$a_0 + \cdots + (a_{j-t-p} - b_0)x^{j-t-p} + \cdots + (a_{j-1} - b_{t+p-1})x^{j-1}.$$

Unless p=0, this is not an R-map, so p=0 and hence $\epsilon \simeq \delta_t$.

A result of M. Loupias [L75] on representations of partially ordered sets implies that C_3 is of infinite representation type; to conclude this chapter we sketch a proof that C_3 is in fact of wild type. Let $S_t = {}^{k[x]}/_{(x^t)}$, and let $\Lambda_{2,t}$ be the ring of lower-triangular 2x2 matrices with entries from S_t. We will exhibit a functor σ which embeds the category $I_{n,m}$ of representations of the diagram

$$\begin{tikzcd} & & & \arrow[d] & & \\ 1 \arrow[r] & \arrow[r] & \cdots \arrow[r] & n+1 \arrow[r] & \cdots \arrow[r] & n+m+1 \end{tikzcd}$$

into $\mathrm{Mod}(\Lambda_{2,n+m+1})$. This will prove our claim, as it is known that for $n,m>3$ the category $I_{n,m}$ is of wild type ([DR76]).

Suppose Z =

$$\begin{tikzcd} & & & W \arrow[d, "\beta"] & & & \\ M_1 \arrow[r, "\alpha_1"'] & M_2 \arrow[r, "\alpha_2"'] & \cdots \arrow[r, "\alpha_n"'] & M_{n+1} \arrow[r, "\gamma_1"'] & M_{n+2} \arrow[r] & \cdots \arrow[r, "\gamma_m"'] & M_{n+m} \end{tikzcd}$$

is in $I_{n,m}$. Let $M=W \oplus \coprod_{i=n+2}^{n+m+1} M_i$ and $N= \coprod_{i=1}^{n+m+1} M_i$. Then M is an S_{n+m+1} module with x acting like β on W and like γ_i on M_{n+i}, $1\leqslant i\leqslant m$, and N is an S_{n+m+1}-module with x acting like α_i on M_i for $1\leqslant i\leqslant n$ and like γ_i on M_i for $n+1\leqslant i\leqslant n+m$. Finally if we let $f:M\longrightarrow N$ be the map sending W to M_{n+1} by β and sending M_{n+i} to M_{n+i} by the identity for $2\leqslant i\leqslant m+3$, then $\sigma Z = (M,N,f)$ is a $\Lambda_{2,n+m+1}$-module.

If $(q,p_1,\ldots,p_{n+m+1}):Z\longrightarrow Z'$ is a morphism in $I_{n,m}$, i.e. if the diagram (*)

$$
\begin{tikzcd}
& & & W' \arrow[d, "\beta'"] & & & \\
M_1' \arrow[r, "\alpha_1'"] & M_2' \arrow[r, "\alpha_2'"] & \cdots \arrow[r, "\alpha_n'"] & M_{n+1}' \arrow[r, "\gamma_1'"] & M_{n+2}' \arrow[r, "\gamma_2'"] & \cdots \arrow[r, "\gamma_m'"] & M_{n+m+1}' \\
& & & W \arrow[uu, bend left, "q"] \arrow[d, "\beta"] & & & \\
M_1 \arrow[r, "\alpha_1"'] \arrow[uu, "\rho_1"] & M_2 \arrow[r, "\alpha_2"'] \arrow[uu, "\rho_2"] & \cdots \arrow[r, "\alpha_n"'] & M_{n+1} \arrow[r, "\gamma_1"'] \arrow[uu, bend right, "\rho_{n+1}"'] & M_{n+2} \arrow[r, "\gamma_2"'] \arrow[uu, "\rho_{n+2}"] & \cdots \arrow[r, "\gamma_m"'] & M_{n+m+1} \arrow[uu, "\rho_{n+m+1}"]
\end{tikzcd}
$$

commutes, we set $p_M=(q,p_{n+2},\ldots,p_{n+m+1}):M\longrightarrow M'$ and $p_N=(p_1,\ldots,p_{n+m+1}):N\longrightarrow N'$. Then the diagram
$$
\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "p_M"'] & N \arrow[d, "p_N"] \\
M \arrow[r, "f'"] & N'
\end{tikzcd}
$$
commutes, so $\sigma(q,p_1,\ldots,p_{n,m+1})= (p_M,p_N):(M,N,f)\longrightarrow(M',N',f')$ in mod $(\Lambda_{2,n+m+1})$. It is easy to see that σ is a functor.

