Geometry of Fractal Squares

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(ABSTRACT)

This paper will examine analogues of Cantor sets, called fractal squares, and some of the geometric ways in which fractal squares raise issues not raised by Cantor sets. Also discussed will be a technique using directed graphs to prove bilipschitz equivalence of two fractal squares.

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Chapter 1

Introduction

Fractal imagery is observed throughout the world - from Norway's fjords and Asia's influenza viruses to cell phone antennae and financial exchange rates [4]. Famous mathematicians, such as Benoit Mandelbrot, made their marks on the world by studying self-similar fractal shapes, and today, biophysicists study the self-similarity of protein chains. Even young mathematicians are exposed to the standard middle third Cantor set, one of the most well-known fractals, in their real analysis foundation course. In this paper, we will study particular kinds of fractals and their equivalence.

This paper addresses a basic question in real analysis: when are two self-similar sets bilipschitz equivalent? The sets I study are two-dimensional analogues of Cantor sets, analogues I call fractal squares. The methods I use are largely geometric and combinatorial. My main result is the identification of conditions sufficient to guarantee the bilipschitz equivalence of a pair of fractal squares.

In basic terms, bilipschitz maps do not expand or contract distances by more than some constant factor and two spaces are bilipschitz equivalent if there is a bilipschitz map from one onto the other. Under a bilipschitz map between two metric spaces, the usual ϵ, δ estimates on one space corresponds to ϵ', δ' estimates on the other space. This indicates that the analysis on the two spaces is the same.

This study of bilipschitz equivalence of self-similar sets began with a question G. David and S. Semmes posed in [1]. They asked if two particular self-similar sets are bilipschitz equivalent. These two sets will be constructed in a manner similar to that of the standard Cantor middle third set. In chapter 3 we will discuss the formation of different Cantor sets, and prove when two general Cantor sets are equivalent. The approach we will use was created by H. Rao, H. J. Ruan and L. F. Xi in [2]. They developed a technique which creates a unique decimal representation for each point in a Cantor set by following paths on a graph. We will prove that if two Cantor sets have the same scale ratio and the same graph representation then they are bilipschitz equivalent.

After completing the Cantor set case I then apply the technique developed in [2] to analogues of Cantor sets formed by using squares in $\mathbb{R}^2$. I call these analogues fractal squares. The study of fractal squares is much more complex than that of Cantor sets. In Cantor sets, two subintervals can be adjacent on the left or the right. However, with fractal squares, two subsquares can be adjacent from the left, right, top, bottom, or corner to corner. Collections of adjacent squares can form intricate subshapes not encountered in the Cantor set case.

However, one aspect of these two cases remains the same. In both cases the Hausdorff dimension is given by a simple formula involving the data used in the construction. The Hausdorff dimension of a fractal square is $\frac{\ln(k)}{\ln(n)}$ where k is the number of squares kept in the first level of construction, i.e., the first set in the nested sequence, and $\frac{1}{n}$ is the side-length of each of the subsquares. We know that Hausdorff dimension is an invariant of bilipschitz equivalence.

Hausdorff dimension tells us less about the bilipschitz equivalence of fractal squares than it does about Cantor sets. We know in the interval case that two Cantor sets are equivalent if they have the same number of subintervals of equal length. In chapter 4 we show that the same is not true, however, in the case of fractal squares. For example, a fractal square can be formed using subsquares that are completely disjoint. Another fractal square can be formed, using the same number of subsquares, with every square on at least one of the two diagonals kept at each set in the construction of the nested sequence forming the fractal square. For example, in a 3×3 fractal square one can choose to keep three isolated squares in the first set in the nested sequence. In another 3×3 fractal square one can keep the top left square, the square in the middle of the second row, and the bottom right square. More specifically, one keeps the three squares in the diagonal from top left to bottom right. In the latter fractal square, there will be a path from one corner to the other; however, in the former fractal square, there will be no path. Since the presence of paths is an invariant of bilipschitz equivalence, we know that these two sets are not equivalent. It also follows that the fractal square with a path could only be represented by an infinite graph.

To begin the application of the graph technique found in [2] to fractal squares I first determined the conditions under which a fractal square is represented by a finite graph in chapter 5. I discovered that when a fractal square has what I call a "complete path," there will be an associated finite graph, and hence, the graph theory technique can be applied. Chapter 6 ends with the discussion of how to form the finite graph.

In chapter 8 I identify a particular type of fractal square that can be represented by a large number of different graphs. I call these squares "separated fractal squares". I have developed a technique that constructs, from many fractal squares with complete paths, associated separated fractal squares. These fractal squares, if they have the same number of subsquares kept at each set in the nested sequence forming the fractal square and the same scaling factor, are all bilipschitz equivalent. Thus, separated fractal squares can be used as convenient representatives of equivalence classes.

I have been able to prove one particular case of the equivalence of fractal squares. Assume that two fractal squares constructed from the same number of subsquares of equal side length can each be associated with a separated fractal square. Assume that both fractal squares only have subshapes, i.e. shapes formed by adjacent squares in the first and second stages of construction, that are rectangular in nature. More specifically, if one views any set in the nested sequence that forms a fractal square, all subshapes in those shapes can be classified as rectangles. If all of the given hypotheses are fulfilled, then those two fractal squares are bilipschitz equivalent.

The proof of the theorem involves many different techniques. I first show that if a complete path is present, one can break the fractal into distinct quadrants. To do this, I develop a piecewise continuous closed path in the plane and prove the fractal can be split into a top-half and a bottom-half using winding numbers. I do the same with a right-half and a left-half and then form quadrants, discussed in chapter 5. In chapter 8, I subsequently show that one can group collections of quadrants together in fractal squares with rectangular subshapes so that each group can be associated with a separated fractal square. Once I have accomplished the collection of quadrants I know how to represent the separated fractal square by the graph of the original fractal square. Thus, I find they are equivalent. Since separated fractal squares are representatives of equivalence classes, I therefore prove my theorem.

The rectangular case is not the end of the story. I will finish my discussion with examples of fractal squares, using the same technique in the rectangular case, that are bilipschitz equivalent. I have also not been able to find an example in which my method can not be implemented. For every fractal square that I have formed, I have been able to prove it is equivalent to a separated fractal square at some level of construction. I have identified the crucial issues involved with proving the equivalence of fractal squares.

Chapter 2

Background Material

To speak of self-similar sets we must first look at at a few definitions and theorems. A more detailed approach is given in [7]. Let's begin by defining bilipschitz equivalence.

Definition 1 *Let* $(M, d(x,y))$ *and* $(N, \rho(u,v))$ *be metric spaces. A mapping* $f : M \to N$ *is said to be* **Lipschitz** *if there is a constant* C *such that* $\rho(f(x), f(y)) \leq Cd(x,y)$ *for all* $x, y \in M$*. We will say* f *is* **C-Lipschitz** *to make the constant explicit. We say* f *is* **bilipschitz** *if there is a* C *such that*

$$C^{-1}d(x,y) \leq \rho(f(x), f(y)) \leq Cd(x,y)$$

for all $x, y \in M$*. We will say* f *is* **C-bilipschitz** *to be more explicit.*

Notice that a Lipschitz map does not expand distances by more than a constant factor. Lipschitz maps are a stronger than uniform continuous maps. Bilipschitz maps do not expand or contract distances by more than some constant factor. Bilipschitz maps are a stronger form of uniform continuity and they have uniformly continuous inverses.

Definition 2 *Two metric spaces (M, d) and (N, ρ) are* **bilipschitz equivalent** *if there exists a bilipschitz map f of (M, d) onto (N, ρ).*

Theorem 1 *Bilipschitz equivalence is an equivalence relation.*

Proof:

Let (M, d) be a metric space. Then $f(x) = x$ is a bilipschitz map of M onto itself with constant 1.

Let $(M, d(x, y))$ and $(N, \rho(u, v))$ be metric spaces. Let f be a bilipschitz map of M onto N with constant C. Then f^{-1} is a bilipschitz map of N onto M with constant C.

Let $(M, d(x, y))$, $(N, \rho(u, v))$ and $(P, \nu(s, t))$ be metric spaces. Let f be a bilipschitz map of M onto N with constant C and g be a bilipschitz map of N onto P with constant K. Then the composition of f and g is a bilipschitz map of M onto P with constant CK.

Thus, we have that bilipschitz equivalence is in fact an equivalence relation. $\square$

Now, take a look at Hausdorff dimension.

Definition 3 *Given a metric space (M, d) and $E \subseteq M$ the* **diameter of** E , diam$E$, *is* *defined to be* $\mathrm{diam}E = \sup\{d(x,y)|x, y \in E\}$.

