A Combinatorially Explicit Relative Möbius Function on Affine Grassmannian Elements and a Proposal for an Affine Infinite Symmetric Group

Michael Ruben Lugo

Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics

Mark Shimozono, Chair
Nicholas Loehr
Leonardo Mihalcea
Daniel Orr

March 25, 2019
Blacksburg, Virginia

Keywords: affine Weyl group, affine Grassmannian, quantum Bruhat graph, Möbius function, infinite affine Weyl group

A Combinatorially Explicit Relative Möbius Function on Affine Grassmannian Elements and a Proposal for an Affine Infinite Symmetric Group

Michael Ruben Lugo

(ABSTRACT)

For an affine Weyl group W, we explicitly determine the elements for which the Möbius function of the subposet of affine Grassmannians under the Bruhat order is non-zero by utilizing the quantum Bruhat graph of the classical Weyl group associated to W. Then we examine embedding stable and consistent statistics on the affine Weyl group of type A which permit the definition of an affine infinite symmetric group.

A Combinatorially Explicit Relative Möbius Function on Affine Grassmannian Elements and a Proposal for an Affine Infinite Symmetric Group

Michael Ruben Lugo

(GENERAL AUDIENCE ABSTRACT)

Similar to the integers, there are groups that have both an infinite number of elements and also a way to partially order those elements. With a partial ordering, we can consider the interval between two elements. When we make a function that sums over an interval of elements, then we can invert the function by using something called the Möbius function. For many groups, the Möbius function is extremely unpredictable and calculating the inverse may require us to consider an infinite number of elements. In this paper, we focus on groups called affine Weyl groups, which are very useful in algebraic geometry. It turns out that most elements in these groups have a very predictable pattern in their Möbius functions which only considers a finite number of elements. The first part of this paper gives very simple rules for calculating it. The second part of this paper focuses on a special type of affine Weyl group: the affine symmetric groups. We provide an attempt at defining a large parent group, which we call the affine infinite symmetric group, that contains all the other affine symmetric groups.

Contents

Introduction **1**

1 Background **3**

1.1 Dynkin Diagrams 3

1.2 Finite Weyl Groups and their Root Systems 5

1.3 Affine Weyl Groups and their Root Systems 6

2 Explicit Relative Möbius Function on Affine Grassmannians **8**

2.1 Relative Möbius Function 8

2.2 Semidirect Product Form for Affine Grassmannian Elements 9

2.3 Quantum Bruhat Graph 10

2.4 Superregular Coverings 10

2.5 Non-minimal Path Weights 12

2.6 Proof of Result 13

2.7 Bounds on Regularity 15

3 An Affine Infinite Symmetric Group **16**

3.1 The $\widetilde{S}_{[a,b]}$ Subgroup 16

3.2 Stability in the Embedding 17

3.3 Root Partitions for $S_{[a,b]}$ 19

3.4 Conclusion on Construction 21

Introduction

We present a specialized construction for the affine flag ind-variety. For a generalized construction, see Kumar [2002]. Let G be a simple Lie group with entries in $\mathbb{C}$, and define the affine Lie group $G_{\mathrm{af}} = G(\!(t)\!)$ to be constructed from the group G by allowing entries in $\mathbb{C}(\!(t)\!)$. As a Lie group, we know there exists a Borel subgroup $B \subset G$. Let the maximal parahoric subgroup $P_{\mathrm{af}} = G[\![t]\!]$ be the group G with entries in $\mathbb{C}[\![t]\!]$, and define the Iwahori subgroup $B_{\mathrm{af}} = \{A \in P_{\mathrm{af}} \mid \mathrm{ev}_0(A) \in B\}$, where $\mathrm{ev}_0(A)$ evaluates the entries of A at $t = 0$. We have $P_{\mathrm{af}} \subset B_{\mathrm{af}} \subset G_{\mathrm{af}}$, and can thus define the affine flag ind-variety $\mathcal{G} = G_{\mathrm{af}}/B_{\mathrm{af}}$ and the affine Grassmannian $\mathcal{G}_0 = G_{\mathrm{af}}/P_{\mathrm{af}}$.

Let $\mathfrak{h}$ and $\mathfrak{h}^*$ be the ambient spaces for the simple coroots $\Delta^\vee$ and simple roots Δ respectively. We can define the Weyl group W to be the subgroup of $\mathrm{Aut}(\mathfrak{h})$ generated by simple reflections (see Kumar [2002], Section 1.3). We identify the corresponding maximal parabolic subgroup $W_0 \subset W$ and the set of minimal length representatives $W^0 = W/W_0$. For $w \in W$ and $v \in W^0$, define the Schubert cells

$$C_w = B_{\mathrm{af}} w B_{\mathrm{af}}/B_{\mathrm{af}} \qquad C_v^{P_{\mathrm{af}}} = B_{\mathrm{af}} v P_{\mathrm{af}}/P_{\mathrm{af}}$$

Let

$$X_w = \overline{C_w} \qquad X_v^{P_{\mathrm{af}}} = \overline{C_v^{P_{\mathrm{af}}}}$$

be the Schubert varities of $\mathcal{G}$ and $\mathcal{G}_0$ respectively. We can consider the Grothendieck groups of the category of coherent sheaves of $\mathcal{O}_{\mathcal{G}}$-modules, $K(\mathcal{G})$ and $K(\mathcal{G}_0)$, which are isomorphic to the cohomology rings $H^*(\mathcal{G})$ and $H^*(\mathcal{G}_0)$ under the tensor product with the respective representation rings.

We can construct the boundary set of a Schubert variety.

$$\partial X_w = \bigsqcup_{v<w} C_w$$

Let $\mathcal{O}_w = [\mathcal{O}_{X_w}]$ be the class of the structure sheaf of a Schubert variety and $\mathcal{I}_w = [\mathcal{O}_{X_w}(-\partial X_w)]$ be the ideal sheaf, which are both classes in $K(\mathcal{G})$. Similarly, we obtain $\mathcal{O}_v^{P_{\mathrm{af}}} = [\mathcal{O}_v^{P_{\mathrm{af}}}]$ and $\mathcal{I}_v^{P_{\mathrm{af}}} = [\mathcal{O}^{P_{\mathrm{af}}}_{X_v^{P_{\mathrm{af}}}}(-\partial X_v^{P_{\mathrm{af}}})]$ are classes in $K(\mathcal{G}_0)$.

Proposition 1 (Kumar [2012])**.** *Both $\{\mathcal{O}_w\}_{w\in W}$ and $\{\mathcal{I}_w\}_{w\in W}$ form bases for $K(\mathcal{G})$. Similarly, $\{\mathcal{O}_v^{P_{\mathrm{af}}}\}_{v\in W^0}$ and $\{\mathcal{I}_v^{P_{\mathrm{af}}}\}_{v\in W^0}$ are bases for $K(\mathcal{G}_0)$.*

Since these sheaves form different bases, then a useful mechanism is to change from one basis to another.

For $w \in W$ and $v \in W^0$, there exist constants $a_{u,w}$, $b_{u,w}$, $c_{u,v}$, and $d_{u,v}$ such that

$$\mathcal{O}_w = \sum_{u \in W} a_{u,w} \mathcal{I}_u \qquad \mathcal{I}_w = \sum_{u \in W} b_{u,w} \mathcal{O}_u$$

$$\mathcal{O}_v^{P_{\text{af}}} = \sum_{u \in W^0} c_{u,v} \mathcal{I}_u^{P_{\text{af}}} \qquad \mathcal{I}_v^{P_{\text{af}}} = \sum_{u \in W^0} d_{u,v} \mathcal{O}_u^{P_{\text{af}}}$$

Much is known of these constants. Initially Brion [2004] showed that in type A, $a_{u,w} = 1$ when $u \leq w$ and $a_{u,w} = 0$ otherwise. Kumar [2019] states that, by combining his Theorem 10.4, Proposition 3.6, Lemma 5.5, Corollary 5.7, and Proposition 6.6 in Kumar [2012], then we obtain the same constants for $a_{u,v}$ where $\mathcal{G}$ is any symmetrizeable Kac-Moody group, which implies the same for the affine flag-ind variety. In the work of Lam et al. [2017] (see Lemma 2), we see that $c_{u,v} = 1$ when $u \leq v$ and 0 otherwise. Due to the incidence relationship from these constants, we can combinatorially recover $b_{u,w}$ as the Möbius function μ on W, and the $d_{u,v}$ as the Möbius function $\widetilde{\mu}$ of the subposet W^0. Deodhar [1997] proved the non-zero values for both the Möbius functions as $\mu(u, w) = (-1)^{\ell(u)+\ell(w)}$ and $\widetilde{\mu}(u, v) = (-1)^{\ell(u)+\ell(v)}$. Furthermore, Deodhar [1997] proved that $b_{u,w} = 0$ if and only if $u \geq b$ and provided a criterion for $d_{u,v} \neq 0$. However this criterion does not explicitly provide the set of u such that $\widetilde{\mu}(u, v) \neq 0$. By extending the result of Deodhar [1997] in the affine Weyl group case and utilizing the quantum Bruhat graph of W_0, we can prove the main result of this paper.

Theorem 1. *Let $x = w' t_{\lambda'}, y = w t_\lambda$ be affine Grassmannian elements of an affine Weyl group where λ is superregular. Then*

$$\widetilde{\mu}(x, y) = \begin{cases} (-1)^{\ell(y)+\ell(x)} & \text{if } \lambda' = \lambda + \mathrm{M}(w, w') \\ 0 & \text{else} \end{cases}$$

where M is the weight of a minimal path in the quantum Bruhat graph from w to w'.