If Z and Z' are in $I_{n,m}$ and $\sigma Z = (M,N,f)$, $\sigma Z'=(M',N',f')$, then if $(\rho,\tau):\sigma Z\longrightarrow\sigma Z'$ then as M,M' and N,N' are sums of the vector spaces W,W', M_i,M_i' in Z and Z', the maps ρ and τ are sums of vector space maps between these. Because of the commutativity condition $f'\rho=\tau f$, these maps make a diagram like (*) commute, and so give a morphism $\delta:Z\longrightarrow Z'$ for which $\sigma\delta=(\rho,\tau)$. Thus σ is full. If (ρ,τ) is an isomorphism, then each of the vector space maps will be an isomorphism, so σ reflects isomorphisms. σ is clearly faithful, so we're done: σ embeds $I_{n,m}$ into $\mathrm{Mod}(\Lambda_{2,n+m+1})$.

Chapter IV. Reduction.

In this chapter we show that there is a full dense functor from the category Ω_n of all representations of A_n by free R-modules to the category $\mathcal{B}_n$ in which the results of chapter II were obtained. Perhaps the best way to explain our strategy is to sketch it for an object of Ω_3.

Let $X = F_2 \xrightarrow{f} F_1 \xrightarrow{g} F_0$ be in Ω_3. Our first step will be to prove that we can split off the kernel F' of f and the free part F'' of the cokernel of g to write X as the direct sum of representations $F' \longrightarrow 0 \longrightarrow F''$ and $X' = F_2' \xrightarrow{f'} F_1' \xrightarrow{g'} F_0'$, where now f' is a monomorphism and g' is full.

We next want to go to a representation in which the maps are both monomorphisms. Let $H_1' = \ker g'$ and $H_2' = \ker(g'f')$; H_1' and H_2' are summands of F_1', F_2' respectively, because $\mathrm{im}(g')$ and $\mathrm{im}(g'f')$ are free. The first part of the reduction is to set $\mu X' = F_2'/H_2' \xrightarrow{\bar{f}'} F_1'/H_1' \xrightarrow{\bar{g}'} F_0'$, where $\bar{f}'$ is the map given by $z + H_2' \longrightarrow f'(z) + H_1'$, and $\bar{g}'(z + H_1') = g'(z)$. Clearly $\mu X'$ is a representation in which both $\bar{f}'$ and $\bar{g}'$ are monomorphisms.

Now let $Y = F_2 \xrightarrow{f} F_1 \xrightarrow{g} F_0$ be such that f and g are both monomorphisms and g is full; this is the type of object that is in the image of μ. We need to go next to an object with the first map full also; this will be an object in $\mathcal{B}_3$. If we let F_1^* be the unique smallest summand of F_1 such that $f(F_2) \subseteq F_1^*$, and F_0^* be the unique smallest summand of F_0 with $g(F_1^*) \subseteq F_0^*$, then $\nu Y = F_2 \xrightarrow{f} F_1^* \xrightarrow{\hat{g}} F_0^*$, where $\hat{g}$ is g restricted to F_1^*, is the desired object in $\mathcal{B}_3$.

In order to do this in general we need to define two categories intermediate between Ω_n and $\mathcal{B}_n$.

Definitions. Let Ω_n' be the full subcategory of Ω_n consisting of those objects $X = F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_1} F_0$ in which f_{n-1} is a monomorphism and f_1 is full. Let $\mathcal{Q}_n$ be the full subcategory of Ω_n' comprising those objects for which each f_i is also a monomorphism.

Recall that $\mathcal{B}_n$ is the full subcategory of $\mathcal{Q}_n$ in which each f_i is also full.