Definition 4 *Given a metric space* (N, ρ) *be a metric space and* $A \subseteq N$ *the* d**-dimensional Hausdorff measure of the set** A *is* $H^d(A) = lim_{\delta\to 0} H^d_\delta(A)$. *For any* $\delta > 0$,

$$H^d_\delta(A) = \inf\{\Sigma \mathrm{diam}(E_j)^d | E_j \subseteq A, A \subseteq \cup_{j=1}^{\infty} E_j, \mathrm{diam}(E_j) < \delta\}$$

Thus, a set E *has* **Hausdorff dimension** d *if* $d = \inf\{s|H^s(E) = 0\} = sup\{s|H^s(E) = \infty\}$.

Hausdorff dimension was shown to be an invariant of bilipschitz equivalent in [7]. Also shown was the presence of paths as an invariant under bilipschitz equivalence.

Proposition 1 *The presence of continuous paths is preserved under bilipschitz equivalence. [7]*

Proof:

Given (M, d) and (N, ρ) bilipschitz equivalent metric spaces, let $f : M \to N$ be a bilipschitz map of M onto N. Let α be a continuous path in M. Then $f \circ \alpha$ is a continuous path in N as desired. $\Box$

This gives that if the number of continuous paths in one self-similar set is different than the number of continuous paths in another self-similar set that they cannot be bilipschitz equivalent. Since Hausdorff dimension is also an invariant, if two self-similar sets have different Hausdorff dimensions they cannot be bilipschitz equivalent. These are the two main courses of action to prove self-similar sets are not bilipschitz equivalent. Our end goal is to construct a method to show to sets are bilipschitz equivalent.

We will now begin to study self-similar sets. As mentioned before, self-similar sets are sets that can be written as a disjoint union of scaled versions of itself with a scaling factor less than one. The scaling factors can be arbitrarily small. This definition is very cursory, a detailed definition will be given after an example, also given in [7].

Example 1 *The most well known self-similar set is the standard middle third Cantor set.*

Begin with the closed unit interval, $[0, 1]$. Now, remove the open middle third, namely $(\frac{1}{3}, \frac{2}{3})$. Thus, we are left with $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Now, remove the open middle third from each of the two subintervals of C_1 to form C_2. Continue this process to form $\cap_1^\infty C_n = C$. C is the standard Cantor middle thirds set. Is it self-similar? Yes! Let $f_1 = \frac{1}{3}x$ and $f_2 = \frac{1}{3}x + \frac{2}{3}$. Since $C = f_1(C) \cup f_2(C)$ we have that the standard middle thirds Cantor set is self-similar.

To give the detailed definition of self-similarity, a few other concepts must be defined and studied. The following was also given in [7].

Definition 5 *A mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is a* **contractive similitude** *also known as a* **contraction** *if $rd(x, y) = d(f(x), f(y))$, in the Euclidean metric, and r is the* **contraction ratio** *or* **similitude ratio** *where $r < 1$.*

Definition 6 *[2] Let $\{f_k\}_{k=1}^N$, mapping $\mathbb{R}^n$ into itself, be a collection of contractive similitudes. A compact, nonempty set E that satisfies $E = \cup_{k=1}^N f_k(E)$ is the* **invariant set** *or the*

self-similar set *of the collection* $\{f_k\}_{k=1}^n$. *If, for all* i, $f_i(E) \cap f_j(E) = \emptyset$ *where* $i \neq j$, *we call* E **dust-like**

It will be shown in Proposition 2 that invariant sets do exist and are unique.

Definition 7 *[3] A collection of contractive maps* $\{\phi_i\}_{i=1}^m$ *is defined to be an* **iterated function scheme**.

Now, as in [7], I will show that the standard middle thirds Cantor set is an invariant set for f_1, f_2 as defined above.

Example 1 Continued We showed that $C = \cup_{i=1}^2 f_i(C)$. For $i = 1, 2$ it is clear that f_i is a contractive similitude with ratio $\frac{1}{3}$. This gives that the standard middle thirds Cantor set is the invariant set for the iterated function scheme $\{f_1, f_2\}$ as desired.

When the standard middle thirds Cantor set is originally introduced to students of mathematics, it is usually written as $\cap_1^\infty C_n = C$. However the definition of self-similar sets involves unions and not intersections. These two notions are, in fact, equivalent.

Lemma 1 *If* $\{f_k\}_{k=1}^n$ *is a collection of contractive similitudes defined on* $\mathbb{R}^n$, *then there exists a compact subset* B *of* $\mathbb{R}^n$ *that satisfies* $f_k(B) \subseteq B$ *for every* k.

Proposition 2 *[4] Let* $\{f_k\}_{k=1}^n$ *be a family of contractive similitudes on* $\mathbb{R}^n$ *and let* D *be a compact set satisfying, for all* k, $F(D) \subseteq D$. *Then* $E = \cap_{n=1}^\infty F^n(D)$ *is a nonempty invariant set for* $\{f_k\}$, *in the sense of definition 6, and* E *is the only set satisfying the definition of invariant set for* $\{f_k\}$.

Refer to [7] for the proofs of the Lemma 1 and Proposition 2.

We wish to study not only the standard middle third Cantor set, but infinitely many different Cantor sets. We need not only consider the construction of the Cantor sets by removing open middle intervals.The standard middle third Cantor set studied earlier is also know as the collection of points in the interval $[0, 1]$ that can be written in in ternary using only the digits 0 and 2. More specifically, the Cantor middle third set is the collection that consists of all points that do not need the digit one in its base three representation. This technique can also be used to form other Cantor sets. We will look at one more construction of the standard middle third Cantor set. This method can be used to form infinitely many more Cantor sets.

Definition 8 *Let $F = \{0, 1, \ldots, n-1\}$. Denote the set of all sequences of elements of F by F^∞. Set $x = x_1x_2x_3\ldots, y = y_1y_2y_3\ldots \in F^\infty$, and let $L(x, y) = n$ denote when $x_i = y_i$ for $1 \leq i \leq n$ and $x_{n+1} \neq y_{n+1}$. Let $0 < a < 1$ and set $d_a(x, y) = a^{L(x,y)}$ where $a^\infty = 0$. This gives that $\{F^\infty, d_a\}$ is a metric space and we call it a* **Cantor set**. *[1]*

In [7] it was shown that $\{F^\infty, d_a\}$ is an ultrametric space and it has no rectifiable paths.

Proposition 3 *[1]*

The Hausdorff dimension of (F^∞, d_a) where $|F| = k$ is $\frac{\ln(k)}{\ln(\frac{1}{a})}$.

This is shown in both [1] and [7].

Theorem 2 *[7] Let* $|F| = 2$ *then* (F^∞, d_a) *is bilipschitz equivalent to a subset of* $[0, 1]$ *if* $0 < a < \frac{1}{2}$*. For the case* $\frac{1}{\sqrt[n-1]{2}} < a < \frac{1}{\sqrt[n]{2}}$ *for* $n > 1$ *then* (F^∞, d_a) *is in fact bilipschitz equivalent to a subset of* $[0, 1]^n$*.*

Refer to [7] for proof.

Chapter 3

Bilipschitz Equivalence of Cantor Sets

In this chapter we will show that two Cantor sets of the same dimension, each with the same number of subintervals and same scale ratio are bilipschitz equivalent. A different proof is given in [8].We shall prove this by using a proof giving by Rao, Ruan, et al in [2]. Some of the exposition follows lines discussed in [7].

To show two self-similar sets, defined in **Background Materials**, are bilipschitz equivalent we must begin by looking at two families of contractive similitudes $\{f_k\}_{k=1}^N$ and $\{g_k\}_{k=1}^N$, each taking $\mathbb{R}^n$ into $\mathbb{R}^n$. Let E and F be invariant sets of each family respectively. We may assume E and F are both dust-like and for each k the contraction ratio for f_k equals the contraction ratio for g_k. Later, we will show that $E \equiv F$.

To begin, a few more definitions are needed. First, let $G = (V, \Gamma)$ be a directed graph. Let $V = \{1, \ldots, N\}$ denote the set of vertices and Γ denote the edges along with their direction.

Assume that each vertex v has at least one edge beginning at v itself. Also, assume that for each edge $e \in \Gamma$ there exists a contractive similitude T_e that maps $\mathbb{R}^n$ to $\mathbb{R}^n$. Let T_e have contraction ratio $\rho_e \in (0,1)$. Let G^* denote the graph $G = (V, \Gamma)$ with the similitudes T_e. Lastly, let $\Gamma_{i,j}$ to be the set of all edges beginning at vertex i and ending at vertex j. [2]

Definition 9 **Graph directed sets on** G^* *are nonempty and unique compact sets* E_i *such that*

$$E_i = \cup_{j=1}^{N} \cup_{e \in \Gamma_{i,j}} T_e(E_j)$$

If this union is disjoint then E_i *is called* **dust-like**. *[2]*

Theorem 3 *Let* $\{E_i\}_{i=1}^n$ *and* $\{F_i\}_{i=1}^n$ *be the graph-directed sets of* G^* *and* H^* *respectively. If*

i) The base graphs coincide, i.e. $G = H$ *and hence*

$$E_i = \cup_{j=1}^{N} \cup_{e \in \Gamma_{i,j}} T_e(E_j)$$

and

$$F_i = \cup_{j=1}^{N} \cup_{e \in \Gamma_{i,j}} S_e(F_j);$$

ii) For each edge e *of* G*, the similitudes* S_e *and* T_e *have the same ratio* ρ_e*;*

iii) $\{E_i\}_{i=1}^n$ *and* $\{F_i\}_{i=1}^n$ *are dust-like.*

Then $E_i \equiv F_i$ *holds for all* $1 \leq i \leq N$. *[2]*

Proof:

[2] Recall that diam$E = sup\{|x-y||x,y \in E\}$. Let the distance between two sets, A, B be dist$(A,B) = \inf\{|x-y||x \in A, y \in B\}$ and set $T_{e_1 \ldots e_k} = T_{e_1} \circ \ldots \circ T_{e_k}$.