This immediately yields the following corollary.

Corollary 1. *Let $y = w t_\lambda \in W^0$ be superregular. Then*

$$\mathcal{I}_{w t_\lambda}^{P_{\text{af}}} = \sum_{u \in W_0} (-1)^{\ell(w,u)} \mathcal{O}_{u t_{\lambda + \mathrm{M}(w,u)}}^{P_{\text{af}}}$$

where $\ell(w, u)$ is the length of a shortest path from w to u in the quantum Bruhat graph.

In the second section, we explore a particular subgroup of the affine Weyl group of type $\widetilde{A}_n$. We develop embedding stable functions on this subgroup, which are consistent with the usual definitions, before defining the limit of this subgroup as the affine infinite symmetric group.

1 Background

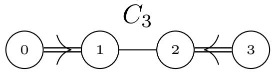
We briefly introduce the basic structures of root systems and their associated reflection groups as they pertain to our topic of affine Weyl groups. Thus, we begin with the Dynkin diagrams associated to finite and affine Weyl groups. Then we define the finite root system and finite Weyl group corresponding with to fixed Dynkin diagram of finite type before discussing properties of both. Finally, we define the structures associated to the affine root system and affine Weyl group associated to a fixed Dynkin diagram of affine type. For further reference, see Humphreys [1993].

1.1 Dynkin Diagrams

The irreducible Weyl groups we are working with are in bijection with certain graphs called *Dynkin diagrams*. The vertices of a Dynkin diagram are joined by either no edge, a single edge, a double edge, or a triple edge. If two vertices are joined with a multiple edge, then the edge is labeled with at least one arrow. Our Dynkin diagrams come in two different types: finite and affine. For the finite types, we label the nodes $1, 2, \ldots, n$, and for the affine types we add a node labeled 0 to a Dynkin diagram of finite type. The irreducible Dynkin diagrams corresponding to Weyl groups covered by this paper are listed in Figure 1. Using the node labels, we can create integers a_{ij} as follows. $a_{ii} = 2$ for all i. If $i \neq j$, then let k be the degree of the edge between nodes i and j. If $k = 0$, then $a_{ij} = a_{ji} = 0$, and if $k = 1$, then $a_{ij} = a_{ji} = -1$. Otherwise there is a multiple edge, so we let a_{ij} satisfy $\min(a_{ij}, a_{ji}) = -k$, if an arrow points to j, then $a_{ij} \neq -1$, and if no arrow points to j, then $a_{ij} = -1$.

Example: Consider the Dynkin diagram corrsponding to $\widetilde{C}_3$ with nodes labeled. Then the corresponding matrix is as follows.

$\widetilde{C}_3$



$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 2 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

This information implies $A = (a_{ij})$ is an integer matrix with the following properties:

- $a_{ii} = 2$ for all $1 \leq i \leq n$
- $a_{ij} \leq 0$ for all $i \neq j$
- $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$

Therefore A is a *generalized Cartan matrix*, and our connected Dynkin diagrams biject with indecomposable Cartan matrices associated to certain Lie algebras. The indecomposable generalized Cartan matrices that

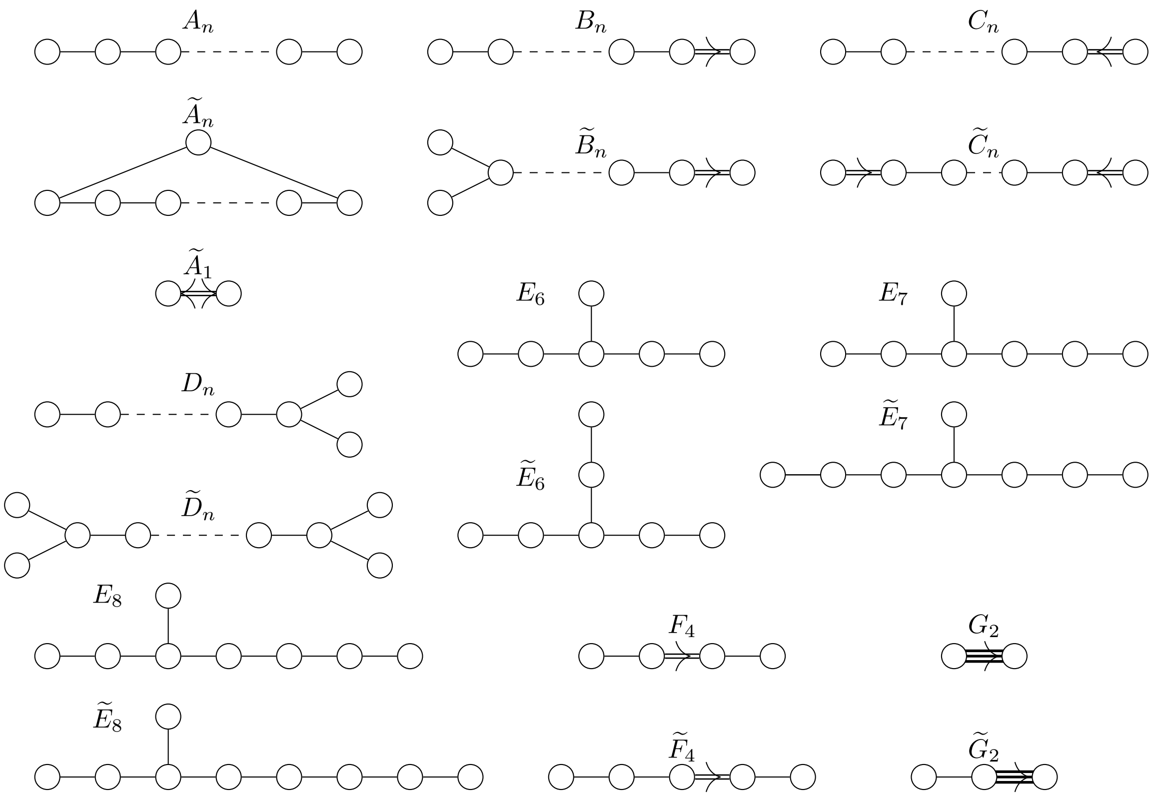


Figure 1: All connected Dynkin diagrams corresponding to unique irreducible finite or untwisted affine Weyl groups up to isomorphism. n is the number of nodes for the associated finite Dynkin diagram. The wide tilde denotes the untwisted affinization of that group achieved by adding an additional node to the corresponding finite diagram.

are positive definite biject with the irreducible finite Weyl groups, while the indecomposable generalized Cartan matrices of corank 1 with all pricipal minors positive definite biject with the irreducible affine Weyl groups.

1.2 Finite Weyl Groups and their Root Systems

Fix indexing set $I_0 = \{1, 2, \ldots, n\}$. If α is a vector, let r_α represent the reflection across the hyperplane orthogonal to α (dependent on a pairing or inner product). A *finite root system* Φ_0 is a finite set of vectors such that for any $\alpha \in \Phi_0$

1. $-\alpha \in \Phi_0$

2. α and $-\alpha$ are the only multiples of α in Φ_0

3. $r_\alpha(\Phi_0) = \Phi_0$

For every finite root system Φ_0, there exists a basis of *simple roots* $\Delta_0 = \{\alpha_i \mid i \in I_0\} \subseteq \Phi_0$ such that if $\alpha \in \Phi_0$, then α's expansion in Δ_0 has either all non-negative or non-positive coefficients, allowing us to decompose $\Phi_0 = \Phi_0^+ \sqcup \Phi_0^-$. Therefore, Δ_0 with a specified pairing completely determines a finite root system.

Fix a generalized Cartan matrix A of finite type. Let $P_0 = \bigoplus_{i \in I_0} \Lambda_i \mathbb{Z}$ be the *weight lattice*. We can create simple roots by defining for any $j \in I_0$

$$\alpha_j = \sum_{i \in I_0} a_{ij} \Lambda_i$$

Since A is positive definite, then $\Delta_0 = \{\alpha_i \mid i \in I_0\}$ is linearly independent in P_0. Let P_0^* to be the dual space to P_0 with dual basis $\Delta_0^\vee = \{\alpha_i^\vee \mid i \in I_0\}$ which satisfy a symmetric bilinear pairing $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$. We can define *simple reflections* which act on $\mu \in P_0$ and $\lambda \in P_0^*$ by

$$s_i(\mu) = \mu - \langle \alpha_i^\vee, \mu \rangle \alpha_i \qquad s_i(\lambda) = \lambda - \langle \lambda, \alpha_i \rangle \alpha_i^\vee \tag{1}$$

By defining the simple reflections corresponding to our simple roots, then we have completely defined a finite root system Φ_0. By property 3 of a root system, we can consider the group $W_0 = \langle s_i \mid i \in I_0 \rangle$, which is called the *reflection group* associated to Φ_0. It is a fact that $W_0(\Delta_0) = \Phi_0$. Since A is positive definite, then W_0 is not only a reflection group, but is actually a *finite Weyl group* (the pairing is crystallographic by design) and W_0 is a subgroup of the automorphism group on P_0.