We first want to show that every object in Ω_n can be written as the direct sum of an object in Ω_n' and one of the form $F' \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow F''$ (where either F' or F'' or both may be zero). Let $X = F_{n-1} \xrightarrow{f_{n-1}} F_{n-2} \longrightarrow \cdots \xrightarrow{f_1} F_0$ be in Ω_n. If $\mathrm{Ker}(f_{n-1}) = F_x'$ is not zero, it is a summand, as $\mathrm{im}(f_{n-1})$ is free; choose a free complement $\hat{F}_x$ to F_x'. Then the diagram

$$\begin{tikzcd}[column sep=large]
F_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "\wr"'] & F_{n-2} \arrow[r, "f_{n-2}"] \arrow[d, "\wr\ \begin{bmatrix}1\\0\end{bmatrix}"'] & \cdots \arrow[r, "f_1"] & F_0 \arrow[d, "\wr\ \begin{bmatrix}1\\0\end{bmatrix}"'] \\
\begin{matrix}\hat{F}_x\\ \oplus \\ F'_x\end{matrix} \arrow[r, "{\begin{bmatrix}f_{n-1}|\hat{F}_x & 0\\ 0 & 0\end{bmatrix}}"'] & \begin{matrix}F_{n-2}\\ \oplus \\ 0\end{matrix} \arrow[r, "{\begin{bmatrix}f_{n-2} & 0\\ 0 & 0\end{bmatrix}}"'] & \cdots \arrow[r, "{\begin{bmatrix}f_1 & 0\\ 0 & 0\end{bmatrix}}"'] & \begin{matrix}F_0\\ \oplus \\ 0\end{matrix}
\end{tikzcd}$$

commutes.

Now suppose that f_1 is not full; then $\operatorname{coker}(f_1) \simeq F \oplus T$, where T is the torsion submodule of $\operatorname{coker}(f_1)$. If $F_1 \xrightarrow{f_1} F_0 \xrightarrow{\pi} F \oplus T \longrightarrow 0$ is exact, let $T^*_x = \pi^{-1}(T)$; then T^*_x is the unique smallest summand of F_0 that contains $\operatorname{im}(f_1)$. Choose a free complement F''_x for T^*_x in F_0; then the diagram

$$\begin{tikzcd}[column sep=large]
F_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "\wr\ \begin{bmatrix}1\\0\end{bmatrix}"'] & F_{n-2} \arrow[r] \arrow[d, "\wr\ \begin{bmatrix}1\\0\end{bmatrix}"'] & \cdots \arrow[r] & F_1 \arrow[r, "f_1"] \arrow[d, "\wr\ \begin{bmatrix}1\\0\end{bmatrix}"'] & F_0 \arrow[d, "\wr"'] \\
\begin{matrix}F_{n-1}\\ \oplus \\ 0\end{matrix} \arrow[r, "{\begin{bmatrix}f_{n-1} & 0\\ 0 & 0\end{bmatrix}}"'] & \begin{matrix}F_{n-2}\\ \oplus \\ 0\end{matrix} \arrow[r] & \cdots \arrow[r, "{\begin{bmatrix}f_2 & 0\\ 0 & 0\end{bmatrix}}"'] & \begin{matrix}F_1\\ \oplus \\ 0\end{matrix} \arrow[r, "{\begin{bmatrix}f_1 & 0\\ 0 & 0\end{bmatrix}}"'] & \begin{matrix}T^*_x\\ \oplus \\ F'_x\end{matrix}
\end{tikzcd}$$

commutes. We can now write X as the direct sum of $\hat{F}_x \longrightarrow F_{n-2} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow T^*_x$ and $F'_x \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow F''_x$.

Given any summand of a free R-module F, it is not hard to show that we can pick a free complement to it in F which has as a basis some subset of the set $\{e_1,\ldots,e_n\}$ of standard basis vectors for R^n. If $\hat{F}_x$ and F_x^* are always chosen this way (where if either one of e_i, e_j is a possible choice, we choose $e_{\min(i,j)}$), passage from Ω_n to Ω_n' becomes functorial. Call this functor σ; σ is clearly full and dense.

We next describe a functor μ from Ω_n' to $\mathcal{Q}_n$.