Since we are given that $\{E_i\}_{i=1}^N$ are dust-like we have that for all $x \in E_i$ there exists a unique infinite path $e_1e_2e_3\ldots$ such that

$$\{x\} = \bigcap_{k=1}^{\infty} T_{e_1e_2\ldots e_k}(E_{i_k}).$$

Note i_k is the vertex at the end of e_k. We let $e_1e_2e_3\ldots$ be defined as a *coding of* x. Let $f : E_i \to F_i$ be given by

$$\{f(x)\} = \bigcap_{k=1}^{\infty} S_{e_1e_2\ldots e_k}(F_{i_k}).$$

The fact that function f is one-to-one as well as onto is clear. If f satisfies the bilipschitz condition we will be done.

Take $x, x' \in E_i$. Let $e_1e_2e_3\ldots$ and $e_1'e_2'e_3'\ldots$ be the unique codings for x and x' respectively. Let $m \in \mathbb{Z}$ be the largest value such that $e_1e_2e_3\ldots e_m = e_1'e_2'e_3'\ldots e_m'$. Since $x, x' \in E_i$ we know

$$|x - x'| \leq \text{diam}T_{e_1e_2\ldots e_m}(E_{i_m}) \leq \Pi_{i=1}^m \rho_{e_i} \text{diam}(E_{i_m}).$$

By choice of m, the largest value with $e_1e_2e_3\ldots e_m = e_1'e_2'e_3'\ldots e_m'$, we have that:

$$|x - x'| \geq d(T_{e_1 e_2 \ldots e_m e_{m+1}}(E_{i_{m+1}}), T_{e_1 e_2 \ldots e_m e'_{m+1}}(E_{i'_{m+1}}) \geq \Pi_{i=1}^{m} \rho_{e_i} min_{(e,e')} d(T_e(E_j), T_{e'}(E_{j'})).$$

Note that the minimum is taken over all pairs of edges, (e, e') which begin at a common vertex, call it i. For (e, e') the ending vertices are denoted by j and j'. Since e and e' both begin at i we have that $T_e(E_j) \cap T_e(E_j) = \emptyset$ and both sets are closed. Thus the minimum is larger than zero.

Now we know that there is some constant $c_1 > 0$ that depends only on G^* where

$$c_1^{-1} \Pi_{i=1}^{m} \rho_{e_i} \leq |x - x'| \leq c_1 \Pi_{i=1}^{m} \rho_{e_i}.$$

In in a similar manner we can find a $c_2 > 0$ where

$$c_2^{-1} \Pi_{i=1}^{m} \rho_{e_i} \leq |f(x) - f(x')| \leq c_2 \Pi_{i=1}^{m} \rho_{e_i}.$$

Now we have that

$$c_1^{-1} c_2^{-1} |x - x'| \leq |f(x) - f(x')| \leq c_1 c_2 |x - x'|.$$

Thus we have that f is a bilipschitz map from E_i onto F_i and hence the two spaces are bilipschitz equivalent, as desired.□

Let's now move on to topics that were not studied in [7].

Definition 10 *Let* $S \subseteq [0, 1]$ *be a Cantor set. Let* $\{S_i\}_1^n$ *be the collection subintervals in the* *first stage of construction of S. If $S_i \cap S_j = \emptyset$ for all $i \neq j$ then S is called an* **isolated Cantor set**.

Theorem 4 *Let $S \subseteq [0,1]$ be an isolated Cantor set and let $M \subseteq [0,1]$ be any Cantor set such that dim(S) = dim(M) with n subintervals of length $\frac{1}{2n-1}$ in the first stage of construction. Then M is bilipschitz equivalent to S.*

Proof:

Let S and M be as described above. Note, that S is a Cantor set with $\frac{1}{2n-1}$ subintervals where the first, third,..., $2n-1^{st}$ subintervals are kept in the first level of construction. This gives that each subinterval in the first stage of construction for S are disjoint and for M the intersection of two subintervals is empty or one single point.

Let $M_2 = M \cup (M+1)$. Let $k = max\{$number of connected intervals in $M_2\}$ Let $M_i = M \cup (M+1) \cup \ldots \cup (M+i-1)$ for all $1 \leq i \leq k$. k is the maximum number of subintervals that appear in each M_i. Note that $M_1 = M$. Let G be a graph with k vertices. Each vertex will correspond with an M_i. To know where to place edges on G one must study each M_i. Take M_j for some $1 \leq j \leq k$. Look at the first stage of M_j. The first stage is made up of copies of different M_i's. For each M_i place an edge connecting M_j to M_i with the arrow facing M_i. There may be multiple directed edges connecting two vertices. Each directed edge will have a corresponding mapping. The mapping with be of the form $\frac{1}{2n-1}x + a$ where a is the distance of the left endpoint of the contracted M_i from the origin. Continue this process until all subintervals in the first stage of construction in M_j are accounted for in the graph. Then repeat for each different vertex. This is your directed graph G^* for the Cantor set M. Note, not all vertices may be used.

Now we need to know if G imposes on S. Let $n_0, n_1, \ldots, n_l$ be the intervals of M with $\Sigma_0^k n_j = n$ and $length(n_j) = \frac{p}{2n-1}$ with $p \in \mathbb{N}$. Now, let $S_j = S \cup (S+2) \cup \ldots \cup (S+2(j-1))$. S_j will represent vertex j, the same vertex representing M_j. If each of the edges coming from j are the same for both S_j and M_j we will know that M and S are represented by the same graph.

Step 1: (S)

There are n disjoint subintervals of S and $\Sigma_0^k n_j = n$ subintervals of M, in order of largest to smallest $n_{1_1}, \ldots n_{1_l}$ Consider the first n_{1_2} intervals a scaled copy of $S_{n_{1_1}}$. Then, consider the next n_{1_1} subintervals as a scaled copy of $S_{n_{1_2}}$. Continue this process. Thus, for every scaled M_j in M, there is a scaled S_j in S and the edges from vertex one are the same for S and for M. We will follow a similar process for each M_L.

Step L: (M_L)

There are :

Number of Subintervals in M_L	Length of the Subinterval
$L-1$	$n_1 + n_l$ length pieces from joining n_1 to n_l
L	$n_2, \ldots, n_{l-1}$
1	n_1
1	n_l

Associate each of the L subintervals of length $n_1 + n_l$ in M_L to a distinct group of separated pieces, i.e. one of translated copies of S which is of the form $S+2r$ in S_L. Also associate the single n_1 and single n_l to one of the translated copies of S in S_L. Thus, we have now used $n_1 + n_l$ subintervals in each of the L copies of S in S_L. Then each of the L $n_2, \ldots, n_{l-1}$ go to a distinct group of separated pieces in each of the L copies of S in S_L. These groupings will correspond to the same graph G which we constructed above for M since we will have the same edges coming from each of the vertices. Again, each map will have the same similitude ratio, $\frac{1}{2n-1}$. $\square$

Example 2 *[2]*

As shown in [2], [7] and discussed in [1] $S = \frac{S}{5} \cup (\frac{S}{5} + \frac{2}{5}) \cup (\frac{S}{5} + \frac{4}{5})$ and $M = \frac{M}{5} \cup (\frac{M}{5} + \frac{3}{5}) \cup (\frac{M}{5} + \frac{4}{5})$ are bilipschitz equivalent.

By definition S and M are self-similar, of the same dimension, and S is isolated.

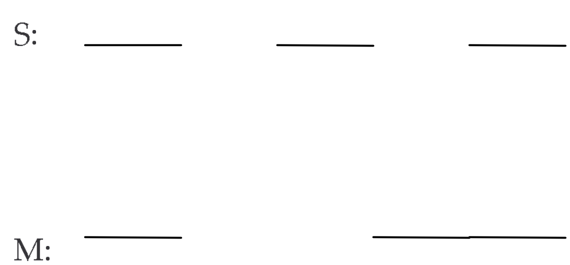


Figure 3.1: First Stage of Self-Similar Set S and Self-Similar Set M

Let $S_1 = S$, $S_2 = S \cup (S+2)$, and $S_3 = S \cup (S+2) \cup (S+4)$. Then we have the following disjoint unions:

$$S_1 = \frac{S}{5} \cup \left(\frac{S}{5} + \frac{2}{5}\right) \cup \left(\frac{S}{5} + \frac{4}{5}\right) = \frac{S_1}{5} \cup \left(\frac{S_2}{5} + \frac{2}{5}\right)$$

$$S_2 = \left(\frac{S_1}{5} + 2\right) \cup \left(\frac{S_3}{5}\right) \cup \left(\frac{S_2}{5} + \frac{12}{5}\right)$$

and

$$S_3 = \left(\frac{S_1}{5} + 4\right) \cup \left(\frac{S_3}{5}\right) \cup \left(\frac{S_3}{5} + 2\right) \cup \left(\frac{S_2}{5} + \frac{22}{5}\right).$$

Form the associated graph, G^*.