Finite Weyl groups have a *length function* $\ell : W_0 \to \mathbb{Z}_{\geq 0}$. If $w \in W_0$, then $\ell(w) = k$ if there is a reduced expression $w = s_{i_1} s_{i_2} \ldots s_{i_k}$ such that s_{i_j} are simple reflections. Furthermore, $\ell(w) = \#\text{Inv}(w)$, where $\mathrm{Inv}(w) = \{\alpha \in \Phi_0^+ \mid w(\alpha) \in \Phi_0^-\}$ is the set of *inversions*. For any finite Weyl group, there exists a unique element of maximal length which we denote w_0. For any $\alpha \in \Phi_0$, we have a unique expansion $\alpha = \sum_{i \in I_0} c_i \alpha_i$. We define the *height* of $\alpha \in \Phi_0^+$ to be $\mathrm{ht}(\alpha) = \sum_{i \in I_0} c_i$. Each finite Weyl group has a unique *highest root* θ which maximizes ht.

Note that the bilinear pairing defined above is W_0-invariant. This allows us to calculate the dual of any root $\alpha \in \Phi_0$. If $\alpha = w(\alpha_i)$, then we may define $\alpha^\vee = w(\alpha_i^\vee)$ which is well defined by invariance. Let $T_0 = \{ws_iw^{-1} \mid i \in I_0, w \in W_0\}$ be the *reflections* of W_0. Since $ws_iw^{-1} = r_{w(\alpha_i)}$, we can see that equivalently $T_0 = \{r_\alpha \mid \alpha \in \Phi_0^+\}$ where r_α has definition similar to Equation 1. Having a length function and set of reflections allows us to define the *Bruhat order* on W_0. Let $u, v \in W_0$. We say that v *covers* u (denoted $u \lessdot v$) if there exists $r \in T_0$ such that $v = ut$ and $\ell(u) + 1 = \ell(v)$. Then the Bruhat order on W_0 is given by the transitive closure of this covering relation. Thus W_0 is a partially ordered set.

Now fix $J \subset I_0$. We define the *parabolic subgroup* $W_J = \langle s_i \mid i \in J\rangle$. It is not immediately obvious, but the length function, reflections, and Bruhat order naturally restrict to the parabolic subgroups. We can then define a *parabolic quotient* $W^J = W_0/W_J$, which is not generally a group. There exists a minimal length coset representative for each coset of W^J, and the set of minimum length coset representatives also form a partially ordered set under the Bruhat order.

1.3 Affine Weyl Groups and their Root Systems

Now we consider the setting of A being of affine type. Fix another indexing set $I = I_0 \sqcup \{0\}$. Let $P = \left(\bigoplus_{i \in I} \Lambda_i \mathbb{Z}\right) \oplus \delta\mathbb{Z}$ be the *affine weight lattice* and P^* be the corresponding dual of P. Fix a set of simple roots $\Delta = \{\alpha_i\}_{i \in I}$ as follows:

$$\alpha_j = \sum_{i \in I} a_{ij} \Lambda_i \text{ if } j \neq 0 \qquad \alpha_0 = \sum_{i \in I} \alpha_{i0} \Lambda_i + \delta$$

Therefore Δ is a linearly independent set in P. Create a symmetric bilinear form $\langle \cdot, \cdot \rangle : P^* \times P \to \mathbb{Z}$ by identifying simple coroots of P^*, $\Delta^\vee := \{\alpha_i^\vee\}_{i \in I}$, to be the elements of P^* such that $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$. Specifying $d \in P^*$ satisfies $\langle d, \delta \rangle = 1$ and $\langle d, \alpha_i \rangle = 0$ for all $i \in I$, we find that $\Delta^\vee \sqcup \{d\}$ forms a dual basis for P^*, and that the bilinear form is completely determined.

Example: For the Weyl group of affine type $\widetilde{C}_3$, we have already calculated the Cartan matrix. The simple roots for this system are specified below.

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 2 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$P = \Lambda_0\mathbb{Z} \oplus \Lambda_1\mathbb{Z} \oplus \Lambda_2\mathbb{Z} \oplus \Lambda_3\mathbb{Z} \oplus \delta\mathbb{Z}$$

$$\alpha_0 = 2\Lambda_0 - 2\Lambda_1 + \delta$$

$$\alpha_1 = -\Lambda_0 + 2\Lambda_1 - \Lambda_2$$

$$\alpha_2 = -\Lambda_1 + 2\Lambda_2 - \Lambda_3$$

$$\alpha_3 = -2\Lambda_2 + 2\Lambda_3$$

Let $\mu \in P$ and $\lambda \in P^*$. Under this construction, we can generate simple reflections $\{s_i\}_{i\in I}$ which act on P and P^* with the same definition as Equation 1. Let W be the affine Weyl group, the subgroup of the automorphism group of P generated by these reflections. Note that W has infinitely many elements. By the construction, $\langle \cdot, \cdot \rangle$ is W-invariant. Let W_0 be the maximal parabolic Weyl group generated by $\{s_i\}_{i\in I_0}$, which is a finite Weyl group. Define $\Phi_0 = \{w(\alpha_i) \mid w \in W_0, i \in I_0\}$ to be the set of *classical roots*, and the non-zero elements $\Phi = \{w(\alpha_i) \mid w \in W, i \in I\}$ to be the set of *affine roots*. We say that $\alpha \in \Phi_0$ is positive ($\alpha > 0$) if α is a non-negative sum of simple roots in $\Delta_0 = \{\alpha_i \in \Delta \mid i \neq 0\}$, and similarly $\alpha \in \Phi$ is positive when it is the non-negative sum of simple roots in Δ. Φ admits an equivalent definition: $\Phi = \{\alpha + k\delta \mid \alpha \in \Phi_0, k \in \mathbb{Z}\}$ where $\alpha + k\delta > 0$ when $k > 0$ or when $k = 0$ and $\alpha > 0$ as a root in Φ_0. Written in this form, $\alpha_0 = -\theta + \delta$.

Let $Q^\vee := \bigoplus_{i\in I_0} \mathbb{Z}\alpha_i^\vee \subset P^*$ be the *coroot lattice*. For $\lambda \in Q^\vee$, let t_λ denote the translation by λ in P^*. If $\eta \in P^*$ and $i \in I_0$, then

$$(s_i t_\lambda s_i)(\eta) = s_i(\lambda + s_i(\eta)) = s_i(\lambda) + \eta = t_{s_i(\lambda)}(\eta)$$

This shows that for any $w \in W_0$ and $\lambda \in Q^\vee$, $wt_\lambda w^{-1} = t_{w(\lambda)}$. Therefore the group $W_0 \ltimes Q^\vee$ is a well defined group, where $Q^\vee$ is identified with its corresponding translation group. It is well known that $W \cong W_0 \ltimes Q^\vee$. Written in this form, $s_0 = s_\theta t_{-\theta^\vee}$. The semidirect form admits a level zero action of W on $P = P_0 \oplus \mathbb{Z}\delta$. If $wt_\lambda \in W$, $\mu \in P_0$ and $n \in \mathbb{Z}$, then

$$wt_\lambda(\mu + n\delta) = w(\mu) + (n - \langle \lambda, \mu \rangle)\delta \tag{2}$$

With this construction the definitions of length, inversions, reflections, Bruhat order, parabolic subgroups, and parabolic quotients naturally extend to W. This paper focuses in particular on the set of affine Grassmannian elements W^0, which is the set of minimum length coset representatives for the maximal parabolic quotient $W^0 = W^{I_0}$.

2 Explicit Relative Möbius Function on Affine Grassmannians

2.1 Relative Möbius Function

For any partially ordered set P whose intervals are always finite, there exists a Möbius function μ in the incidence algebra on P such that if $f, g : P \times P \to \mathbb{Z}$ where

$$g(x, y) = \sum_{z \in [x,y]} f(x, z)$$

then we can recover f from knowing only g by using μ.

$$f(x, y) = \sum_{z \in [x,y]} \mu(x, z) g(z, y)$$

Since W is a locally finite partially ordered set, then there exists a Möbius function. Furthermore, since the Bruhat order naturally produces a partially ordered set for each parabolic quotient, one can ask to know the Möbius function $\widetilde{\mu}$ for the partially ordered set W^0. The problem of the relative Möbius function is not new, and the calculation of its values is already known. For $u, v \in W^J$, Deodhar [1997] defined the sets $R(u, v) = \{w \in W^J \mid u \leq w \leq v\}$ and $K(u, v) = \{w \in R(u, v) \mid \text{ if } w \leq z \leq v, \text{ then } z \in W^J\}$. Using these sets, we get the following theorem.

Theorem 2.1.1 (Deodhar [1997], Theorem 1.2)**.** *For $u, v \in W^J$ with $u \lneq v$, then*

$$\widetilde{\mu} = \begin{cases} (-1)^{\ell(u)+\ell(v)} & K(u, v) = R(u, v) \\ 0 & \textit{else} \end{cases}$$

Further, $K(u, v) = R(u, v) \Leftrightarrow$ there is no $i \in J$ such that $us_i \leq v$.

Note that $K(u, v) \subseteq R(u, v)$. If $w \in R(u, v) \backslash K(u, v)$, then there exists a $z \notin W^J$ such that $u \leq w \leq z \leq v$, implying $[u, v] \not\subset W^J$. Similarly, if $[u, v] \not\subset W^J$, then $u \notin K(u, v)$. Since $\widetilde{\mu}(y, y) = 1$ for any $y \in W^0$, combining these facts allows us to redo the notation as follows:

Theorem 2.1.2 (Deodhar [1997], Theorem 1.2)**.** *For $u, v \in W^J$ with $u \leq v$, then*

$$\widetilde{\mu} = \begin{cases} (-1)^{\ell(u)+\ell(v)} & [u,v] \subset W^J \\ 0 & \textit{else} \end{cases}$$

Further, if $u \lneq v$, then $[u,v] \subset W^J \Leftrightarrow$ there is no $i \in J$ such that $us_i \leq v$.