<u>Definition</u>. Let $X=F_{n-1}\xrightarrow{f_{n-1}}F_{n-2}\longrightarrow\cdots\xrightarrow{f_1}F_0$

and $X'=F_{n-1}'\xrightarrow{f_{n-1}'}F_{n-2}'\longrightarrow\cdots\xrightarrow{f_1'}F_0'$ be in Ω_n', and for each $i=1,2,\ldots n-1$ set $H_i = \mathrm{Ker}(f_1\circ\cdots\circ f_i)$. The object μX in $\mathcal{Q}_n$ is the representation

$F_{n-1}/_{H_{n-1}}\xrightarrow{\bar{f}_{n-1}}F_{n-2}/_{H_{n-2}}\longrightarrow\cdots\xrightarrow{\bar{f}_2}F_1/_{H_1}\xrightarrow{\bar{f}_1}F_0$, where the $\bar{f}_i$ are the induced maps $z+H_i\longmapsto f_i(z)+H_{i-1}$. H_i is a summand for each i, so μX is a representation of A_n by free R-modules; as $\bar{f}_1\circ\cdots\circ\bar{f}_i$ is a monomorphism for each i, each $\bar{f}_i$ is a monomorphism, so this representation is in $\mathcal{Q}_n$. If $\alpha=(\alpha_{n-1},\ldots,\alpha_0):X\longrightarrow X'$ in Ω_n', then $\mu\alpha=(\alpha_{n-1}^{\mu},\ldots,\alpha_0^{\mu}):\mu X\longrightarrow\mu X'$ is given by $\alpha_i^{\mu}(z+H_i)=\alpha_i(z)+H_i'$ for $1\leqslant i\leqslant n-1$, and $\alpha_0^{\mu}=\alpha_0$. It is tedious but routine to check that this actually does give a morphism from μX to $\mu X'$. Another tedious but routine chore is to check that μ preserves compositions; as $\mu 1=1$ is obvious, μ is a functor.

$\mathcal{Q}_n$ is contained in Ω_n', and $\mu X=X$ for X in $\mathcal{Q}_n$, so μ is dense. μ is clearly not faithful, but it is full as the following shows. Suppose $\beta=(\beta_{n-1},\ldots,\beta_0):\mu X\longrightarrow\mu X'$ in $\mathcal{Q}_n$, and write $F_i={}^{F_i}/_{H_i}\oplus H_i$, etc., for each i (we take $H_0 = 0$). Then if we choose $\alpha_i = \begin{bmatrix}\beta_i & 0\\ 0 & 0\end{bmatrix}$ for each i, then $\mu\alpha = \beta$, so μ is full.

We now want to describe a functor ν from $\mathcal{Q}_n$ to $\mathcal{B}_n$. If $f:F\longrightarrow F'$ is a map of free R-modules, let $f(F)^*$ be the unique smallest summand of F' that contains $f(F)$. If $X=F_{n-1}\xrightarrow{f_{n-1}}F_{n-2}\longrightarrow\cdots\xrightarrow{f_1}F_0$ and $X'=F_{n-1}'\xrightarrow{f_{n-1}'}F_{n-2}'\longrightarrow\cdots\xrightarrow{f_1'}F_0'$ are in $\mathcal{Q}_n$, let $f_{n-1}(F_{n-1})^*=G_{n-2}$, $f_{n-2}(f_{n-1}(F_{n-1})^*)^*=G_{n-3}$, etc. Then νX is the object $F_{n-1}\xrightarrow{f_{n-1}}G_{n-2}\xrightarrow{\hat{f}_{n-2}}G_{n-3}\longrightarrow\cdots\xrightarrow{\hat{f}_1}G_0$, where $\hat{f}_i$ is the restriction of f_i to G_i. νX is certainly in $\mathcal{B}_n$. If $\alpha=(\alpha_{n-1},\ldots,\alpha_0):X\longrightarrow X'$ in $\mathcal{Q}_n$, then define α_i^ν to be the restriction of α_i to G_i; then it is clear that $\nu\alpha=(\alpha_{n-1},\alpha_{n-2}^\nu,\ldots,\alpha_0^\nu)$ is a morphism from νX to $\nu X'$ in $\mathcal{B}_n$. Again it is routine to check that ν is a functor.

