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9 Appendix A: Summary of Section 3, The Lotfi Model

Variable Was	Now	Description
i	omit	Part Type
j	omit	Manufacturing Step
m, n	omit	Machinery Type
t	t	Time Period
T	T	Planning or Prediction Horizon
z'_{ijm}, z''_{ijn}	omit	Manufacturing Capability Incidence Matrix Element
d'_{ijm} , d''_{ijn}	d' , d''	Old Machine and New Module Variable Costs
e'_{ijm} , e''_{ijn}	0	Old Machine and New Module Fixed Element Cost
p'_{ijm}, p''_{ijn}	omit	Old Machine and New Module Processing Time
ST'_{ijm}, ST''	omit	Old Machine and New Module Setup
SC'_{ijm}, SC''_{n}	omit	Old Machine and New Module Setup Cost
L_t	omit	Number of Operating Cycles in a Period
$g(L_t)$	omit	Cost function Associated with Operating Cycles
P_{it}	P_t	Price
D_{it}	D_t	Quantity Demanded
MC_{it}	MC_t	Marginal Cost
$C''_{n,t}$	C''_t	Capital Cost for New Modules
$S_{m,t}$	S_t	Salvage Value
r_t	r_t	Discount Factor
TR_t,	omit	Tax-Rate
$IC_{n,t}$	omit	Investment Tax Credit
$?_t$	r_t	Discount factor
h_{it}	h_t	Unit Inventory Carrying Cost
K	omit	Hours of Available Capacity
g	omit	Capacity Slack
x_t	x_t	Efficiency, Marginal Product
$M_{m,t}$	M_t	Number of Old Machines
$N_{n,t}$	N_t	Number of New Machines
X_{it}	X_t	Quantity of Parts produced on Old Machines
Y_{it}	Y_t	Quantity of Parts produced on New modules
w_n	omit	Binary Variable tracking Implementation of New Modules
s_{it}.	omit	Number of Setups on Current Machines
v_t,	omit	Binary Variable tracking Use of Old Machines
t_m	T	Last period by which a machine type must be phased out.
$\sum_n y_{ijn,t}$	Y_t	Quantity of Parts produced on New modules

Objective 1:

$$\max\ f_1 = -\sum_t \sum_n \left(N_{n,t} - N_{n,t-1}\right) C''_{n,t} \left(1 - IC_{n,t}\right) r_t + \sum_t \sum_m \left(M_{m,t} - M_{m,t-1}\right) S_{m,t} r_t$$

$$-\left[\sum_t \sum_i \sum_j \sum_n d''_{ijn} y_{ijn,t} h_t + \sum_t \sum_i \sum_j \sum_m d'_{ijm} X_{it} h_t\right]$$

$$-\left[\sum_t \sum_i \sum_j \sum_n e''_{ijn} N_{n,t} h_t + \sum_t \sum_i \sum_j \sum_m e'_{ijm} M_{m,t} h_t\right]$$

$$-\sum_t \left[L_t \sum_n SC''_n N_{nt} + \sum_i \sum_j \sum_m SC'_{ijm} s_{it}\right] h_t - \sum_t \left[\sum_i \left(\frac{\left(D_{it}/L_t\right)}{2}\right) h_{it} + g\left(L_t\right)\right] h_t$$

Becomes:

$$\max\ f_1 = \sum_t \left(P_t - MC_t\right) D_t r_t - \sum_t \left(N_t - N_{t-1}\right) C''_t r_t + \sum_t \left(M_t - M_{t-1}\right) S_t r_t$$

$$-\left[\sum_t d'' Y_t r_t + \sum_t d' X_t r_t\right] - \sum_t \frac{D_t}{2} h_t r_t$$

Objective 2

$$\text{Max}\ \ f_2 = -\max\left[\sum_n \left(N_{nt} - N_{n,t-1}\right) C''_{nt} - \sum_m \left(M_{m,t-1} - M_{mt}\right) S_{mt}\right]$$

subject to

$$\sum_n \left(N_{nt} - N_{n,t-1}\right) C''_{nt} - \sum_m \left(M_{m,t-1} - M_{mt}\right) S_{mt} \leq$$

$$\max\left[\sum_n \left(N_{nt} - N_{n,t-1}\right) C''_{nt} - \sum_m \left(M_{m,t-1} - M_{mt}\right) S_{mt}\right] \quad \forall t$$