We seek to apply this theorem when $W^J = W^0$. An inconvenience with Theorem 2.1.2 is that, for a fixed $w \in W^0$, the set $\{v \in W^0 \mid \widetilde{\mu}(v,w) \neq 0\}$ is not explicit. This section reveals how to calculate such a set for elements in W^0 which satisfy a regularity condition.

2.2 Semidirect Product Form for Affine Grassmannian Elements

First, we must be able to describe elements of W^0 in the semidirect product form $W \cong W_0 \ltimes Q^\vee$. We say that $\lambda \in Q^\vee$ is *antidominant* if $\langle \lambda, \alpha_i \rangle \leq 0$ for all $i \in I_0$, and denote the set of such elements by $\tilde{Q}$. If $\lambda \in \tilde{Q}$ and $\alpha > 0$, then it is straightforward to show that $\langle \lambda, \alpha \rangle \leq 0$ and $\langle \lambda, -\alpha \rangle \geq 0$.

Proposition 2.2.1. *$wt_\lambda \in W^0$ if and only if $\lambda \in \tilde{Q}$ and if $\langle \lambda, \alpha_i \rangle = 0$ for some $i \in I_0$, then $w(\alpha_i) > 0$.*

Proof. $wt_\lambda \in W^0$ if and only if $\ell(wt_\lambda) < \ell(wt_\lambda s_i)$ for all $i \in I_0$ if and only if $wt_\lambda(\alpha_i) > 0$ for all $i \in I_0$. Thus $wt_\lambda(\alpha_i) = w(\alpha_i) - \langle \lambda, \alpha_i \rangle \delta > 0$. To ensure positivity, then either $\langle \lambda, \alpha_i \rangle < 0$ or $\langle \lambda, \alpha_i \rangle = 0$ and $w(\alpha_i) > 0$. □

Therefore, we have an exact description of elements in W^0 in terms of the semidirect product. Another property of a translation element that we require is regularity. We say that $\lambda \in Q^\vee$ is *regular* if $\langle \lambda, \alpha_i \rangle \neq 0$ for all $i \in I_0$. This implies that the only $w \in W_0$ such that $w(\lambda) = \lambda$ is the identity element. We obtain a characterizing property of such λ that are both regular and antidominant.

Lemma 2.2.1. *Let λ and $v\lambda$ be regular and antidominant. Then $v = 1$.*

Proof. If v is not 1, then there exists an $\alpha > 0$ such that $v^{-1}(\alpha) < 0$. So if λ and $v\lambda$ are antidominant, then $0 \geq \langle v\lambda, \alpha \rangle = \left\langle \lambda, v^{-1}(\alpha) \right\rangle \geq 0$. So $\langle v\lambda, \alpha \rangle = \left\langle \lambda, v^{-1}(\alpha) \right\rangle = 0$, implying λ is not regular. □

In the work of Lam and Shimozono [2007], they further required that certain elements were superregular, i.e. $|\langle \lambda, \alpha_i \rangle| \gg 0$ for all $i \in I_0$. We say $y = wt_\lambda$ is superregular if λ is superregular. The question of sufficient regularity to qualify as superregular will be addressed in Section 2.7.

2.3 Quantum Bruhat Graph

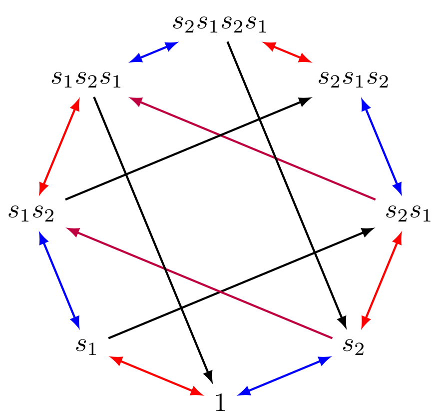
By construction, $\{\Lambda_i\}_{i\in I}$ are the *fundamental weights*, i.e. $\langle \alpha_i^\vee, \Lambda_i\rangle = \delta_{ij}$ where δ_{ij} is the Kronecker delta. Designate $\rho = \sum_{i\in I_0} \Lambda_i = \frac{1}{2}\sum_{\alpha\in\Phi_0^+}\alpha$. For the group W_0, we can generate the *quantum Bruhat graph* G. The vertices of G are the elements of W_0. For any $\alpha \in \Phi_0^+$ and $w \in W_0$, a directed edge $w \to wr_\alpha$ exists if one of the following is true:

1. $\ell(wr_\alpha) = \ell(w) + 1$. Such edges are called *non-quantum edges*.
2. $\ell(wr_\alpha) = \ell(w) - \langle \alpha^\vee, 2\rho\rangle + 1$. Such edges are called *quantum edges*.

For each edge in G, we associate both a label and a weight. If $w \to wr_\alpha$ is an edge, then the label of the edge is α. Assign non-quantum edges a weight of 0 and quantum edges a weight of the dual for its label $\alpha^\vee$. If P is a path in G, then we can generate the label sequence of P as the ordered sequence of its edge labels. Further, we say the weight of P is the sum of the weight of its edges and denote it by $\mathrm{wt}(P)$, and the length of the path by $\ell(P)$. For any $u, v \in W_0$, define $\mathrm{M}(u, v)$ to be the weight of any shortest path in G from u to v. The following result shows that M is well defined.

Lemma 2.3.1 (Postnikov [2002])**.** *For any $u, v \in W_0$, every shortest path from u to v in G has the same weight.*

Example: The Quantum Bruhat Graph for type C_2 is pictured below. The edges are color coded with weights $\alpha_1^\vee$, $\alpha_2^\vee$, $\alpha_1^\vee + 2\alpha_2^\vee$, $\alpha_1^\vee + \alpha_2^\vee$. With this graph, we can fix an element and check the weighted paths from, say, $s_2s_1s_2$.



y	$\mathrm{M}(s_2s_1s_2, y)$
1	$\alpha_1^\vee + 2\alpha_2^\vee$
s_1	$\alpha_1^\vee + 2\alpha_2^\vee$
s_2	$\alpha_1^\vee + \alpha_2^\vee$
s_1s_2	$\alpha_1^\vee + \alpha_2^\vee$
s_2s_1	$\alpha_2^\vee$
$s_1s_2s_1$	$\alpha_2^\vee$
$s_2s_1s_2$	0
$s_2s_1s_2s_1$	0

2.4 Superregular Coverings

The condition for Theorem 1 holding is dependent on when the following proposition holds.

Proposition 2.4.1 (Lam and Shimozono [2007], Proposition 4.1)**.** *Let $\lambda \in \tilde{Q}$ be antidominant and superregular, and let $y = wt_{v\lambda}$. Then $x = yr_{v\alpha+n\delta} \lessdot y$ if and only if one of the following conditions hold:*

1. $\ell(wv) = \ell(wvr_\alpha) - 1$ *and* $n = \langle \lambda, \alpha \rangle$*, giving* $x = wr_{v\alpha}t_{v\lambda}$.

2. $\ell(wv) = \ell(wvr_\alpha) + \langle \alpha^\vee, 2\rho \rangle - 1$ *and* $n = \langle \lambda, \alpha \rangle + 1$*, giving* $x = wr_{v\alpha}t_{v(\lambda+\alpha^\vee)}$.

3. $\ell(v) = \ell(vr_\alpha) + 1$ *and* $n = 0$*, giving* $x = wr_{v\alpha}t_{vr_\alpha(\lambda)}$

4. $\ell(v) = \ell(vr_\alpha) - \langle \alpha^\vee, 2\rho \rangle + 1$ *and* $n = -1$*, giving* $x = wr_{v\alpha}t_{vr_\alpha(\lambda+\alpha^\vee)}$.

Using the notation of paths in G, we can condense these four cases to two. In the terminology introduced by Lam and Shimozono [2007], we call the first two cases near, since the chamber of the translation component remains fixed, and the last two cases far.

Corollary 2.4.1. *Let $\lambda \in \tilde{Q}$ be antidominant and superregular, and let $y = wt_{v\lambda}$. Then $x = yr_{v\alpha+n\delta} \lessdot y$ if and only if one of the following conditions hold:*

1. $wv \to wvr_\alpha$ *is an edge E in G and* $x = wr_{v\alpha}t_{v(\lambda+\mathrm{wt}(E))}$. *We call this a* near covering.

2. $vr_\alpha \to v$ *is an edge E in G and* $x = wr_{v\alpha}t_{vr_\alpha(\lambda+\mathrm{wt}(E))}$. *We call this a* far covering.

However, with a little elbow grease and superregularity implying that a saturated chain consists of superregular elements, we can condense these two cases into one.

Lemma 2.4.1. *Suppose $x < y = wt_{v\lambda}$ with λ antidominant and superregular, $k = \ell(y) - \ell(x)$, and*

$$x \xrightarrow{\beta_k} z_{k-1} \xrightarrow{\beta_{k-1}} \dots z_1 \xrightarrow{\beta_1} y$$

is a saturated chain from x to y in the Bruhat order, where $\beta_i \in \Phi_0^+$ is the root corresponding to α in Proposition 2.4.1. Let $1 \leq f_1 < f_2 < \cdots < f_i \leq k$ be the ordered subindices relating to far coverings and $1 \leq n_1 < n_2 < \cdots < n_j \leq k$ be the ordered subindices relating to near coverings. Define $r_f = r_{\beta_{f_1}} r_{\beta_{f_2}} \dots r_{\beta_{f_i}}$ and $r_n = r_{\beta_{n_1}} r_{\beta_{n_2}} \dots r_{\beta_{n_j}}$. Let P_f be the associated path $vr_f \to v$ (as described in Corollary 2.4.1) given by far coverings and P_n be the associated path $wv \to wvr_n$ given by near coverings. Then

$$x = wvr_n(vr_f)^{-1} t_{vr_f(\lambda+\mathrm{wt}(P_n)+\mathrm{wt}(P_f))}$$

Proof. We proceed by induction. The base case for $k = 1$ is given explicitly by Proposition 2.4.1. Suppose the result is true for any fixed length difference $k \geq 1$ and that $\ell(y) - \ell(x) = k+1$. By induction, there are elements r'_n, r'_f and associated paths $P'_n : wv \to wvr'_n, P'_f : vr'_f \to v$ such that $z_k = wvr'_n(vr'_f)^{-1} t_{vr'_f(\lambda+\mathrm{wt}(P'_n)+\mathrm{wt}(P'_f))}$.