As before, $\mathcal{B}_n$ is contained in $\mathcal{Q}_n$ and $\nu X=X$ for X in $\mathcal{B}_n$, so ν is dense. We will show that ν is full, i.e. that given $\beta=(\beta_{n-1},\ldots,\beta_0):\nu X\longrightarrow \nu X'$ in $\mathcal{B}_n$ we can find a map $\alpha=(\alpha_{n-1},\ldots,\alpha_0):X\longrightarrow X'$ in $\mathcal{Q}_n$ for which $\nu\alpha=\beta$. We must have $\alpha_{n-1}=\beta_{n-1}$. For $0\leqslant i\leqslant n-2$, let Q_i be a free complement for G_i, and define $\alpha_i:G_i \oplus Q_i \longrightarrow G_i' \oplus Q_i'$ by $\alpha_i=\begin{bmatrix}\beta_i & 0\\ 0 & \rho_i\end{bmatrix}$, where

$(\rho_{n-1},\ldots,\rho_0)$ is any morphism from

$Q_{n-2}\xrightarrow{f_{n-2}}Q_{n-3}\xrightarrow{f_{n-3}}\cdots\xrightarrow{f_1}Q_0$ to

$Q_{n-2}'\xrightarrow{f_{n-2}'}Q_{n-3}'\longrightarrow\cdots\xrightarrow{f_1'}Q_0'$ in $\mathcal{Q}_n$. It is then clear that $\nu\alpha=\beta$. Also clear from this is that ν cannot be faithful.

The composition $\nu\mu\sigma$ is the promised full dense functor from Ω_n' to $\mathcal{B}_n$.

Chapter V. A result on indecomposability of short exact sequences.

For this chapter we will let Λ be an artin algebra of finite representation type, and $\mathrm{mod}(\Lambda)$ the category of finitely generated (left) Λ-modules. Our result says that if C is in $\mathrm{mod}(\Lambda)$, then there are only finitely many modules A in $\mathrm{mod}(\Lambda)$ for which there is a short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ which is indecomposable. To prove it we need the following version of a result of M. Auslander, in which Tr and D are the usual artin algebra transpose and dual (see Auslander [A76]), and in which we write (X,C) for $\mathrm{Hom}_\Lambda(X,C)$. The submodule $P(X,C)$ of (X,C) consists of those

maps $f:X \longrightarrow C$ for which there exists a factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & C \\ & P \arrow[ur, "\beta"'] & \end{tikzcd}$$ with P projective.

<u>Theorem A</u>. Let C be $\mathrm{mod}(\Lambda)$. Let $A_1,\ldots,A_m$ be a complete list of all non-injective indecomposable modules in $\mathrm{mod}(\Lambda)$ and let $X_i = \mathrm{TrD}A_i$. For each i, $(X_i,C)/_{P(X_i,C)}$ is an $(\mathrm{End}\, X_i)^{op}$-module of finite length. Let $S_{i_1},\ldots,S_{i_{d_i}}$ be a complete set of non-isomorphic simple $(\mathrm{End}\, X_i)^{op}$-modules, and for each $(\mathrm{End}\, X_i)^{op}$-submodule H of (X_i,C) containing $P(X_i,C)$ let $n_1(A_i,H),\ldots,n_{d_i}(A_i,H)$ be the uniquely

determined non-negative integers so that the (End $X_i)^{op}$-socle of ${}^{(X_i,C)}/_{H}$ is isomorphic to to $\coprod_{j=1}^{d_i} S_{ij}^{n_j(A_i,H)}$. Finally let $n(A_i)=\max(n_j(A_i,H))$ as j runs through $1,2,\ldots,d_i$ and as H runs through all (End $X_i)^{op}$-submodules of (X_i,C) containing $P(X_i,C)$. Then

(a) $n(A_i)$ is finite; and

(b) if $k > n(A_i)$ and $0 \longrightarrow A_i^k \xrightarrow{g} B \longrightarrow C \longrightarrow 0$ is exact, then A_i^k contains a submodule A' (isomorphic to $A_i^{k-n(A_i)}$) such that $g(A')$ is a summand of B.