Becomes:

$$\text{Max}\ \ f_2 = -\max\left[\left(N_t - N_{t-1}\right) C''_t - \left(M_{t-1} - M_t\right) S_t\right]$$

subject to

$$(N_t - N_{t-1})C_t'' - (M_{t-1} - M_t)S_t \le \max\left[(N_t - N_{t-1})C_t'' - (M_{t-1} - M_t)S_t\right] \quad \forall t$$

Objective 3

$$f_3 = -\sum_n w_n$$

subject to:

$$\sum_t N_{nt} \le V w_n \quad \forall t$$

Becomes:

Nothing, Completely Omittted

Objective 4

Max $f_4 = -f_4'$

subject to:

$$\sum_m \left(M_{m,t-1} - M_{m,t}\right) \le f_4' \quad \forall t$$

Becomes:

Max $f_4 = -f_4'$

subject to:

$$\left(M_{t-1} - M_t\right) \le f_4' \quad \forall t$$

Objective 5

Max $f_5 = -f_5'$

subject to:

$$\sum_n \left(N_{n,t} - N_{n,t-1}\right) \le f_5' \quad \forall t$$

Becomes:

Max $f_5 = -f_5'$

subject to:

$$\left(N_t - N_{t-1}\right) \le f_5' \quad \forall t$$

Constraint 1

$$M_{m,t} \le M_{m,t-1} \quad \forall m,t$$

$$N_{n,t} \ge N_{n,t-1} \quad \forall n,t$$

$$M_{m,t} = 0 \quad \forall m, t = t_m$$

Becomes:

$$M_t \le M_{t-1} \quad \forall t$$

$$N_t \ge N_{t-1} \quad \forall t$$

$$M_T = 0$$

Constraint 2

$$Y_{it} + X_{it} = D_{it} \quad \forall i,t$$

$$y_{ijn,t} \le z''_{ijn} D_{it} \quad \forall i,j,n,t$$

$$\sum_n y_{ijn,t} = Y_{it} \quad \forall i,j,t$$

$$\sum_i \sum_j p''_{ijn} y_{ijn,t} + L_t ST'' N_{n,t} \le KgN_{n,t} \quad \forall n,t$$

Becomes:

$$Y_t + X_t = D_t \quad \forall t$$

Constraint 3

$$\sum_i \sum_j p'_{ijm} X_{it} + \sum_i \sum_j ST'_{ijm} s_{it} \le Kx_t M_{m,t} \quad \forall m,t$$

$$X_{it} \le Vv_t \quad \forall i,t$$

$$s_{it} = L_t v_t \quad \forall i,t$$

Becomes:

$$X_t \le x_t M_t \quad \forall t$$

10 Appendix B: Summary of Section 4, Lotfi Critique and Modifications

Variable Was (In Sec 3)	Now	Description
t	t	Time Period
T	T	Planning or Prediction Horizon
d', d''	MC', MC''	Variable Cost, now absorbed into Marginal Costs
P_t	P_t	Price
D_t	D_t	Quantity Demanded
	Q_t	Introduced Variable: Quantity of Part Produced
MC_t	MC', MC''	Marginal Cost, now absorbed into two Marginal Costs
C''_t	C''_t	Capital Cost for New Modules
S_t	S_t	Salvage Value for Old Modules
r_t	r_t	Discount Factor
h_t	MC', MC''	Holding Cost, now absorbed into Marginal Costs
	MC'	Marginal Cost of producing one part on an old machine
	MC''	Marginal Cost of producing one part on a new machine
x_t	x', x''	Marginal Product for Old Machines and New Modules
M_t	M_t	Number of old machines
N_t	N_t	Number of new machines
$M_t - M_{t-1}$	ΔM_{t-1}	Change in the number of old machines
$N_t - N_{t-1}$	ΔN_{t-1}	Change in the number of new modules
X_t	X_t	Production of parts from old machines
Y_t	Y_t	Production of parts from new modules

Objective 1:

$$\max\ f_1 = \sum_t (P_t - MC_t)(Y_t + X_t)r_t - \sum_t (N_t - N_{t-1})C''_t r_t + \sum_t (M_{t-1} - M_t)S_t r_t$$
$$-\left[\sum_t d''Y_t r_t + \sum_t d'X_t r_t\right] - \sum_t (Y_t + X_t)\frac{h_t}{2} r_t$$