Let $\beta = \beta_{k+1}$. We will perform cases based on whether the covering $x \lessdot z_k$ is a near or far covering. Note that in either case, we observe the following is the resulting classical Weyl component of x:

$$wvr'_n(vr'_f)^{-1}r_{vr'_f(\beta)} = wvr'_n(vr'_f)^{-1}(vr'_f)r_\beta(vr'_f)^{-1} = wvr'_nr_\beta(vr'_f)^{-1}$$

If $x \lessdot z_k$ by a near covering, then $r'_f = r_f$, $r'_nr_\beta = r_n$, and $P'_f = P_f$. The covering yields an edge E from $wvr'_n(vr'_f)^{-1}(vr'_f)^{-1} = wvr'_n \to wvr'_nr_\beta = wvr_n$ and shows that $P'_n + E = P_n$ and that $\mathrm{wt}(P'_n) + \mathrm{wt}(E) = \mathrm{wt}(P_n)$. Therefore

$$\begin{aligned} x &= wvr'_nr_\beta(vr'_f)^{-1}t_{vr'_f(\lambda+\mathrm{wt}(P'_n)+\mathrm{wt}(P'_f)+\mathrm{wt}(E))} \\ &= wvr'_nr_\beta(vr_f)^{-1}t_{vr_f(\lambda+\mathrm{wt}(P'_n)+\mathrm{wt}(P_f)+\mathrm{wt}(E))} \\ &= wvr_n(vr_f)^{-1}t_{vr_f(\lambda+\mathrm{wt}(P_n)+\mathrm{wt}(P_f))} \end{aligned}$$

If $x \lessdot z_k$ by a far edge, then $r'_n = r_n$, $r'_fr_\beta = r_f$, and $P'_n = P_n$. The covering yields an edge E from $vr'_fr_\beta = vr_f \to vr'_f$ and shows that $E + P'_f = P_f$ and $\mathrm{wt}(P'_f) + \mathrm{wt}(E) = \mathrm{wt}(P_f)$. Therefore

$$\begin{aligned} x &= wvr'_nr_\beta(vr'_f)^{-1}t_{vr'_fr_\beta(\lambda+\mathrm{wt}(P'_n)+\mathrm{wt}(P'_f)+\mathrm{wt}(E))} \\ &= wvr_nr_\beta {r'_f}^{-1}v^{-1}t_{vr'_fr_\beta(\lambda+\mathrm{wt}(P_n)+\mathrm{wt}(P'_f)+\mathrm{wt}(E))} \\ &= wvr_nr_f^{-1}v^{-1}t_{vr_f(\lambda+\mathrm{wt}(P_n)+\mathrm{wt}(P_f))} \\ &= wvr_n(vr_f)^{-1}t_{vr_f(\lambda+\mathrm{wt}(P_n)+\mathrm{wt}(P_f))} \end{aligned}$$

So by induction, the result is true for all superregular elements $x < y$. □

2.5 Non-minimal Path Weights

Now we briefly discuss observations uncovered by Postnikov [2002]. In his joint work in Brenti et al. [1998], they proved a strong relationship between label sequences of certain paths and reflection orderings. A *reflection ordering* on W_0 is a map $\varphi : T_0 \to \{1, 2, \ldots, \#T_0\}$ such that for any two $a, b \in T$, the sequence $\varphi(a), \varphi(aba), \varphi(ababa), \ldots, \varphi(babab), \varphi(bab), \varphi(b)$ is either increasing or decreasing. Note that this is equivalent to a positive root ordering, and for $\alpha, \beta \in \Phi_0^+$ we denote $\alpha < \beta$ if $\varphi(r_\alpha) < \varphi(r_\beta)$.

Lemma 2.5.1 (Brenti et al. [1998], Lemma 6.7)**.** *Fix a reflection ordering on the roots associated to W_0. Assume that $u, x, v \in W_0$, $u \xrightarrow{\alpha} x \xrightarrow{\beta} v$, and $\alpha > \beta$. Then there exists a unique $y \in W_0$ such that $u \xrightarrow{\gamma} y \xrightarrow{\epsilon} v$ with $\beta < \epsilon > \gamma < \alpha$.*

Theorem 2.5.1 (Brenti et al. [1998], Theorem 6.4)**.** *For any reflection ordering φ on a [finite] Weyl group W_0.*

1. *For any pair of elements $u, v \in W_0$, there is a unique path from u to v in G such that the associated root sequence of the path is strictly increasing (resp., strictly decreasing).*

2. *The unique label-increasing (resp., label-decreasing) path from u to v in G has the smallest possible length. Moreover, it is lexicographically minimal (resp., lexicographically maximal) among all shortest paths from u to v.*

Postinkov makes the observation that Lemma 2.5.1 is weight preserving on the length 2 paths. Furthermore, if we look at the associated label sequence of any path, Lemma 2.5.1 allows us to change a descent in the reflection ordering at a position to an ascent. This flip is not guaranteed to lessen the number of descents, but the benefit is that the resulting reflection sequence is lexicographically smaller in the reflection ordering. Therefore, if we take any path, we can successively apply Lemma 2.5.1 to the path until the resulting path has the same weight, is still between the original endpoints, and has no descents. If the resulting reflection sequence is strictly increasing, then it is minimal by Theorem 2.5.1. Otherwise, it has a pair of adjacent, identical reflections in the sequence that is associated with a length 2 loop in G. Eliminating all such pairs (loops) results in a path that is minimal by Theorem 2.5.1. This observation allows us to prove the result for this paper, by uncovering what such loops associate to in the Bruhat order for W.

Corollary 2.5.1 (Postnikov [2002],Lugo)**.** *If $P : w \to w'$ is a non-minimal path in G, then P is weight equivalent to a path P' containing a length 2 loop where P and P' share the same start and end vertices. Further, if y is superregular and we obtain x by descending in the Bruhat order by interpreting P as a strictly near or strictly far path (using Proposition 2.4.1), then the resulting saturated chain obtained by descending in the Bruhat order by interpreting P' as the same type of path (strictly near or strictly far) is contained in $[x, y]$ as well.*

2.6 Proof of Result

By Lemmas 2.2.1 and 2.4.1, we get the following corollary when restricted to elements of W^0.

Corollary 2.6.1. *Suppose $x, y \in W^0$, $x < y = wt_\lambda$ with λ superregular, $k = \ell(y) - \ell(x)$, and*

$$x \xrightarrow{\beta_k} z_{k-1} \xrightarrow{\beta_{k-1}} \ldots z_1 \xrightarrow{\beta_1} y$$

is a saturated chain from x to y in the Bruhat order, where β_i, $r_f, r_n, P_f : 1 \to 1, P_n : w \to wr_n$ are defined *as in Lemma 2.4.1. Then*

$$x = wr_n t_{\lambda+\mathrm{wt}(P_n)+\mathrm{wt}(P_f)}$$

Proof. Since $y \in W^0$, then v in Lemma 2.4.1 is the identity by Lemma 2.2.1. Furthermore, since $x \in W^0$, then $vr_f = r_f = 1$. This explains why P_f is a loop at the identity, and completes the proof. □

This gives a strong correspondence between paths in G and saturated chains in the Bruhat order specifically for affine Grassmannian elements. Next, we note two facts about the Bruhat order.

Theorem 2.6.1 (Björner and Brenti [2005], Chain Property)**.** *If $u < w$ in W^J, then there exist elements $w_i \in W^J$, $\ell(w_i) = \ell(u) + i$, for $0 \leq i \leq k$, such that $u = w_0 < w_1 < \cdots < w_k = w$.*

Lemma 2.6.1 (Björner and Brenti [2005], Lemma 2.7.3)**.** *If $x < y$ in W, with $\ell(y) = \ell(x) + 2$, then $[x, y] = \{x, u, z, y\}$ where $x < u < y$ and $x < z < y$.*

These results give intimate details on the structure of intervals between elements differing by two in length. If $x < y$ are in W^0 and $\ell(x) + 2 = \ell(y)$, then by Theorem 2.6.1, we can assume, without loss of generality, that $z \in W^0$. Note that any loop of length 2 in G has a simple root as the label for both of its edges (apply both quantum and non-quantum edge conditions to α). With this information, we associate loops of length two with $[x, y] \not\subset W^0$.