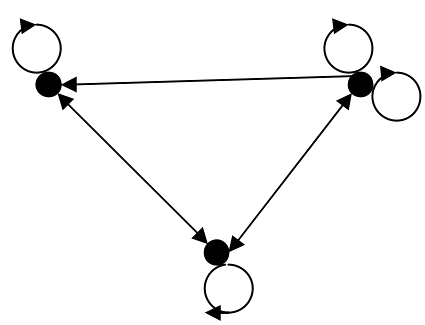


Figure 3.2: Directed Graph G^*

Label the vertices $1, 2, 3$ in the counterclockwise direction with vertex 1 at the top left. The associated similitudes must be constructed. For vertex 1, the similitude of the loop is $\frac{1}{5}x$ and the edge connecting vertex 1 to vertex 2 is has similitude $\frac{1}{5}x + \frac{2}{5}$. For vertex 2, the loop is matched with $\frac{1}{5}x + \frac{12}{5}$, the edge connecting vertex 2 to vertex 3 has $\frac{1}{5}x$ and the edge connecting vertex 2 to vertex 1 has similitude $\frac{1}{5}x+2$. For vertex 3, one loop has $\frac{1}{5}x$, the other is paired with $\frac{1}{5}x+2$, the edge connecting vertex 3 with vertex 2 has similitude $\frac{1}{5}x+\frac{22}{5}$ and the edge connecting vertex 3 with vertex 1 has $\frac{1}{5}x+4$.

Now, S_1, S_2, S_3 are dust-like invariant sets of G^*. Also note that each similitude has similitude ratio $\frac{1}{5}$ as desired.

Consider $M_1 = M$, $M_2 = M \cup (M+1)$, and $M_3 = M \cup (M+1) \cup (M+2)$. Then we have the following disjoint unions:

$$M_1 = \frac{M_1}{5} \cup \left(\frac{M_2}{5} + \frac{3}{5} \right)$$

$$M_2 = \frac{M_1}{5} \cup \left(\frac{M_3}{5} + \frac{3}{5} \right) \cup \left(\frac{M_2}{5} + \frac{8}{5} \right)$$

and

$$M_3 = \frac{M_1}{5} \cup \left(\frac{M_3}{5} + \frac{3}{5} \right) \cup \left(\frac{M_3}{5} + \frac{8}{5} \right) \cup \left(\frac{M_2}{5} + \frac{13}{5} \right).$$

The similitudes for M are: $\frac{1}{5}x, \frac{1}{5}x+\frac{3}{5}, \frac{1}{5}x+\frac{3}{5}, \frac{1}{5}x+\frac{8}{5}, \frac{1}{5}x+\frac{3}{5}, \frac{1}{5}x+\frac{8}{5}$ and $\frac{1}{5}x+\frac{13}{5}$.

$\{M_1, M_2, M_3\}$ are also dust-like invariant sets of G^*. As above, each similitude has similitude ratio $\frac{1}{5}$.

Thus, both $\{S_1, S_2, S_3\}$ and $\{M_1, M_2, M_3\}$ are dust-like and each similitude has ratio $\frac{1}{5}$. By theorem 3 we have that M_1 is equivalent to S_1 and hence M is equivalent to S as desired.

As previously discussed in [2] and [7], this answers the question posed by David and Semmes in [1].

Now, the following proof also discusses the equivalence of Cantor sets. However, instead of using isolated Cantor sets as we did above, we use a different kind of Cantor set which we call separated. We will extend the idea of a separated Cantor set to a separated fractal square in chapter 8. Separated fractal squares will be the ideal representatives for equivalence classes in our study of fractal squares.

Theorem 5 *Let F and M be Cantor sets of the same dimension. Assume in their first stage of construction the interval is split into n equal subintervals where $b < n$ subintervals are kept. Then $F \equiv M$.*

Proof:

We will show that both F and M are equivalent to a new Cantor set, S. Let S be the Cantor set of the same dimension constructed as follows. Take the unit interval and divide into n equal subintervals. For its first stage of construction keep the first b subintervals. So in the first stage there is just one subinterval of length $b \cdot n$. Note that in the second stage there are b^2 disjoint subintervals of length $\frac{1}{n^2}$. The fact that all subintervals are disjoint is crucial. Now we will show that $M \equiv S$.

Form the directed graph for M, call it G^*, as in Theorem 4. There are b adjacent subintervals of S in the first stage of construction, however, when looking at the second stage you can see there are actually b disjoint copies of the first stage of construction. Let M_j be as defined in theorem 4. Let $S_j = S \cup (S+1) \cup \ldots \cup (S+j)$. We can follow the techniques used in theorem 4 and see that G^* can be imposed on S. This is actually easier than in Theorem 4 since we do not need to worry about order. Instead, we can just match corresponding subintervals of M of any length to corresponding subintervals of the same length in S. We need not worry about spacing, as we did before, because of the placement of the subintervals in S. This means that one must just take a subinterval of M, say it has l sub-subintervals of length n, and map it to the first l subintervals of S. Then choose another subinterval of M, say of length l' and map it to the next l' subintervals of S. Continue this process until it ends. It will end because M has a finite number of subintervals and S have the same number of subintervals as M. This gives our vertices for S and thus the graph of G imposes on the set S. Thus we have that $M \equiv S$.

Similarly, we can also impose the graph representing F onto S and have $F \equiv S$. By the transitivity property of equivalence relations we have that $F \equiv M$. $\square$

Example 3 *The following two Cantor sets, represented by their first stage of construction, are equivalent:*

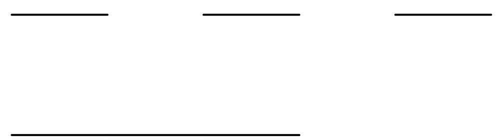


Figure 3.3: First Stage of two Cantor sets

Call the first Cantor set E and the second F. Each can be represented by a graph with one vertex and three edges. The similitudes for E are $T_1(x) = \frac{1}{5}x, T_2(x) = \frac{1}{5}x + \frac{2}{5}$ and $T_3(x) = \frac{1}{5}x + \frac{4}{5}$. For F the similitudes are $S_1(x) = \frac{1}{5}x, S_2(x) = \frac{1}{5}x + \frac{1}{5}$ and $S_3(x) = \frac{1}{5}x + \frac{2}{5}$.

Since $T_i(E) \cap T_j(E) = \emptyset$ for all $i \neq j$ and $\cup_1^3 T_j(E) = E$ we have that E is dust-like. Similarly F is also dust-like.

We will show the equivalence following the proof of theorem 3.

Since E is dust-like we know that for every $x \in E$ there exists a unique coding $e_1e_2e_3\ldots$ such that $\{x\} = \cap_1^\infty T_{e_1e_2\ldots e_k}(E)$. Let $f : E \to F$ be defined as $\{f(x)\} = \cap_1^\infty S_{e_1e_2\ldots e_k}(F)$. Clearly f is one-to-one and onto. Is f bilipschitz?

Let $x, x' \in E$. Let $e_1e_2e_3\ldots$ and $e'_1e'_2e'_3\ldots$ be codings of x and x' respectively. Let m be the largest integer with $e_1e_2\ldots e_m = e'_1e'_2\ldots e'_m$. This gives that $|x - x'| \leq diam(T_{e_1e_2\ldots e_m}(E)) = \frac{1}{5^m} diam(E) = \frac{1}{5^m}$.

Now, by maximality of m we have

$$|x - x'| \geq d(T_{e_1e_2\ldots e_{m+1}}(E), T_{e'_1e'_2\ldots e'_{m+1}}(E)) \geq \frac{1}{5^m} min(d(T_e(E), T'_e(E)) \geq \frac{1}{5^{m+1}}$$

where the minimum is taken over all pairs of edges (e, e').

This gives that $\frac{1}{5^{m+1}} \leq |x - x'| \leq \frac{1}{5^m}$.

Now we will follow the same process with F. Now, $f(x)$ and $f(x')$ have the same codings as x and x' so the maximum where $e_1e_2\dots e_m = e'_1e'_2\dots e'_m$ is the same m as before. This gives that $|f(x) - f(x')| \leq diam(S_{e_1e_2\dots e_m}(F)) = \frac{1}{5^m}diam(F) = \frac{1}{5^m}\frac{3}{5} = \frac{3}{5^{m+1}}$.