Becomes:

$$\max\ f_1 = \sum_t (Px''N_t r_t - MC''x''N_t r_t) + \sum_t (Px'M_t r_t - MC'x'M_t r_t)$$
$$-\sum_t \Delta N_{t-1} C'' r_t - \sum_t \Delta M_{t-1} S r_t$$

subject to:

$$\Delta N_{t-1} = N_t - N_{t-1}$$

$$\Delta M_{t-1} = M_t - M_{t-1}$$

Objective 2

$$\text{Max} \;\; f_2 = -\max\left[(N_t - N_{t-1})C_t'' - (M_{t-1} - M_t)S_t\right]$$

subject to

$$(N_t - N_{t-1})C_t'' - (M_{t-1} - M_t)S_t \leq \max\left[(N_t - N_{t-1})C_t'' - (M_{t-1} - M_t)S_t\right] \quad \forall t$$

Becomes:

Nothing, as it is Completely Omitted

Objective 3

Nothing, Omitted in section three

Objective 4

$$\text{Max} \;\; f_4 = -f_4'$$

subject to:

$$(M_{t-1} - M_t) \leq f_4' \quad \forall t$$

Becomes:

$$\text{Min} \;\; f_4 = \max(-\Delta M_{t-1}) \quad \forall t$$

Objective 5

$$\text{Max} \;\; f_5 = -f_5'$$

subject to:

$$\left(N_t - N_{t-1}\right) \leq f_5' \quad \forall t$$

Becomes:

$$\text{Min} \quad f_5 = \max\left(\Delta N_{t-1}\right) \quad \forall t$$

Constraints:

$$M_t \leq M_{t-1} \quad \forall t$$
$$N_t \geq N_{t-1} \quad \forall t$$
$$M_T = 0$$
$$Y_t + X_t = D_t \quad \forall t$$
$$X_t \leq x_t M_t \quad \forall t$$

Becomes:

$$x'' N_t + x' M_t = Q_t \quad \forall t$$

and **Objective Six**:

$$\text{Min} \;\; f_6 = \sum_t \left(Q_t - D_t\right)$$

11 Appendix C: Summary of Section 5, The Lotfi Model and its Simplification

Variable Was (In Sec 4)	Now	Description
t	t	Time Period
T	T	Planning or Prediction Horizon
P_t	P_t	Price
D_t	D_t	Quantity Demanded
Q_t	Q_t	Introduced Variable: Quantity of Part Produced
C_t''	C_t''	Capital Cost for New Modules
S_t	S_t	Salvage Value for Old Modules
r_t	r_t	Discount Factor
MC'	MC'	Marginal Cost of producing one part on an old machine
MC''	MC''	Marginal Cost of producing one part on a new machine
x'	x'	Marginal Product for Old Machines
x''	x''	Marginal Product for New Modules
M_t	M_t	Number of old machines
N_t	N_t	Number of new machines
ΔM_{t-1}	ΔM_{t-1}	Change in the number of old machines
ΔN_{t-1}	ΔN_{t-1}	Change in the number of new modules
X_t	X_t	Production of parts from old machines
Y_t	Y_t	Production of parts from new modules

Objective 1:

$$\max\ f_1 = \sum_t \left(Px''N_t r_t - MC''x''N_t r_t\right) + \sum_t \left(Px'M_t r_t - MC'x'M_t r_t\right)$$
$$- \sum_t \Delta N_{t-1} C'' r_t + \sum_t \Delta M_{t-1} S r_t$$

Becomes:

$$\text{Min}\ f_1 = \sum_t TC_t^2$$

subject to:

$$TC_t = MC''x''N_t r_t + MC'x'M_t r_t + \Delta N_{t-1} C'' r_t + \Delta M_{t-1} S r_t$$

Objective 2

Omitted

Objective 3

Omitted

Objective 4

$$\text{Min } f_4 = \max \sum_t \left(-\Delta M_{t-1}\right)$$

Becomes:

$$\text{Min } f_4 = \sum_t \Delta M_{t-1}^2$$

Objective 5

$$\text{Min } f_5 = \max \sum_t \Delta N_{t-1}$$

Becomes:

$$\text{Min } f_5 = \max \sum_t \Delta N_{t-1}$$

Objective 6

$$\text{Min } f_6 = \sum_t \left(Q_t - D_t\right)$$

Becomes:

$$\text{Min } f_6 = \sum_t \left(Q_t - Q_{PMax}\right)^2$$

Constraints:

$$x'' N_t + x' M_t = Q_t \quad \forall t$$
$$\Delta N_{t-1} = N_t - N_{t-1}$$
$$\Delta M_{t-1} = M_t - M_{t-1}$$