Lemma 2.6.2. *Let $x, y \in W^0$ with y superregular, $\ell(y) = \ell(x) + 2$ and $x < y$. Then $[x, y] \not\subset W^0$ if and only if the near path associated to $x \to z \to y$ is a 2-loop.*

Proof. Let $y = wt_\lambda$. Suppose $u \notin W^0$. Then by Lemma 2.4.1, $x \to u \to y$ is a strictly far covering path that is a loop at the identity, $u = ws_\alpha t_{s_\alpha(\lambda)}$, and $x = wt_{\lambda+\alpha^\vee}$ for $\alpha \in \Delta_0$. Therefore by Corollary 2.6.1, the associated near path for $x < z < y$ is a loop of length 2 at w. Now suppose the near path associated to $x \to z \to y$ is a loop $P_n : w \to ws_\alpha \to w$ ($\alpha \in \Delta_0$). Then $x = wt_{\lambda+\alpha^\vee}$ by Corollary 2.6.1. If $u \in W^0$, this and Lemma 2.6.1 would imply a different $\beta^\vee$ such that

$$wt_{\lambda+\beta^\vee} = x = wt_{\lambda+\alpha^\vee} \Rightarrow \alpha = \beta$$

But this contradicts $u \neq z$. □

Proof of Theroem 1. By Theorem 2.1.2, our result is equivalent to showing that for $x = w't_{\lambda'}, y = wt_\lambda \in W^0$, $[x, y] \subset W^0$ if and only if $\lambda' = \lambda + \mathrm{M}(w, w')$. Suppose $[x, y] \subset W^0$. Then there exists a saturated chain in W^0 from x to y, which by Corollary 2.6.1 implies $x = w't_{\lambda+\mathrm{wt}(P_n)}$ for the associated near path. By the assumption $[x,y] \subset W^0$, Lemma 2.6.2, and Corollary 2.5.1, then P_n cannot be path equivalent to any path containing a simple loop, which by Theorem 2.5.1 implies P_n is minimal, i.e. $\mathrm{wt}(P_n) = \mathrm{M}(w, w')$.

Suppose $\lambda' = \lambda + \mathrm{M}(w, w')$. Then for any path $P : w \to w'$, $\mathrm{wt}(P) - \mathrm{M}(w, w')$ must be a sum of positive coroots by Corollary 2.5.1. If $[x, y] \not\subset W^0$, then there exists a saturated chain from x to y which utilizes far coverings. Corollary 2.6.1 supplies that $x = w' t_{\lambda + \mathrm{wt}(P_n) + \mathrm{wt}(P_f)}$ where P_f is a loop at the identity and $P_n : w \to w'$. But this implies $\mathrm{wt}(P_n) - \mathrm{M}(w, w') = -\mathrm{wt}(P_f)$ with $\mathrm{wt}(P_f) \neq 0$, which is a contradiction. $\square$

Example:

Let $y = wt_\lambda = s_2 s_1 s_2 t_{-2\alpha_1^\vee - 3\alpha_2^\vee}$. With the power of Sage, we know that the elements x such that $[x, y] \subset W^0$ are in the following table. Note the translation differences match the path weights in a previous example. This explicitly yields the expansion of $\mathcal{I}_y^{P_{\mathrm{af}}}$ in terms of the structure sheaves below.

$x = w' t_{\lambda'}$	$\lambda' - \lambda = \mathrm{M}(w, w')$
$t_{-\alpha_1^\vee - \alpha_2^\vee}$	$\alpha_1^\vee + 2\alpha_2^\vee$
$s_1 t_{-\alpha_1^\vee - \alpha_2^\vee}$	$\alpha_1^\vee + 2\alpha_2^\vee$
$s_2 t_{-\alpha_1^\vee - 2\alpha_2^\vee}$	$\alpha_1^\vee + \alpha_2^\vee$
$s_1 s_2 t_{-\alpha_1^\vee - 2\alpha_2^\vee}$	$\alpha_1^\vee + \alpha_2^\vee$
$s_2 s_1 t_{-2\alpha_1^\vee - 2\alpha_2^\vee}$	$\alpha_2^\vee$
$s_1 s_2 s_1 t_{-2\alpha_1^\vee - 2\alpha_2^\vee}$	$\alpha_2^\vee$
$s_2 s_1 s_2 t_{-2\alpha_1^\vee - 3\alpha_2^\vee}$	0
$s_2 s_1 s_2 s_1 t_{-2\alpha_1^\vee - 3\alpha_2^\vee}$	0

$$\begin{aligned}\mathcal{I}_{wt_\lambda}^{P_{\mathrm{af}}} = &\mathcal{O}_{wt_\lambda}^{P_{\mathrm{af}}} \\ &- \mathcal{O}_{ws_2 t_{\lambda+\alpha_2^\vee}}^{P_{\mathrm{af}}} - \mathcal{O}_{ws_1 t_\lambda}^{P_{\mathrm{af}}} \\ &+ \mathcal{O}_{ws_1 s_2 t_{\lambda+\alpha_2^\vee}}^{P_{\mathrm{af}}} + \mathcal{O}_{ws_2 s_1 t_{\lambda+\alpha_1^\vee+\alpha_2^\vee}}^{P_{\mathrm{af}}} \\ &- \mathcal{O}_{ws_2 s_1 s_2 t_{\lambda+\alpha_1^\vee+2\alpha_2^\vee}}^{P_{\mathrm{af}}} - \mathcal{O}_{ws_1 s_2 s_1 t_{\lambda+\alpha_1^\vee+\alpha_2^\vee}}^{P_{\mathrm{af}}} \\ &+ \mathcal{O}_{ws_2 s_1 s_2 s_1 t_{\lambda+\alpha_1^\vee+2\alpha_2^\vee}}^{P_{\mathrm{af}}}\end{aligned}$$

2.7 Bounds on Regularity

This general theory has currently glossed over the necessary conditions of superregularity. Initially, Lam and Shimozono [2007] left a large bound of superregular for Proposition 2.4.1 to be $|\langle \lambda, \alpha \rangle| > 2|W_0| + 2$ for all $\alpha \in \Delta_0$. This bound was found to be exceedingly extravagent by Milićević [2016] (see Proposition 4.2), who found that one only required $|\langle \lambda, \alpha \rangle| \geq 2\ell(w_0)$ if $W_0 \neq G_2$ (change to $3\ell(w_0)$ in the case of G_2) for all such $\alpha \in \Delta_0$. There has been further refinement by Welch [2019], who showed Proposition 2.4.1 holds when W_0 is simply laced and $|\langle \lambda, \alpha \rangle| \geq 3$ for all $\alpha \in \Delta_0$. Furthermore, this hints at a similar, smaller bound for the non-simply laced cases using a similar proof to Milićević [2016].

For simplicity sake, suppose the regularity bound for Proposition 2.4.1 is k and suppose the maximal length of a shortest path between two vertices in G is m. Since Theorem 1 depends on Proposition 2.4.1 for each step of a shortest path, then sufficient superregularity for Theorem 1 can be expressed as $|\langle\lambda,\alpha_i\rangle| \geq k(m-1)$. One extravagent upper bound on m is $2\ell(w_0)-1$. However, note that the shortest path from 1 to w_0 utilizes only non-quantum edges, so the path weight is 0. Going in the other direction, a path from w_0 to 1 may have a shorter length than w_0, but must necessarily have a non-trivial weight. So the question of sufficient regularity for Theorem 1 deserves further study.

3 An Affine Infinite Symmetric Group

The problem with the definition of an affine infinite symmetric group is the fact that there is no easy way to embed $\widetilde{A}_n$ into $\widetilde{A}_{n+1}$, which can been seen by trying to embed the associated Dynkin diagrams. If we attempted an embedding which allowed W_0 to embed naturally, then we must unnaturally disturb the affine simple reflection s_0. The alternative is to insert nodes away from s_0, which may fix s_0, but does not yield stable measures such as length and root height in either W or W_0. We attempt the second strategy here by considering a subgroup $\widetilde{S}_{[a,b]}$ of W that is of type $\widetilde{A}_n$. Then we define new measures on $\widetilde{S}_{[a,b]}$ which are stable under embeddings, and thus permits the consideration of the limit of $\widetilde{S}_{[a,b]}$.

3.1 The $\widetilde{S}_{[a,b]}$ Subgroup

First we clarify the classical structure of type $\widetilde{A}_n$. Let $X=\bigoplus_{i\in I} x_i\mathbb{Z}$ be the *root space* and $X^*=\bigoplus_{i\in I} e_i\mathbb{Z}$ the *coroot space* which is canonically dual (i.e. $\langle e_i,x_j\rangle=\delta_{ij}$, where δ_{ij} is the Kronecker delta and $x_i^\vee=e_i$). The classical simple roots are $\Delta_0=\{\alpha_i=x_i-x_{i+1}\mid i\in I_0\}$ with full set of classical roots $\Phi_0=\{x_{ij}=x_i-x_j\mid i,j\in I\}$. Positivity of roots is easily determined by $x_{ij}>0\Leftrightarrow i<j$. The classical Weyl group is generated by $W_0=\langle s_i\mid i\in I_0\rangle$ and is isomorphic to S_n. The *coroot lattice* can be identified as $Q^\vee=\{\lambda\in X^*\mid \sum_{i\in I}\lambda_i=0\}$, and we recover the classical identification $W\cong S_n\ltimes Q^\vee$, where $\theta=x_1-x_n$ is the longest root and $s_0=s_\theta t_{-\theta^\vee}$. The action of $w\in W_0$ on X and X^* is $w(x_i)=x_{w(i)}$ and $w(e_i)=e_{w(i)}$. We write $W\cong\widetilde{S}_n$ as the *affine symmetric group*.