Again, by maximality of m we have

$|f(x) - f(x')| \geq d(S_{e_1e_2\dots e_{m+1}}(F), S_{e'_1e'_2\dots e'_{m+1}}(F)) \geq \frac{1}{5^m}min(d(S_e(F), S'_e(F)) \geq \frac{1}{5^m}\frac{2}{25} = \frac{2}{5^{m+1}}$

where the minimum is taken over all pairs of edges (e, e').

This gives that $\frac{2}{5^{m+2}} \leq |x - x'| \leq \frac{3}{5^{m+1}}$.

Combining our two inequalities we have that

$\frac{2}{25}|x - x'| \leq |f(x) - f(x')| \leq 3|x - x'|$

If $c = \frac{25}{2}$ we have that $c^{-1}|x - x'| \leq |f(x) - f(x')| \leq c|x - x'|$ and $E \equiv F$. □

Note that if the scaling factors weren't the same then proof will necessarily hold. Look at the following base graph:

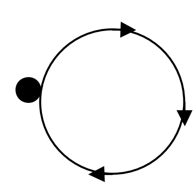


Figure 3.4: Graph representing F and M

This is a graph for the Cantor set with three subintervals in the first stage of construction with either scale factor $\frac{1}{5}$ or scale factor $\frac{1}{4}$. These two Cantor sets have different dimension and therefore cannot be bilipschitz equivalent.

Thus, we have shown that two Cantor sets with the same number of subintervals and the same scale ratio are equivalent. We have worked through a few examples and have completed our study on the bilipschitz equivalence of two distinct general Cantor sets.

Chapter 4

Fractal Squares

Now, we want to expand our technique. Instead of looking at self-similar sets formed using intervals, we will form self-similar sets using squares. A self-similar fractal square can be defined in the following way. To form a self-similar fractal square one will take the intersection of a sequence of nested sets, as in the definition of the Cantor Middle Third set. One begins with a single unit square and divides it into n^2 subsquares. Choose k of these subsquares. These will be known as the defining data. The k chosen subsquares will sometimes be referred to as "filled in" subsquares and the remaining $n^2 - k$ subsquares will sometimes be referred to as empty or blank subsquares. The m^{th} set in the sequence is the union of k^m closed squares chosen from the n^{2m} squares of side length $\frac{1}{n^m}$ whose union is the unit square.

The sets in the sequence are constructed as follows. The first set is the union of the unit squares k subsquares, chosen from the unit squares n^2 subsquares of equal size chosen previously in the defining data. The second set is defined by decomposing each of those k subsquares into n^2 subsquares of equal size and, in each of the k subsquares chosing the k subsquares as in the defining data. The second set is the union of these k^2 subsquares. Thereafter each set in the sequence is formed by decomposing each square in the preceding set into n^2 subsquares using the defining data to choose k subsquares from each of these arrays of n^2 squares, and taking the union of the subsquares chosen in this manner.

Example 4

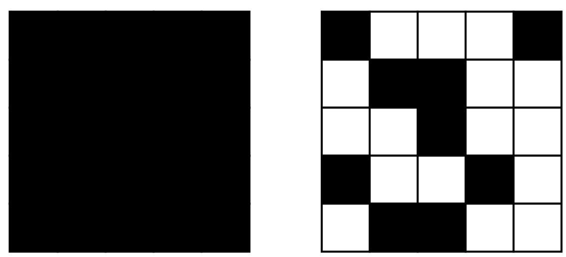


Figure 4.1: Unit square and the first level of construction of a fractal square

As discussed before this example begins with the filled in unit square. Then the unit square was broken into 25 subsquares and 9 subsquares were kept and 16 were removed. The picture of the first stage of construction is shown to the right of the unit square. By discussion above, repeating this process indefinitely forms a fractal square. Note, that fractal squares, as defined above, are self-similar. They can also be seen as the union of k -scaled versions of itself, as needed in showing equivalence of self-similar sets in theorem 3.

Calculating the dimension of fractal squares is very similar to calculating the dimension of the self-similar sets formed using intervals. Let a denote the number of squares remaining after the first iteration. Let b denote the side length of the squares after the first iteration. Then the dimension is $\frac{\ln(a)}{\ln(\frac{1}{b})}$ [1]. The above example has dimension $\frac{\ln(9)}{\ln(5)}$. In [7] the dimension of the Sierpinski Carpet was shown. The proof of the dimension of any fractal square can be shown in a similar manner.

Generally, when working with fractal squares, one works with a level of construction of that fractal square and not the actual self-similar set. We call the m^{th} set in the sequence which is used to define the fractal square the m^{th} level of construction of the fractal square. We now give many definitions associated with fractal squares. Most of these definitions look at the fractal square at a particular level of construction and are in reference to that specific level of construction. An important result follows.

Lemma 2 *Any stage of the construction of a fractal square may be iterated to form the same self-similar fractal square.*

Proof:

To begin choose $k \in \mathbb{N}$. We will show that the fractal square created by iterating the first stage is the same fractal square created by iterating the k^{th} stage. Note that when iterating using the k^{th} stage instead of replacing each filled in square with a contracted first stage each filled in square is replaced by a contracted k^{th} stage.

Let F designate the fractal square generated by iterating the first stage of the construction, and K the fractal square generated by iterating the k^{th} stage. Let F_n be the set formed by iterating the first stage n times, and similarly K_n is the set formed by iterating the k^{th} stage n times. Now, $F = \bigcap_{n=1}^{\infty} F_n$ and $K = \bigcap_{n=1}^{\infty} K_n$.

Now, we will show that $F = K$. Let $x \in F$. Then $x \in F_n$ for all $n \in \mathbb{N}$. This gives that $x \in K_n$ for all $n \in \mathbb{N}$ because each $K_n = F_{k^n}$. Thus $x \in \bigcap_{n=1}^{\infty} K_n$ and $x \in K$.

Now, assume that $x \in K$. This gives that $x \in K_n$ for all $n \in \mathbb{N}$. Now, $K_n = F_{k^n}$ for all $n \in \mathbb{N}$. We also know that for all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ so that $m \leq k^n$. This gives that $x \in F_m$ for all $m \in \mathbb{N}$ since $F_{K^n} \subseteq F_m$. Hence $x \in \bigcap_{n=1}^{\infty} F_n = F$. Thus $K = F$ and we have our result. $\square$

Definition 11 *Let I_d represent a fractal square of dimension d in which the filled in blocks, i.e. the k squares chosen out of the n^2 subsquares in the defining data for I_d, of its first stage of construction are completely isolated, more specifically, no two of them share an edge or a corner. We call I_d an* **isolated fractal square** *of dimension d.*

Example 5

The following is an example of the first stage of construction of an isolated fractal square.

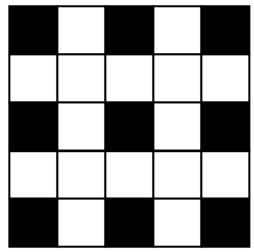


Figure 4.2: Isolated Fractal Square of Dimension $\frac{\ln(9)}{\ln(5)}$

Recall, for the following definitions, we consider a fractal square at a particular level of construction, not after the limiting process of levels of construction that forms the self-similar fractal square.

Definition 12 *Two filled-in squares at a particular level of construction are called* **adjacent** *if they share an edge or corner.*

Definition 13 *A* **concatenation** *of squares is a collection of squares, $S_1, S_2, \ldots, S_n$ such that S_i is adjacent to S_{i+1} for each $1 \leq i \leq n-1$.*

Definition 14 *A* **separation** *of a collection of squares is a decomposition of the collection into two nonempty subcollections with the property that no square from one subcollection is equal or adjacent to any square from the other subcollection.*

Definition 15 *A* **chain** *is a collection of squares that admits no separation.*

The following is an example of a chain.

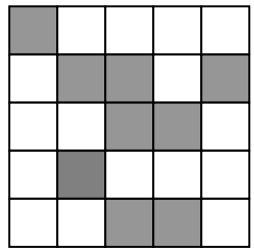


Figure 4.3: Chain of Length 9

Lemma 3 *Supposed that S and T are different squares in a chain. Then there is a concatenation $S_1, \ldots, S_n$ of squares in the chain with $S_1 = S$ and $S_n = T$*

Proof:

Let C be a chain containing more than one square. Let S be a square in C. Let C' be the subcollection of C consisting of S and of all the squares reachable by a concatenation beginning with S. If $C - C'$ is nonempty, then C' and $C - C'$ give a separation of C because any square adjacent to a square S' in C' is reachable via a concatenation involving adding one step to the concatentation reaching S'! $\square$

Definition 16 *A* **subchain** *is a subset of a chain that is itself a chain.*

Definition 17 *The* **length** *of a chain is the number of squares in the chain.*

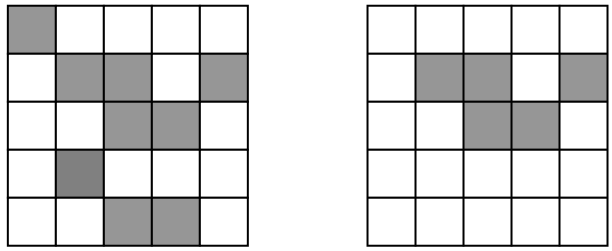


Figure 4.4: Subchain of Length 5

The length of the above chain is 9 and the length of its subchain is 5.