Become:

$$TC_t = MC''x''N_t r_t + MC'x'M_t r_t + \Delta N_{t-1}C''r_t + \Delta M_{t-1}Sr_t$$
$$Q_t = x''N_t + x'M_t$$
$$\Delta N_{t-1} = N_t - N_{t-1}$$
$$\Delta M_{t-1} = M_t - M_{t-1}$$

And the final Transversal model is:

$$\text{Min } F = \sum_t \left(TC_t - TC_f\right)^2 + \sum_t \left(Q_t - Q_{PMax}\right)^2 + \sum_t \Delta N_{t-1}^2 + \sum_t \Delta M_{t-1}^2$$

subject to:

$$\left(TC_t - TC_f\right) = MC''x''\left(N_t - N_f\right)r_t + MC'x'\left(M_t - M_f\right)r_t + \Delta N_{t-1}C''r_t - \Delta M_{t-1}Sr_t$$
$$\left(Q_t - Q_{PMax}\right) = x''\left(N_t - N_f\right) + x'\left(M_t - M_f\right)$$
$$\Delta N_{t-1} = N_t - N_{t-1}$$
$$\Delta M_{t-1} = M_t - M_{t-1}$$

12 Appendix D: A Least-Squares Numerical Example

12.1 Brute-Force: The Least-Squares Solution

The least-squares method is a direct, brute-force approach to solving the Linear Quadratic problem. The least-squares approach is the matrix equivalent of taking the derivative of an equation, setting it equal to zero, and solving for the minimum value. However, the least-squares approach involves not-so-simply deriving a matrix equation that is itself made up of other matrices (a super-matrix). This one super-matrix equation describes all of Lotfi's constraints for all time periods. The super-matrix is then substituted into the Linear Quadratic problem, the matrix derivative is taken, set equal to a zero matrix, and then the optimal inputs are calculated.

The super-matrix objective function is written as:

$$\mathbf{f} = \begin{bmatrix} \mathbf{y}_0^{\mathbf{T}} & \dots & \mathbf{y}_{T-1}^{\mathbf{T}} \end{bmatrix} \begin{bmatrix} \mathbf{Q} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{Q} \end{bmatrix} \begin{bmatrix} \mathbf{y}_0 \\ \vdots \\ \mathbf{y}_{T-1} \end{bmatrix} + \begin{bmatrix} \mathbf{u}_0^{\mathbf{T}} & \dots & \mathbf{u}_{T-1}^{\mathbf{T}} \end{bmatrix} \begin{bmatrix} \mathbf{R} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{u}_0 \\ \vdots \\ \mathbf{u}_{T-1} \end{bmatrix}$$

and, since **D** is a zero matrix, the constraints are written as:

$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_T \end{bmatrix} = \begin{bmatrix} \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^5 \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_{T-1} \end{bmatrix} + \begin{bmatrix} \mathbf{CB} & 0 & \dots & 0 \\ \mathbf{CAB} & \mathbf{CB} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{CA}^4\mathbf{B} & \mathbf{CA}^3\mathbf{B} & \dots & \mathbf{CB} \end{bmatrix} \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{T-1} \end{bmatrix}$$

Note, the entire sequence of unknowns is stacked into a vector of vectors, or "super-vector",

$$\hat{\mathbf{u}} = \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{T-1} \end{bmatrix}$$

Now, define the super-matrices,

$$\hat{\mathbf{Q}} = \begin{bmatrix} \mathbf{Q} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{Q} \end{bmatrix} \qquad \hat{\mathbf{R}} = \begin{bmatrix} \mathbf{R} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{R} \end{bmatrix}$$

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^5 \end{bmatrix} \qquad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{CB} & 0 & \dots & 0 \\ \mathbf{CAB} & \mathbf{CB} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{CA}^4\mathbf{B} & \mathbf{CA}^3\mathbf{B} & \dots & \mathbf{CB} \end{bmatrix}$$