In this section, for $x<y\in\mathbb{Z}$, let $[x,y]=\{i\in\mathbb{Z}\mid x\leq i\leq y\}$. For our work, we interpret each $i\in I$ as a representative of $i \pmod n$, so $s_i=s_{i+n}$, $e_i=e_{i+n}$, etc. Fix $a,b\in\mathbb{Z}$ such that $a\leq 0<b$ and choose $n>b-a+1$. We define the subgroup $\widetilde{S}_{[a,b]}$ of $\widetilde{S}_n$ as follows. Let $Q^\vee_{[a,b]}=\bigoplus_{i\in[a,b]}\alpha_i^\vee\mathbb{Z}$, where we identify $\alpha_0^\vee=e_0-e_1$ for this definition. Thus $\theta^\vee=e_1-e_0\in Q^\vee_{[a,b]}$. Define $\widetilde{S}_{[a,b]}=\langle s_i,t_\lambda\mid i\in[a,b],\lambda\in Q^\vee_{[a,b]}\rangle$. We find that $s_\theta=s_0t_\theta\in\widetilde{S}_{[a,b]}$, which allows us to identify $\widetilde{S}_{[a,b]}\cong S_{[a,b]}\ltimes Q^\vee_{[a,b]}$, where $S_{[a,b]}=\langle s_\theta,s_i\mid i\in[a,b],i\neq 0\rangle$. One crucial difference in our reindexing is that our definition of positive roots must change.

We say $x_{ij} \in \Phi$ is positive when $i \prec j$, where the $\prec$ ordering on $\mathbb{Z}$ is

$$1 \prec 2 \prec 3 \prec \cdots \prec -2 \prec -1 \prec 0$$

3.2 Stability in the Embedding

We seek independence of $\widetilde{S}_{[a,b]}$ from n. The generators chosen for $\widetilde{S}_{[a,b]}$ are already independent from n, but the associated calculations like length and pairings are dependent on n. Inspired by the ϵ-length function utilized in Muthiah and Orr [2016], we develop stable $\mathbb{Z}^2$ statistics by considering the changes in $S_{[a,b]}$ and $Q^\vee_{[a,b]}$ under the embedding. Let $\widetilde{S}_{[a,b]}$ be a subgroup of $\widetilde{S}_n$ for a sufficiently large n. Choose n_0, n_1 such that $n_0 \leq a$, $b < n_1$, and $n_1 - n_0 = n$. For an embedding $\widetilde{S}_n \hookrightarrow \widetilde{S}_{n+1}$, let k be the new modulo class such that $k \notin [n_0, n_1]$. For a fixed n, define $\Phi_{0,\Delta}$ to be the new positive classical roots introduced by the embedding. It is clear that

$$\Phi_{0,\Delta} = \{x_{1,k}, x_{2,k}, \ldots, x_{n_1-1,k}, x_{n_1,k}, x_{k,n_0}, x_{k,n_0-1}, \ldots, x_{k,-1}, x_{k,0}\}$$

and there are $n+1$ such new roots. Let $w \in S_{[a,b]}$ and $m = \#\{i \in [1, b+1] \mid w(i) \in [a, 0]\}$. Since $S_{[a,b]}$ can be considered as a finitely generated group $S_{[a,b]} \cong \langle s_\theta, s_i \mid a \leq i \leq b, i \neq 0\rangle$, then we know $m = \#\{i \in [a, 0] \mid w(i) \in [1, b+1]\}$. Define

$$L(w) = (2m, \ell(w) - 2nm) \tag{3}$$

Proposition 3.2.1. *$L(w)$ is stable under the natural embedding, with $2m$ being the number of new inversions of w under the natural embedding.*

Proof. First, we consider the number of new inversions w accomplishes on $\Phi_{0,\Delta}$. Consider $x_{i,k} \in \Phi_{0,\Delta}$. $w(x_{i,k}) \in \mathrm{Inv}(w)$ if and only if $w(i) < 0$, and the number of such i is m. Similarly, we receive another m inversions from new roots of the form $x_{k,j}$. This shows the first component is fixed under the natural embedding. But our proof has shown that $2m$ is the linear growth rate of our inversion set with respect to n, implying that for any further embedding, $\ell(w) = 2mn + c$ for some constant c. Therefore $L(w)$ is fixed in both components. □

Corollary 3.2.1. *For $u, v \in S_{[a,b]}$, $L(u) < L(v)$ if and only if there exists an N such that for all $n > N$, the embedding of u, v in $\widetilde{S}_n$ satisfies $\ell(u) < \ell(v)$.*

We now turn our attention to length of translations elements in $Q^\vee_{[a,b]}$. To consider antidominance in this setting, we must make a stronger definition. We say that $\lambda \in Q^\vee_{[a,b]}$ is *antidominant* if

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{b+1} \leq 0 \leq \lambda_a \leq \cdots \leq \lambda_0$$

The strengthening of this definition enables an antidominant λ to remain antidominant in the traditional sense under the embedding. For $\lambda \in Q^\vee_{[a,b]}$, let $\lambda^- = v\lambda$ where $v \in S_{[a,b]}$ satisfies $v\lambda$ is antidominant. Note that if we sum the roots of $\Phi_{0,\Delta}$, we obtain an expression of the form

$$\sum_{x \in \Phi_{0,\Delta}} x = \sum_{1 \leq i \leq n_1} x_i - \sum_{n_0 \leq i \leq 0} x_i + (n_1 + n_0 + 1)x_k \tag{4}$$

Let $\rho_\infty = \frac{1}{2}\sum_1^\infty x_i - x_{1-i}$ and $P : Q^\vee_{[a,b]} \to \mathbb{Z}^2$ satisfy

$$P(\lambda) = (\langle \lambda, \rho_\infty \rangle, \langle \lambda, \rho_n \rangle - n \langle \lambda, \rho_\infty \rangle) \tag{5}$$

Proposition 3.2.2. *P is stable under the natural embedding, with $\langle \lambda^-, -2\rho_\infty \rangle$ being the number of new inversions of t_λ under the natural embedding.*

Proof. Let $\lambda \in Q^\vee_{[a,b]}$ and ρ_n, ρ_{n+1} be defined for the respective ranks. Since $\langle \lambda, x_k \rangle = 0$, then it is clear that

$$\langle \lambda, \rho_{n+1} \rangle = \langle \lambda, \rho_n \rangle + \left\langle \lambda, \sum_{\alpha \in \Phi_{0,\Delta}} \alpha \right\rangle = \langle \lambda, \rho_n \rangle + \left\langle \lambda, \sum_{1 \leq i \leq n_1} x_i - \sum_{n_0 \leq i \leq 0} x_i \right\rangle = \langle \lambda, \rho_n \rangle + \langle \lambda, \rho_\infty \rangle$$

As before, since $\langle \lambda, \rho_\infty \rangle$ captures the linear growth rate of $\langle \lambda, \rho \rangle$, then P is stable under the natural embedding. Let v be the minimal element which satisfies $v\lambda = \lambda^-$. The fact that $\ell(t_\lambda) = \langle \lambda^-, -2\rho \rangle$ is clear from Equation 2 since

$$\ell(t_\lambda) = \sum_{\alpha \in \Phi^+} |\langle \lambda, \alpha \rangle| = \sum_{w^{-1}(\alpha) \in \Phi^+} |\langle \lambda, w^{-1}(\alpha) \rangle| = \sum_{\alpha \in \Phi^+} \langle \lambda^-, -\alpha \rangle = \langle \lambda^-, -2\rho \rangle$$

Thus by growth rate analysis of P, the result is clear. □

Corollary 3.2.2. *For $\lambda, \mu \in Q^\vee_{[a,b]}$, $P(\lambda^-) < P(\mu^-)$ if and only if there exists an N such that for all $n > N$, $\ell(t_\lambda) < \ell(t_\mu)$.*

Since this paper is interested in the affine Grassmannians, we consider $W^{0'} \subset \widetilde{S}_{[a,b]}$, which has the same definition as before, but with the stronger condition on antidominance. By the previous propositions, then the following is immediate.

Corollary 3.2.3. *For $wt_\lambda, w't_{\lambda'} \in W^{0'}$, then $-2P(\lambda) - L(w) < -2P(\lambda') - L(w')$ if and only if there exists an N such that for all $n > N$, $\ell(wt_\lambda) < \ell(w't_{\lambda'})$ when embedded in $\widetilde{S}_n$.*

Example: Consider $s_0 = s_\theta t_{-\theta^\vee} \in W^{0'}$. $L(s_\theta) = (2, -1)$ while $P(-\theta^\vee) = (-1, 0)$. Thus

$$-2P(-\theta^\vee) - L(s_\theta) = (0, 1)$$

3.3 Root Partitions for $S_{[a,b]}$

The function P yields additional partitions of the roots. It is well known that for this construction in type A, we have $x_i^\vee = e_i$. For $x \in \Phi_0^+$, if $P(x^\vee) = (a, b)$, then let $\Phi_\infty^+ = \{x \in \Phi_0^+ \mid a = 1\}$ and $\Phi_{\text{fin}}^+ = \Phi_0^+ \setminus \Phi_\infty^+$. As the name implies, Φ_∞ contains roots whose height grows by 1 for each natural embedding. This yields a partition $\Phi_0 = \Phi_\infty^+ \sqcup \Phi_{\text{fin}}^+ \sqcup \Phi_\infty^- \sqcup \Phi_{\text{fin}}^-$. An equivalent defintion for Φ_∞^+ is $\{x_{ij} \in \Phi_0 \mid j < i, i \prec j\}$. What is not immediately obvious is that these roots correlate strongly to L.