Definition 18 *A fractal square is of* **bounded chain length** *if there exists an* $M \in \mathbb{N}$ *which is a bound on the length of every chain in every level of construction of the fractal square.*

Definition 19 *A fractal square has* **unbounded chain length** *or an* **infinite chain** *if it is not of bounded chain length.*

More specifically, given a fractal square F, list the sequence of levels of construction, L_i, forming F. If there exists a sequence of chains C_i in L_i such that the collection of points in the union of squares in C_i is contained in the collection of points in the union of squares in C_{i+1} and no $M \in \mathbb{N}$ is a bound for all C_i then F is a fractal square with an **infinite chain**.

Note, that when looking at the progression of the sequence of sets that form a fractal square F, then one can consider an infinite chain a "chain" such that for every $M \in \mathbb{N}$ there exists a set in the sequence forming F that has a chain of length at least M which contains the infinite "chain" in that set. Sometimes the set involves multiple copies of our fractal square. We call this an $n \times n$ block.

Definition 20 *For any $n \in \mathbb{N}$, an $n \times n$* **block** *is formed by creating a larger square with n^2 copies of your k^{th} stage of construction for some k and some n.*

The following result, although seemingly trivial, is very important to our studies.

Lemma 4 *The number of adjacencies in a collection of squares cannot decrease by the addition of squares.*

Proof:

The addition of squares keeps the original adjacencies intact, although it may increase the total number of adjacencies. This is because the original set of squares is a subset of the new collection of squares. □

Now, chains are not the only collection of adjacencies that we will study. We will also study a notion of adjacency that requires squares to share an edge in order to be considered adjacent.

Definition 21 *Two filled-in squares at a particular level of construction are called* **edge-adjacent** *if they share an edge.*

Definition 22 *An* **edge-concatenation** *of squares is a collection of squares, $S_1, S_2, \ldots, S_n$ such that S_i is edge-adjacent to S_{i+1} for each $1 \le i \le n-1$.*

Definition 23 *An* **edge-separation** *of a collection of squares is a decomposition of the collection into two nonempty subcollections with the property that no square from one subcollection is equal or edge-adjacent to any square from the other subcollection.*

Definition 24 *An* **edge-connected set** *is a collection of blank squares that admits no edge-separation.*

Lemma 5 *If S and T are different squares in an edge-connected set then there is an edge-concatenation $S_1, \ldots, S_n$ of squares in the edge-connected set with $S_1 = S$ and $S_n = T$.*

The proof of this is similar to the proof of lemma 3.

Definition 25 *A* **path** *is an edge-concatenation that begins at one square, S, and ends at another square, T.*

Note, one can "travel" the path in the reverse order by reversing the order of the edge-concatenation.

Definition 26 *A* **non-repeating path** *is a path in which no square is repeated.*

Lemma 6 *Given a path in a fractal square F there exists a subset of the path which is a non-repeating path with the same first square and the same last square.*

Proof:

Let S_1 and S_n be connected by an edge-concatenation $S_1 \ldots, S_n$. Let S_{k_1} be the last appearance of S_1. Replace by $S_{k_1}, \ldots S_n$. More specifically, remove $S_1, \ldots, S_{k_1-1}$. Let S_{k_2} be the last appearance of S_{k_1+1}. If $k_2 > k_1 + 1$, remove $S_{k_1+1}, \ldots S_{k_2-1}$ to get $S_{k_1}, \ldots S_{k_2}, \ldots S_n$. If $k_2 = k_1 + 1$ then do nothing. Continue this process which ends after no more than n steps. The subset of squares which remain are a non-repeating path. $\square$

Definition 27 *A* **complete path** *is a union of two non-repeating paths of blank squares that satisfy the following: One non-repeating path begins with a square in column j in the top row and ends with a square in the bottom row of column j for some j; the other non-repeating path begins with a square in row i in the left-most column and ends with a square in the right-most column of row i for some i; and the two paths share at most one square or one edge concatenation.*

Lemma 7 *A collection of squares in a fractal square F that is the union of two edge-concatenations that satisfy the following has a subset which is a complete path: One edge-concatenation contains a square in column j in both the top row and bottom row for some j; and the other edge-concatenation contains a square in row i in both the left-most column and right-most column for some i.*

Remark: We will see later that these two edge-concatenations will share at least one square. However, we will not be able to prove this until chapter 5.

Proof:

By lemma 6 we can form a non-repeating path from the vertical portion of the complete path from the bottom row of the fractal square in column j to the top row of the fractal square in column j. Call this path P_v. Similarly, we can form a non-repeating path from the left-most column in row i to the right-most column in row i. Call this path P_h.

Note that the squares in P_h are in a non-repeating edge-concatenation, and hence ordered. Let S be the first square in P_h that is also in P_v. Let T be the last square in P_h that is also in P_v. If there are squares in P_h between S and T then replace them in P_h by the squares in P_v between S and T. □

Example 6 *In gray, there is a complete path:*

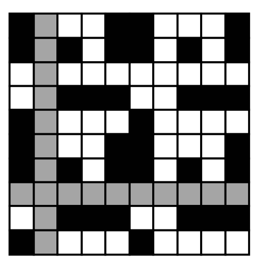


Figure 4.5: Complete Path

Example 7

In gray is another complete path:

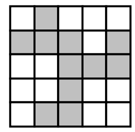


Figure 4.6: Complete Path

Now, look at the figures below.

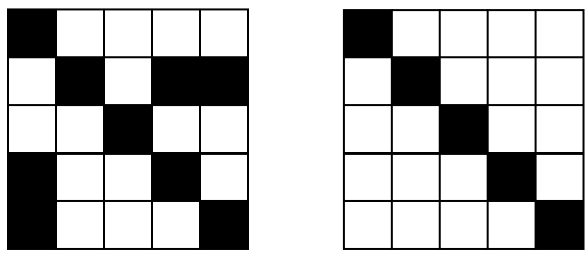


Figure 4.7: No Complete Path

On the left is the first stage of construction of a fractal square. Now, on the right we have removed all but the five squares on the diagonal. Looking at the figure on the right, one can easily see that no matter what stage of construction one chooses the diagonal will always exist. Hence, the diagonal will always exist in the figure on the left, and no complete path can be formed.

It can be shown that if a fractal square is isolated its k^{th} level of construction the $k+1^{st}$ level of construction will have a complete path. The above example cannot be equivalent to the completely isolated fractal square of the same dimension. This is due to the fact that the isolated square has no rectifiable paths, however, the fractal square formed by the example above will have a straight line connecting the top left corner to the bottom right corner.

Since the presence of rectifiable paths is preserved under the bilipschitz equivalence relation we know that these two fractal squares cannot be equivalent.

Also note, that in the above example the fractal square will not be represented by a finite graph. By the end of the next chapter will we see how to form the graph which represents a fractal square. The concept is similar to that of Cantor sets. A vertex will represent a collection of adjacent filled in blocks. However, in the above example, at the first level of construction we see we have 5 squares connected by corners. At the next level of construction we will have 25 filled in squares connected by bottom right corner to top left corner. At the k^{th} stage of construction we will have 5^k squares connected corner-to-corner. Therefore, we will have a vertex for each level of construction, representing the 5^k adjacent squares for each level of construction. Thus, the graph would be an infinite graph! The complete path will be the key to forcing a finite graph.

Chapter 5

Fractal Squares and Paths

In this chapter we will learn the tools in order to show how to form a graph that represents a fractal square. In the following proof we will need a definition of closed path. The idea of a closed path is the following: A **closed path** is a path of empty squares in an $n \times n$ block which creates two disjoint subsets of the $n \times n$ block where one subset, the "inside" subset, remains unchanged by the addition of more $n \times n$ blocks. Thus, there is a collection of filled in squares from which no new adjacencies can be formed since the collection has at a minimum distance of $\frac{1}{n}$ from any filled in square in the first row, first column, n^{th} row and n^{th} column.

Example 8 *A closed path is in gray:*

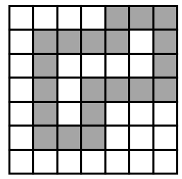


Figure 5.1: Closed Path

To form the precise definition of the "inside" and "outside" of a closed path many new definitions are needed.