Substituting the model into the objective function will result in a quadratic equation in $\hat{\mathbf{u}}$, for which the derivative can be taken, set equal to zero and solved. Substitute:

$$\mathbf{F} = \left(\hat{\mathbf{x}}_0^{\mathbf{T}}\hat{\mathbf{A}}^{\mathbf{T}} + \hat{\mathbf{u}}^{\mathbf{T}}\hat{\mathbf{B}}^{\mathbf{T}}\right)\hat{\mathbf{Q}}\left(\hat{\mathbf{A}}\hat{\mathbf{x}}_0 + \hat{\mathbf{B}}\hat{\mathbf{u}}\right) + \hat{\mathbf{u}}\hat{\mathbf{R}}\hat{\mathbf{u}}$$

$$\vdots$$

$$\mathbf{F} = \hat{\mathbf{u}}^{\mathbf{T}}\hat{\mathbf{B}}^{\mathbf{T}}\hat{\mathbf{Q}}\hat{\mathbf{B}}\hat{\mathbf{u}} + \hat{\mathbf{u}}\hat{\mathbf{R}}\hat{\mathbf{u}} + \hat{\mathbf{u}}^{\mathbf{T}}\hat{\mathbf{B}}^{\mathbf{T}}\hat{\mathbf{Q}}\hat{\mathbf{A}}\hat{\mathbf{x}}_0 + \hat{\mathbf{x}}_0^{\mathbf{T}}\hat{\mathbf{A}}^{\mathbf{T}}\hat{\mathbf{Q}}\hat{\mathbf{B}}\hat{\mathbf{u}} + \hat{\mathbf{x}}_0^{\mathbf{T}}\hat{\mathbf{A}}^{\mathbf{T}}\hat{\mathbf{Q}}\hat{\mathbf{A}}\hat{\mathbf{x}}_0$$

Now, take the derivative, set equal to zero, and solve for $\hat{\mathbf{u}}$:

$$\frac{d\mathbf{F}}{d\hat{\mathbf{u}}} = 2\hat{\mathbf{B}}^{\mathbf{T}}\hat{\mathbf{Q}}\hat{\mathbf{B}}\hat{\mathbf{u}} + 2\hat{\mathbf{R}}\hat{\mathbf{u}} + \hat{\mathbf{B}}^{\mathbf{T}}\hat{\mathbf{Q}}\hat{\mathbf{A}}\hat{\mathbf{x}}_0 + \hat{\mathbf{B}}^{\mathbf{T}}\hat{\mathbf{Q}}\hat{\mathbf{A}}\hat{\mathbf{x}}_0 = 0$$

$$\vdots$$

$$\hat{\mathbf{u}} = -\left(\hat{\mathbf{B}}^{\mathbf{T}}\hat{\mathbf{Q}}\hat{\mathbf{B}} + \hat{\mathbf{R}}\right)^{-1}\hat{\mathbf{B}}^{\mathbf{T}}\hat{\mathbf{Q}}\hat{\mathbf{A}}\hat{\mathbf{x}}_0$$

Finally, substituting the values used in section 6.3 for marginal cost, salvage value, etc. and solving for $\hat{\mathbf{u}}$ using the Matlab program in the next section yields:

Table 12.1: Lotfi's Least-Squares Solution

Time	DN	DM
0	5	-11
1	5	-8
2	3	-5
3	2	-3
4	2	-1
5	1	0

Which is identical to the optimal control moves calculated by the Riccati iteration (see the ΔN and ΔM columns below from the Riccati calculation), thus verifying the Riccati solution.

Table 12.2: Riccati Optimal Solution of the Lotfi Problem

Time	ΔN	N	ΔM	M	TC	Q	P	MR	MC'	MC"	Profit
						5800	144	28	28	19.8	
0	5	0	-11	28	147,000	5880	142.4	24.8	28	19.8	690,312
1	5	5	-8	17	108,786	5220	155.6	51.2	28	19.8	703,446
2	3	10	-5	9	114,411	5190	156.2	52.4	28	19.8	696,267
3	2	13	-3	4	100,904	5130	157.4	54.8	28	19.8	706,558
4	2	15	-1	1	97,518	5160	156.8	53.6	28	19.8	711,570
5	1	17	0	0	116,239	5610	147.8	35.6	28	19.8	712,919
6	0	18	0	0	116,618	5940	141.2	22.4	28	19.8	722,110
						6005	139.9	19.8	28	19.8	