Proposition 3.3.1. *Let $w \in S_{[a,b]}$ and m defined as in Equation 3. Then*

$$\#\{x \in \Phi_\infty^+ \mid w(x) \in \Phi_\infty^-\} = m^2$$

Proof. This is clear combinatorially. If $x = x_{ij}$ is in the specified set, then $1 \leq i$ with $w(i) \leq 0$ and independently $j \leq 0$ with $w(j) \geq 1$. There are m ways to map i, and another independent m ways to map j, thus m^2 possible inversions of this type. □

This implies that the growth component of L is consistent with the number of positive infinite to negative infinite inversions. Let us define $L_\infty(w) = m^2$.

Lemma 3.3.1. *Let $w \in S_{[a,b]}$ and m defined as in Equation 3 when embedded in $\widetilde{S}_n$. Then*

- $\#\{x \in \Phi_\infty^+ \mid w(x) \in \Phi_\infty^+\} = (1 - n_0 - m)(n_1 - m)$
- $\#\{x \in \Phi_{\text{fin}}^+ \mid w(x) \in \Phi_{\text{fin}}\} = \binom{n_1 - m}{2} + 2\binom{m}{2} + \binom{1 - n_0 - m}{2}$

Proof. We follow the same style as the proof for Proposition 3.3.1. For the first, we simply note that for such an x_{ij}, i must map to a positive index (leaving $n_1 - m$ such choices) while j must independently map to a negative index (leaving $1 - n_0 - m$ choices), and this yields the given expression. For the second, we could send two positive indicies to positive indices (the first summand), two positive indicies to negative indices or vice versa (middle summand), or two negative indicies to negative indicies (the final summand). □

Define $L_{\text{fin}}(w) = \#\{x \in \Phi^+_{\text{fin}} \mid w(x) \in \Phi^-_{\text{fin}}\} - \#\{x \in \Phi^+_{\text{fin}} \mid w(x) \in \Phi^+_{\infty}\} - \#\{x \in \Phi^+_{\infty} \mid w(x) \in \Phi^+_{\text{fin}}\}$. Then we recover the following.

Proposition 3.3.2. *Let $w \in S_{[a,b]}$ and m be defined as in Equation 3 when embedded in $\widetilde{S}_n$. Then*

$$\ell(w) = 2mn + 2m - 3m^2 + L_{\text{fin}}(w) = 2n\sqrt{L_\infty(w)} + 2\sqrt{L_\infty(w)} - 3L_\infty(w) + L_{\text{fin}}(w)$$

Proof. We consider where all $\binom{n+1}{2}$ positive roots of Φ_0^+ are sent by a fixed w. We will define many constants to facilitate the calculation. Let $C(A,B) = \#\{x \in A \mid w(x) \in B\}$. Note that $n_1 - n_0 = n$. Let

$$K_1 = (1 - n_0 - m)(n_1 - m) \qquad K_2 = \binom{n_1 - m}{2} + 2\binom{m}{2} + \binom{1 - n_0 - m}{2}$$

$$L^1_{\text{fin}} = C(\Phi^+_{\text{fin}}, \Phi^-_{\text{fin}}) \qquad L^2_{\text{fin}} = C(\Phi^+_{\text{fin}}, \Phi^-_{\infty}) + C(\Phi^+_{\infty}, \Phi^-_{\text{fin}}) \qquad L_\infty = C(\Phi^+_{\infty}, \Phi^-_{\infty}) = m^2$$

and clearly $L_{\text{fin}} = L^1_{\text{fin}} - L^2_{\text{fin}}$. Then the calculuation follows:

$$\begin{aligned}
\text{Inv}(w) &= L^1_{\text{fin}} + L_\infty + C(\Phi^+_{\text{fin}}, \Phi^-_{\infty}) + C(\Phi^+_{\infty}, \Phi^-_{\text{fin}}) \\
&= L^1_{\text{fin}} + L_\infty + \left[\binom{n+1}{2} - \left(L^1_{\text{fin}} + L_\infty + C(\Phi^+_{\text{fin}}, \Phi^+_{\text{fin}}) + L^2_{\infty} + C(\Phi^+_{\infty}, \Phi^+_{\infty})\right)\right] \\
&= \binom{n+1}{2} - \left(C(\Phi^+_{\text{fin}}, \Phi^+_{\text{fin}}) + L^2_{\text{fin}} + C(\Phi^+_{\infty}, \Phi^+_{\infty})\right) \\
&= \binom{n+1}{2} - \left(\left[K_1 - L^1_{\text{fin}}\right] + L^2 + K_2\right) \\
&= \binom{n+1}{2} - K_1 - K_2 + L_{\text{fin}} \\
&= 2nm - 3m^2 + 2m + L_{\text{fin}}
\end{aligned}$$

□

Clearly, we see these equations are not aestetically pleasing. However, the point is that if we define $L_{\Phi_0}(w) = (L_\infty(w), L_{\text{fin}}(w))$, then we get consistency similar to that seen in Proposition 3.2.1.

Corollary 3.3.1. *Let $u, v \in S_{[a,b]}$. Then $L_{\Phi_0}(u) < L_{\Phi_0}(v)$ if and only if there exists an N such that for all $n > N$, $\ell(u) < \ell(v)$ when u, v are embedded into $\widetilde{S}_n$.*

3.4 Conclusion on Construction

With the definitions presented, we can easily define an affine infinite symmetric group $\widetilde{S}_\infty$ as

$$\widetilde{S}_\infty = \lim_{\substack{a \to -\infty \\ b \to \infty}} \widetilde{S}_{[a,b]}$$

The previous sections have shown there is a consistent length on its associated classical Weyl group $S_\mathbb{Z}$ (finitely generated in $\{s_\theta, s_i \neq s_0\}$) and a consistent ρ equivalent on its associated coroot lattice $\{\lambda \in \bigoplus_{i \in \mathbb{Z}} e_i\mathbb{Z} \mid \sum \lambda_i = 0\}$. There are antidominant elements and affine Grassmannian elements which have familiar properties with L and P. However, there is still work to be done. For example, one thing lacking is compatibility of "insertion" permutations. For $i \prec j \in \mathbb{Z}$, consider $w_{ij} \in S_\mathbb{Z}$

$$w_{ij}(k) = \begin{cases} i & \text{if } k = j \\ k+1 & \text{if } i \preceq k \prec j \\ k & \text{else} \end{cases}$$

If $j \leq 0 < i$, then this element is not finitely generated, but we still wish for compatibility. It does not naturally fit into $S_{[a,b]}$ for any fixed n since it must utilize simple reflections outside of $S_{[a,b]}$. In this particular case, we can overcome this obstacle in the limit by considering growth dictated by $m_+ = \#\{i \in \mathbb{Z} \mid 1 \leq i, w(i) \leq 0\}$ and $m_- = \#\{i \in \mathbb{Z} \mid i \leq 0, 1 \leq w(i)\}$ and considering growth rate as $m_+ + m_-$. In the finitely generated case, $m_+ = m_-$, both of which we called m in the previous section and this explains the growth rate of $2m$ for elements in $S_{[a,b]}$. For such an insertion permutation, m_+ and m_-can differ. If $j \leq 0 < 1 \leq i$, then we have $m_+ = 1$ and $m_- = 0$ in the limit, which does capture the growht rate of the associated insertion permutations in $\widetilde{S}_n$. This shows that the theory developed in this paper does leave room for improvement in order to overcome such shortcomings.

References

Anders Björner and Francesco Brenti. *Combinatorics of Coxeter Groups*. Springer Science+Business Media, Inc., 2005.

Francesco Brenti, Sergey Fomin, and Alexander Postnikov. Mixed bruhat operators and yang-baxter equations for weyl groups. *International Mathematics Research Notices*, 1998.

Michel Brion. Lectures on the geometry of flag varieties. *arXiv Mathematics e-prints*, art. math/0410240, Oct 2004.

Vinay Deodhar. Some characterizations of bruhat ordering on a coxeter group and determination of a relative möbius function. *Inventiones mathematicae*, pages 187–198, 1997.

James E. Humphreys. *Reflection Groups and Coxeter Groups*. Cambridge University Press, 1993.

Shrawan Kumar. *Kac-Moody Groups and Flag Varieities and Representation Theory*, volume 204 of *Progress In Mathematics*. Springer Science+Business New York, 2002.

Shrawan Kumar. Positivity in T-Equivariant K-theory of flag varieties associated to Kac-Moody groups. *arXiv e-prints*, art. arXiv:1209.6422, Sep 2012.

Shrawan Kumar. Personal correspondence, 2019. Email messages between Shrawan Kumar and Mark Shimozono.

Thomas Lam and Mark Shimozono. Quantum cohomology of G/P and homology of affine Grassmannian. *arXiv e-prints*, art. arXiv:0705.1386, May 2007.

Thomas Lam, Changzheng Li, Leonardo C. Mihalcea, and Mark Shimozono. A conjectural Peterson isomorphism in K-theory. *arXiv e-prints*, art. arXiv:1705.03435, May 2017.

Elizabeth Milićević. Maximal Newton points and the quantum Bruhat graph. *arXiv e-prints*, art. arXiv:1606.07478, Jun 2016.

Dinakar Muthiah and Daniel Orr. On the double-affine Bruhat order: the $\epsilon = 1$ conjecture and classification of covers in ADE type. *arXiv e-prints*, art. arXiv:1609.03653, Sep 2016.

Alexander Postnikov. Quantum Bruhat graph and Schubert polynomials. *arXiv Mathematics e-prints*, art. math/0206077, Jun 2002.

Mandy Welch. *Double Affine Bruhat Order*. PhD thesis, Virginia Tech, May 2019.