Now we will discuss what it means for a fractal square to be broken into four quadrants. These quadrants will be a key aspect in showing bilipschitz equivalence of fractal squares. The detailed definition will be given later in this chapter. We will begin with a fractal square that has a complete path at level k, for some k. An example of one fractal square F is given below:

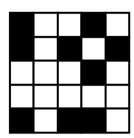


Figure 5.2: F

The complete path of F is shown in gray below:

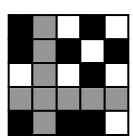


Figure 5.3: F with the complete path in gray

Definition 28 *A* **curve constructed from a path** *is the curve created by forming the line segment between the midpoints of a pair of edge-adjacent squares for each pair of edge-adjacent squares in the path.*

Now, extend the vertical path as follows:

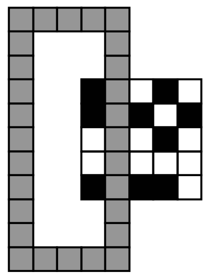


Figure 5.4: Vertical Extension

Now, in each of these extensions form a curve by connecting the midpoints of each of the squares with a line segment. Since the subset of the extended vertical path in F is non-repeating we know that linear path formed from the extended vertical path is a simple closed path.

Definition 29 *The* **left half** *of the fractal square consists of all squares such that the curve formed from the extended vertical path has winding number one with respect to the midpoint of the square. The squares such that the curve has winding number zero with respect the midpoints are those that consist of the* **right half** *of the fractal square.*

We will also need to keep the following in mind from [6]. When Hatcher discusses the Jordan curve theorem and Schoenflies theorem he mentions a few ideas that we need to keep

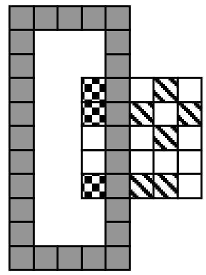


Figure 5.5: Left and Right Halves of F

in mind. He discusses both theorems in terms of $S^1 \to S^2$, however this is just the one-point compactification manifestation of a simple curve extended horizontally or vertically to infinity in $\mathbb{R}^2$. In particular, the half-spaces defined by such curves are homeomorphic to open disks, hence are simply-connected. Consider the top, left and bottom edge of a square with an extension to infinity that extends from both the upper right corner as well as the lower right corner of the square. Then we have that the definition of the left half is independent of the extension of the vertical portion outside the square, as long as the extension stays in the region above, below or to the left of the square. The independence also extends to the following definition of top half and bottom half.

To define the top and bottom half, extend the horizontal path as follows:

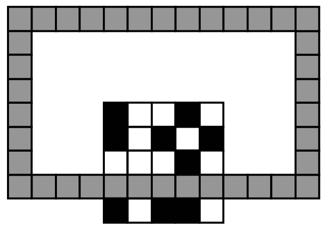


Figure 5.6: Horizontal Extension

Similarly to the vertical case, the non-repeating horizonal path forms a simple, closed linear path.

Definition 30 *The* **top half** *of the fractal square consists of all the squares such that the curve formed from the extended horizontal path has winding number one with respect to the midpoints of the squares and* **the bottom half** *consists of all the squares such that the curve has winding number zero with respect to the midpoints of the squares.*

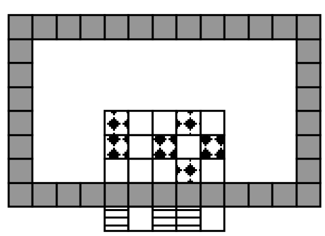


Figure 5.7: Top and Bottom Halves of F

Remark: In the above definitions we did not need the restrictive definition of complete path to form the vertical or horizontal extensions. We only needed a vertical non-repeating path and a horizontal non-repeating path.

Note, we could have formed different vertical extensions so that the right half had winding number one and the left half winding number zero. Also, we could have formed different horizontal extensions so that the top half had winding number zero and the bottom half had winding number one. All following results will be the same and can be shown in a similar manner.

Now, we will use a complete path to show that a fractal square with a complete path can be broken up into quadrants.

Definition 31 *Let F be a fractal square with a complete path at level k, for some k. The first quadrant of F consists of all the squares in both the top half of F and the right half of F. The second quadrant consists of all the squares in both the right half and the bottom half of F. The third quadrant consists of all the squares in both the bottom half and the left half of F. Lastly, the fourth quadrant consists of all the squares in both the left half and the top half of F. We call the collection of these four sets the* **quadrants** *of F.*

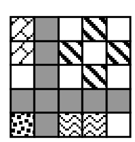


Figure 5.8: Quadrants of F

By lemma 2 we usually consider $k = 1$. The formation of quadrants may not be a unique one, however, when using a complete path the quadrants are, in fact, unique. Also, an empty set can be considered as a quadrant. Note that the union of all four quadrants contains every filled in square of F in the k^{th} stage of construction.

We will show that in F one cannot form a chain from one quadrant into another without using at least one square from the complete path. Then we will show that in the 2×2 block of copies of F the complete path forms a closed path, to be defined later, which holds each of the four quadrants in which one cannot form a path from a filled in square in that collection of quadrants to any other filled in square. This information will be very useful in showing the bilipschitz equivalence of two fractal squares.

Lemma 8 *An edge-concatenation determines a continuous, piecewise linear function from some interval* $[a, b]$ *to* $\mathbb{R}^2$ *with image a subset of the edge-concatenation.*

Proof:

The squares of an edge concatenation can be listed as a finite sequence $\{a_1, a_2, a_3, \ldots, a_n\}$ of squares such that a_i shares an edge with a_{i+1} for all $1 \leq 1 \leq n-1$. Let m_i be the midpoint of the square a_i for $1 \leq i \leq n$. Join m_i to m_{i+1} with a horizontal or vertical line segment, call it l_i. (The line segment is horizontal if the two squares share a vertical edge. The line segment is vertical if the two squares share a horizontal edge.) Note, that by definition l_i ends at m_{i+1} and l_{i+1} begins at m_{i+1}, for $1 \leq i \leq n-1$. Thus, the curve formed by the union of the segments $l_1, l_2, \ldots, l_{n-1}$ is a continuous, piecewise linear function with range extending from m_1 to m_2. Call this function ξ. I want to show that the image of ξ does not intersect any square $S \notin \{a_1, a_2, a_3, \ldots a_n\}$.

Let S be a square. Suppose there exists $x \in \xi \cap S$. Then there exists j with $1 \leq j \leq n$ such that $x \in l_j \subset a_{j-1} \cup a_j$. Recall that l_j is either a horizontal or vertical line segment. Without loss of generality, suppose l_j is a horizontal line segment. Now l_j lies in the interior of $a_{j-1} \cup a_j$. Thus, $x \in \xi$ lies in the interior of $a_{j-1} \cup a_j$. This gives that x lies in the interior of a_j, the interior of a_{j-1} or is in the midpoint of the shared vertical line If x is the midpoint of the vertical line connecting a_j with a_{j-1} then $S = a_j$ or $S = a_{j-1}$. Otherwise, x is in the interior or a_j or a_{j-1}. Without loss of generality, assume $x \in a_j$. This gives that $S = a_j$. Thus, ξ lies entirely in $\{a_1, a_2, \ldots a_n\}$. □

Lemma 9 *A concatenation determines a continuous, piecewise linear function from some interval* $[a, b]$ *to* $\mathbb{R}^2$ *with the image a subset of the concatenation.*

Proof:

The piecewise linear path is formed in a similar manner as above. The squares of a concatenation can be listed as a finite sequence $\{a_1, a_2, a_3, \ldots, a_n\}$ of squares such that a_i shares an edge or corner with a_{i+1} for all $1 \leq 1 \leq n-1$. Let m_i be the midpoint of the square a_i for $1 \leq i \leq n$. Join m_i to m_{i+1} with a diagonal, horizontal or vertical line segment, call it l_i. (The line segment is horizontal if the two squares share a vertical edge. The line segment is vertical if the two squares share a horizontal edge. The line segment is diagonal if the two squares share a corner.) Note, that by definition l_i ends at m_{i+1} and l_{i+1} begins at m_{i+1}, for $1 \leq i \leq n-1$. Thus, the curve formed by the union of the segments $l_1, l_2, \ldots, l_{n-1}$ is the image of a continuous, piecewise linear function with image extending from m_1 to m_n. Thus it only needs to be shown that the image of this function, call the function ρ, is a subset of the concatenation.

There are two cases to show that ρ is a subset of the concatenation, call it C. If a_i and a_{i+1} share an edge, then we are done by the previous lemma. If a_i and a_{i+1} share a corner then l_i has midpoint at the corner where the two squares intersect. Up until the midpoint l_i is a subset of a_i. After the midpoint l_i is a subset of a_{i+1}. Since the corner is both in a_i and a_{i+1} we have that $l_i \subseteq a_i \cup a_{i+1}$ and we have that the ρ lies entirely in the concatenation C, as desired. $\square$

Note, one may intersect the continuous, piecewise linear path of one chain with the continuous piecewise linear path of another chain without the two chains sharing a block. For example: two squares from one chain are drawn with their path, the other chain is not shown, however, the path formed from the chain is shown.