12.2 MATLAB: The Least-Squares Program

```
clear

MCd=19.8;
Xid=330;
r= 1/1.12;
Cd=13000;
MCs=28;
Xis=210;
S=6890;

x0=[-18; 0; 28; 0];

Q=[1 0 0 0
0 1 0 0
0 0 1 0
0 0 0 1];

R=zeros(2);
A=[1 0 0 0
0 0 0 0
0 0 1 0
0 0 0 0];
B= [1 0
```

```
1 0
0 1
0 1];
C= [MCd*Xid*r Cd*r MCs*Xis*r S*r
Xid 0 Xis 0
0 1 0 0
0 0 0 1];
C=[1/1000 0 0 0; 0 1/100 0 0; 0 0 1 0; 0 0 0 1]*C;
D=zeros(4,2);

Q_hat=[Q zeros(4) zeros(4) zeros(4) zeros(4)
zeros(4) Q zeros(4) zeros(4) zeros(4)
zeros(4) zeros(4) Q zeros(4) zeros(4)
zeros(4) zeros(4) zeros(4) Q zeros(4)
zeros(4) zeros(4) zeros(4) zeros(4) Q];

R_hat=[R zeros(2) zeros(2) zeros(2) zeros(2)
zeros(2) R zeros(2) zeros(2) zeros(2)
zeros(2) zeros(2) R zeros(2) zeros(2)
zeros(2) zeros(2) zeros(2) R zeros(2)
zeros(2) zeros(2) zeros(2) zeros(2) R];

A_hat=[C*A; C*A^2; C*A^3; C*A^4; C*A^5 ];
B_hat=[C*B zeros(4,2) zeros(4,2) zeros(4,2) zeros(4,2)
C*A*B C*B zeros(4,2) zeros(4,2) zeros(4,2)
C*A^2*B C*A*B C*B zeros(4,2) zeros(4,2)
C*A^3*B C*A^2*B C*A*B C*B zeros(4,2)
C*A^4*B C*A^3*B C*A^2*B C*A*B C*B];

u0 = -inv(B_hat'*Q_hat*B_hat+R_hat)*B_hat'*Q_hat*A_hat*x0;
round(u0(1:2))
x1=A*x0+B*round(u0(1:2));
u1 = -inv(B_hat'*Q_hat*B_hat+R_hat)*B_hat'*Q_hat*A_hat*x1;
round(u1(1:2))
x2=A*x1+B*round(u1(1:2));
u2 = -inv(B_hat'*Q_hat*B_hat+R_hat)*B_hat'*Q_hat*A_hat*x2;
round(u2(1:2))
x3=A*x2+B*round(u2(1:2));
u3 = -inv(B_hat'*Q_hat*B_hat+R_hat)*B_hat'*Q_hat*A_hat*x3;
round(u3(1:2))
x4=A*x3+B*round(u3(1:2));
u4 = -inv(B_hat'*Q_hat*B_hat+R_hat)*B_hat'*Q_hat*A_hat*x4;
round(u4(1:2))
x5=A*x4+B*round(u4(1:2));
u5 = -inv(B_hat'*Q_hat*B_hat+R_hat)*B_hat'*Q_hat*A_hat*x5;
round(u5(1:2))
```

13 Appendix E: Proof of Equivalency – Linear Quadratic and Matrix Presentations of the Lotfi Model

This section will demonstrate that the matrix presentation of Lotfi's Linear Quadratic problem is equivalent to the non-matrix presentation. Section five presented the Lotfi Problem:

$$\text{Min}\, F = \sum_t \left(TC_t - TC_f\right)^2 + \sum_t \left(Q_t - Q_{PMax}\right)^2 + \sum_t \Delta N_{t-1}^2 + \sum_t \Delta M_{t-1}^2$$

subject to:

$$TC_t = MC''x''N_t r_t + MC'x'M_t r_t + \Delta N_{t-1}C''r_t - \Delta M_{t-1}Sr_t$$

$$Q_t = x''N_t + x'M_t$$

$$\Delta N_{t-1} = N_t - N_{t-1}$$

$$\Delta M_{t-1} = M_t - M_{t-1}$$

Section five also presented the equivalent matrix formulation (for Riccati):

$$\min_{\{\mathbf{u}_t\}} f = \sum_t \left(\overline{\mathbf{y}}_t^{\mathbf{T}}\mathbf{Q}\overline{\mathbf{y}}_t + \overline{\mathbf{u}}_t^{\mathbf{T}}\mathbf{R}\overline{\mathbf{u}}_t\right)$$