Using the notation of theorem A we have

<u>Theorem</u> <u>V.1</u>. Fix C in mod(Λ). Then there are only a finite number of modules A in mod(Λ) such that $0 \longrightarrow A \xrightarrow{g} B \longrightarrow C \longrightarrow 0$ is indecomposable as a short exact sequence. In fact, if $A \simeq \coprod_{i=1}^{m} A_i^{p_i}$, then if $p_i > n(A_i)$ for some i, the sequence decomposes.

<u>Proof</u>. We first mention that if A has an injective summand then the sequence decomposes, so we have lost nothing by not including these in the list of A_i's. Suppose $p > n(A_i)$, and form the pushout diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "g"] \arrow[d, "surj"] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A_i^{p_i} \arrow[r, "h'"'] & D \arrow[r] & C \arrow[r] & 0. \end{tikzcd}$$

Because $p_i > n(A_i)$, $A_i^{p_i}$ has a submodule A' for which $h'(A')$ is a summand of D (so A' is actually a summand of $A_i^{p_i}$) by theorem A. Let A'', A''' be such that $A' \oplus A'' = A$, $A' \oplus A''' = A_i^{p_i}$. Then we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A' \oplus A'' \arrow[r, "g"] \arrow[d, "\begin{bmatrix} 1 & 0 \\ 0 & \rho \end{bmatrix}"] & B \arrow[r] \arrow[d, "\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}"] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A' \oplus A''' \arrow[r, "\begin{bmatrix} \alpha & 0 \\ 0 & h \end{bmatrix}"'] & \alpha(A') \oplus B' \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

in which $\alpha = h'|_{A'}$ is an isomorphism and $h = h'|_{A'''}$. Then $A' \xrightarrow{1} A' \xrightarrow{g} g(A')$ is a monomorphism which is split by $B \xrightarrow{\mu_1} \alpha(A') \xrightarrow{\alpha^{-1}} A'$, so $g(A')$ is a summand of B. Thus $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ decomposes as a short exact sequence. ■

<u>Remark</u>. In mod(R) it is easy to calculate that $n(V_j, C) = s(C)$ for each V_j. Let $s(C) = t$, and let $p > 0$ be an integer. Then theorem V.1 implies that there are no more than $1+p+p^2+\cdots+p^t = \frac{1+p^{t+1}}{1-p}$ non-isomorphic indecomposable short exact sequences $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of torsion R-modules in

which the length of the longest indecomposable summand of A is at most p. In chapter III we exhibited $\min(r,p)+1$ non-isomorphic indecomposable short exact sequences $0 \longrightarrow V_p \longrightarrow V_{r+p-t} \oplus V_t \longrightarrow V_r \longrightarrow 0$; if $r=p$, then $\min(r,p)+1=1+p=\frac{1+p^2}{1-p}$, so this bound is the best possible.

__Corollary V.2.__ Let $X = R^n \xrightarrow{f} R^n \xrightarrow{g} R^n$ be in C_3 and let $\lambda X = 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ (so $f,g \in M_n(R)$). Suppose a diagonal form of f is $\begin{pmatrix} x^{a_1}\mu_1 & & 0 \\ & \ddots & \\ 0 & & x^{a_n}\mu_n \end{pmatrix}$ where each μ_i is a unit. Then if any exponent a_i appears more than $s(C)$ times, X must decompose. In particular if we set $d(f)=\max(a_i \mid i=1,\ldots,n)$ then if $d(f)\cdot s(C)<n$ then X must decompose.

__Proof.__ If a_i appears more than $s(C)$ times than the module A has more than $s(C)$ summands V_{a_i}. ∎

Certainly $\deg(\det(f)) \geqslant d(f)$, so this corollary implies that if $\deg(\det(f))\cdot s(C)<n$ then X decomposes. If g is an isomorphism then $\xrightarrow{f}\xrightarrow{g}$ certainly decomposes. If g is not an isomorphism then $s(C) \geqslant 1$, so if $\deg(\det(f))<n$ then $\xrightarrow{f}\xrightarrow{g}$ decomposes.

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