subject to:

$$\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1} + \mathbf{B}\mathbf{u}_{t-1}$$

$$\mathbf{y}_t = \mathbf{C}\mathbf{x}_t + \mathbf{D}\mathbf{u}_t$$

The goal is to prove that these two sets of equations are identical when:

$$\mathbf{y}_t = \begin{bmatrix} TC_t \\ Q_t \\ \Delta N_{t-1} \\ \Delta M_{t-1} \end{bmatrix} \quad \mathbf{u}_t = \begin{bmatrix} \Delta N_t \\ \Delta M_t \end{bmatrix} \quad \mathbf{x}_t = \begin{bmatrix} N_t \\ \Delta N_{t-1} \\ M_t \\ \Delta M_{t-1} \end{bmatrix}$$

$$\bar{\mathbf{y}}_t = \begin{bmatrix} TC_t - TC_f \\ Q_t - Q_{PMax} \\ \Delta N_{t-1} \\ \Delta M_{t-1} \end{bmatrix} \qquad \bar{\mathbf{x}}_t = \begin{bmatrix} N_t - N_f \\ \Delta N_{t-1} \\ M_t - M_f \\ \Delta M_{t-1} \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} MC''x''r_t & C''r_t & MC'x'r_t & -Sr_t \\ x'' & 0 & x' & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

13.1 Approach: Substitute and Solve

To demonstrate that the matrices presented in section six are equivalent to Lotfi's problem in section five, I will simply substitute the vectors and matrices **A**, **B**, **C**, **D**, **Q**, $\mathbf{u}_t$, $\bar{\mathbf{y}}_t$, and $\bar{\mathbf{x}}_t$ into the Linear Quadratic problem and constraints, and multiply. First the Linear Quadratic summation:

$$\min_{\{\mathbf{u}_t\}} f = \sum_t \left(\mathbf{y}_t^{\mathrm{T}} \mathbf{Q} \mathbf{y}_t + \mathbf{u}_t^{\mathrm{T}} \mathbf{R} \mathbf{u}_t \right)$$

Substituting,

$$\min_{\{\mathbf{u}_t\}} f = \sum_t \left(\begin{bmatrix} TC_t - TC_f \\ Q_t - Q_{PMax} \\ \Delta N_{t-1} \\ \Delta M_{t-1} \end{bmatrix}^{\mathbf{T}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} TC_t - TC_f \\ Q_t - Q_{PMax} \\ \Delta N_{t-1} \\ \Delta M_{t-1} \end{bmatrix} + \begin{bmatrix} \Delta N_t \\ \Delta M_t \end{bmatrix}^{\mathbf{T}} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta N_t \\ \Delta M_t \end{bmatrix} \right)$$

$$\min_{\{\mathbf{u}_t\}} f = \sum_t \left(\begin{bmatrix} TC_t - TC_f & Q_t - Q_{PMax} & \Delta N_{t-1} & \Delta M_{t-1} \end{bmatrix} \begin{bmatrix} TC_t - TC_f \\ Q_t - Q_{PMax} \\ \Delta N_{t-1} \\ \Delta M_{t-1} \end{bmatrix} \right)$$

$$\min_{\{\mathbf{u}_t\}} f = \sum_t \left((TC_t - TC_f)^2 + (Q_t - Q_{PMax})^2 + \Delta N_{t-1}^2 + \Delta M_{t-1}^2 \right)$$

$$\min_{\{\mathbf{u}_t\}} f = \sum_t (TC_t - TC_f)^2 + \sum_t (Q_t - Q_{PMax})^2 + \sum_t \Delta N_{t-1}^2 + \sum_t \Delta M_{t-1}^2$$

Which is equivalent to Lotfi's formulation:

$$\text{Min}\, F = \sum_t (TC_t - TC_f)^2 + \sum_t (Q_t - Q_{PMax})^2 + \sum_t \Delta N_{t-1}^2 + \sum_t \Delta M_{t-1}^2$$

The Constraints:

$$\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1} + \mathbf{B}\mathbf{u}_{t-1}$$
$$\mathbf{y}_t = \mathbf{C}\mathbf{x}_t + \mathbf{D}\mathbf{u}_t$$

One at a time:

$$\mathbf{x}_k = \mathbf{A}\mathbf{x}_{k-1} + \mathbf{B}\mathbf{u}_{k-1}$$

$$\begin{bmatrix} N_t - N_f \\ \Delta N_{t-1} \\ M_t - M_f \\ \Delta M_{t-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} N_{t-1} - N_f \\ \Delta N_{t-2} \\ M_{t-1} - M_f \\ \Delta M_{t-2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta N_{t-1} \\ \Delta M_{t-1} \end{bmatrix}$$

$$\begin{bmatrix} N_t - N_f \\ \Delta N_{t-1} \\ M_t - M_f \\ \Delta M_{t-1} \end{bmatrix} = \begin{bmatrix} N_{t-1} - N_f \\ 0 \\ M_{t-1} - M_f \\ 0 \end{bmatrix} + \begin{bmatrix} \Delta N_{t-1} \\ \Delta N_{t-1} \\ \Delta M_{t-1} \\ \Delta M_{t-1} \end{bmatrix}$$

$$\begin{bmatrix} N_t \\ \Delta N_{t-1} \\ M_t \\ \Delta M_{t-1} \end{bmatrix} = \begin{bmatrix} N_{t-1} + \Delta N_{t-1} \\ \Delta N_{t-1} \\ M_{t-1} + \Delta M_{t-1} \\ \Delta M_{t-1} \end{bmatrix}$$

$$N_t - N_{t-1} = \Delta N_{t-1}$$
$$M_t - M_{t-1} = \Delta M_{t-1}$$

Which matches the constraints:

$$\Delta N_{t-1} = N_t - N_{t-1}$$
$$\Delta M_{t-1} = M_t - M_{t-1}$$

Now, the second constraint:

$$\mathbf{y}_t = \mathbf{C}\mathbf{x}_t + \mathbf{D}\mathbf{u}_t$$

$$\begin{bmatrix} TC_t - TC_f \\ Q_t - Q_{PMax} \\ \Delta N_{t-1} \\ \Delta M_{t-1} \end{bmatrix} = \begin{bmatrix} MC''x''r_t & C''r_t & MC'x'r_t & Sr_t \\ x'' & 0 & x' & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} N_t - N_f \\ \Delta N_{t-1} \\ M_t - M_f \\ \Delta M_{t-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta N_t \\ \Delta M_t \end{bmatrix}$$

$$\begin{bmatrix} TC_t - TC_f \\ Q_t - Q_{PMax} \\ \Delta N_{t-1} \\ \Delta M_{t-1} \end{bmatrix} = \begin{bmatrix} MC''x''r_t\left(N_t - N_f\right) + C''r_t\Delta N_{t-1} + MC'x'r_t\left(M_t - M_f\right) + Sr_t\Delta M_{t-1} \\ x''\left(N_t - N_f\right) + x'\left(M_t - M_f\right) \\ \Delta N_{t-1} \\ \Delta M_{t-1} \end{bmatrix}$$

$$TC_t - TC_f = MC''x''r_t\left(N_t - N_f\right) + C''r_t\Delta N_{t-1} + MC'x'r_t\left(M_t - M_f\right) + Sr_t\Delta M_{t-1}$$
$$Q_t - Q_{PMax} = x''\left(N_t - N_f\right) + x'\left(M_t - M_f\right)$$

$$TC_t = MC''x''r_t N_t + C''r_t\Delta N_{t-1} + MC'x'r_t M_t + Sr_t\Delta M_{t-1}$$
$$Q_t = x''N_t + x'M_t$$

Which is equivalent to:

$$TC_t = MC''x''N_t r_t + MC'x'M_t r_t + \Delta N_t C'' r_t + \Delta M_t S r_t$$
$$Q_t = x''N_t + x'M_t$$

14 Vita

Benjamin C. Mull, son of Ron and Linda Mull, was born in Findlay, Ohio on 11 January 1970. Benjamin graduated Summa Cum Laude and Phi Beta Kappa from the University of Michigan in Chemical Engineering. He also holds an M.S. in Chemical Engineering from The University of Texas, a Masters of Engineering Administration from Virginia Tech, and an Industrial Certificate in Nuclear Reactor Engineering from the Bettis Nuclear Power Laboratory. Benjamin worked in Nuclear Submarine Acoustics for the U.S. Navy for five years before becoming a Program Manager at Cisco Systems. Benjamin and wife Allison currently reside in San Jose, California where Benjamin is pursuing an MBA from the Wharton School of Business.