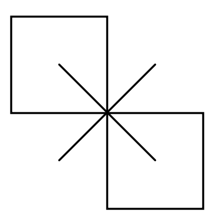


Figure 5.9: Path Intersection the Path Formed by a Chain

Lemma 10 *If a piecewise linear path ξ_1 formed by a chain, C, intersects a piecewise linear path ξ_2 formed by an edge concatenation, E, then C and E intersect in at least one block.*

Proof:

By lemma 8 we know that ξ_2 is a subset of the inside of E. If there exists $x \in \xi_1 \cap \xi_2$ then $x \in C$ and x is in the interior of E. Let S be the square of E that contains x. There are two cases, either x is in the interior of S or x is the midpoint of the shared edge connecting S to a neighboring square in the edge concatenation, call this square S_0. If x is in the interior of S then we know that C intersects a point in the interior of S and hence S is a square in the chain C. If x is the midpoint of the shared edge between S and S_0 in the edge concatenation then C contains either S or S_0. Thus, E and C must share a square. $\square$

Lemma 11 *Let F be a fractal square with a complete path at the k^{th} level of construction. If the horizontal path used in forming the complete path is not entirely on the top edge of F then there exists a top half of F.*

Proof: [5]

Choose S, a square not a part of the horizontal path, in the top row. Make a cut that runs horizontally to infinity and horizontally to negative infinity. Let α_1 be the function that represents the net change of the angle of the curve defined on the top half plane with respect to the horizontal line. Let α_2 be the function that represents the net change of the angle of the curve on the bottom half plane with respect to the horizontal line. Let $\alpha = \alpha_1 + \alpha_2$. Then the winding number can be calculated with respect to S and equals $\frac{\alpha}{2\pi} = 1$, as desired. Hence there is a top half of F. $\square$

Similarly, one can show the following.

Lemma 12 *Let F be a fractal square with a complete path at the k^{th} level of construction.*

If the horizontal path used in forming the complete path is not entirely on the bottom edge of F then there exists a bottom half of F.

Using the same technique as lemma 11, one can show that F has a left half and a right half.

Lemma 13 *Let F be a fractal square with a complete path at the k^{th} level of construction. If the vertical path used in forming the complete path is not entirely on the left edge of F then there exists a left half of F.*

Lemma 14 *Let F be a fractal square with a complete path at the k^{th} level of construction. If the vertical path used in forming the complete path is not entirely on the right edge of F then there exists a right half of F.*

Now, I wish to show that one cannot form a chain from the top half to the bottom half, or the left half and the right half, without using a square in the complete path.

Proposition 4 *Let F be fractal square with a complete path at the k^{th} level of construction for some k. If F has both a top half and a bottom half then there is no chain from the top half to the bottom half, or vice versa, without using a square from the horizontal path given by the complete path.*

Proof:

[5] Suppose there is chain from the top half of F to the bottom half of F. Form ρ, the continuous, piecewise linear curve associated with the chain. Then there is a midpoint of a

square in the range of ρ in the top half such that the curve constructed from the horizontal extension of the complete path has winding number one with respect to the point. There is also a midpoint of a square in the range of ρ in the bottom half which has winding number zero. Since winding numbers remain constant on connected components we have that ρ joins connected components, and hence, ρ crosses the continuous, piecewise linear path that is formed from horizontal portion of the complete path. Thus, the chain shares a square with the horizontal portion of the complete path. $\square$

Similarly, we can show that one cannot form a chain from the left half to the right half without using a square from the vertical subset of the complete path.

Proposition 5 *Let F be fractal square with a complete path at the k^{th} level of construction for some k. If F has both a right half and a left half then there is no chain from the left half to the right half, or vice versa, without using a square from the vertical path given by the complete path.*

Now, we can show that if we have a fractal square with a complete path, the fractal square can be broken into four disjoint quadrants!

Theorem 6 *Let F be a fractal square F with a complete path at some level of construction. Then one cannot form a chain from one quadrant into another quadrant without using a square from the complete path.*

Proof:

Let C be a chain which has at least one square from quadrant A and one square from quadrant B where $1 \leq A < B \leq 4$. If both A and B are both in the top half of F or both in the bottom half of F then by proposition 5 there is no chain from A to B without using a square from the complete path. Similarly, if A and B are in the right half or both in the left half we know that a square from the complete path must be used by proposition 4. If A and B are the top left and bottom right or the top right and the bottom left then we can use either proposition 4 or 5 to show that a square from the complete path must be a square in the chain. Thus, the quadrants are separated by the complete path. □

Now, we will define a closed path. Then we will show, using the above lemmas and propositions that a complete path forms a closed path. The closed path and the quadrants of F with be essential in showing two fractal squares, of a certain type, are bilipschitz equivalent.

Definition 32 *A* **closed path** *in an $n \times n$ block of a fractal square F is an edge-concatenation that satisfies the following properties: it is non-repeating except that the first square must equal the last square; and the piece-wise linear curve formed from the edge-concatenation is a closed curve in the usual sense.*

Note, that the piecewise linear path will be formed from empty squares $a_1, a_2, \ldots a_n = a_1$ such that the path begins and ends at the midpoint of a_1. Also, when referring to a winding number of a curve around a square, we actually mean the winding number of a curve around the midpoint of a square. Note, I will denote a closed path in $\mathbb{R}^2$, in the usual sense of the word, as a loop.

Definition 33 *Given an $n \times n$ block of a fractal square F with a closed path, the* **inside subset** *of the closed path is the collection of squares in the block such that the path has winding number 1 with respect to the each square.*

Definition 34 *Given an $n \times n$ block of a fractal square F with a closed path, the* **outside subset** *of the closed path is the collection of squares in the block such that the path has winding number 0 with respect to each square.*

Example 9 *A closed path is in gray:*

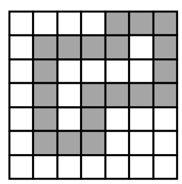


Figure 5.10: Closed Path

Theorem 7 *Let F be a fractal square with a complete path in its k^{th} stage of construction. Then there is a closed path in the 2×2 block of copies of F.*

Proof:

Form the 2×2 block of copies of F and extend the complete path as follows. Starting at the left edge of the lower right square the 2×2 block follow the horizontal path right until it meets the vertical path, then follow the vertical path up until it meets the horizontal path in the top right square, follow the horizontal path left until it meets the vertical path in the top left square, follow the vertical path down until it meets the horizontal path in the bottom left square and finally follow the horizontal path right until it meets the original square. By creation this path is an edge-concatenation. Note, that by design the curve formed from this closed path is, in fact, a closed curve. We matched squares in each column from the original two horizontal portions of the complete path to form the top and bottom of the closed path and analogues from the vertical portions of the complete path for the left and right parts of the closed path □

Now, we will show that in the 2×2 construction the closed path contains each of the four quadrants.

Corollary 1 *Let F be a fractal square with a complete path in its k^{th} stage of construction. Then there is a closed path in the 2×2 block of copies of F, as formed in theorem 7, that contains one copy of each of the four quadrants of F.*

Proof:

To see that this is a closed path such that its closed curve (i.e. join the midpoints of all the squares with line segments) has winding number one around every square (i.e. midpoint) in one copy of each of the four quadrants, it suffices to check that its part in the lower right square extended above and left (independent of extension as before) has winding number one around quadrant 4 in the lower right copy of F, and to check, by similar arguments, that the closed curve has winding number one around quadrant 1 in the lower left copy of F, quadrant 2 in the upper left copy of F and quadrant 3 in the upper right copy of F.

Assume the square (i.e. its midpoint) is in quadrant 4, more specifically in the top half and the left half of the lower right square. "The" curve, call it ρ, defining the top half winds around it once. This curve can be represented by the closed curve plus a curve that runs along the vertical in the square, runs through squares right of the vertical and runs through squares staying along the vertical path or to the right of the vertical path. By Schoenflies theorem this second curve, in relevant right half plane, is homotopically trivial, so its winding number about the given point in quadrant 4 is zero. Thus the closed curve plus the curve with winding number zero has winding number one. Hence the closed curve has winding number one for each square in our designated quadrant 4. Now, ρ has winding number zero with respect to the two squares in the bottom half of the lower right copy of F. Thus we have that the closed curve has winding number zero with respect to each of the midpoints of the squares in the bottom half of the lower right copy of F because ρ has winding number zero and the remaining portion of the curve is homotopically trivial. A similar argument can be made with "the" curve that defines the left half of the lower right copy of F with the added curve in lower left portion of the lower right copy of F. This argument will show that the closed curve has winding number zero with respect to the right half of the lower right copy of F. Thus, the closed curve has winding number one for only the midpoints of the squares in quadrant 4 of the lower right square. Similar results follow for each of the other three designated quadrants. □

Now, we have all the definitions needed to identify conditions that allow us to form a finite graph that represents a fractal square. Once we learn how to form a graph, then our plan will be to show that the same conditions allow us to form an associated fractal square, which we will call a separated fractal square. We will use quadrants to show that, under certain additional conditions, the original fractal square is equivalent to its associated separated fractal square. This result is a powerful tool in proving bilipschitz equivalence because we also show that separated fractal squares with the same initial data (i.e. same scale ratio and same number of filled in squares at level one) are equivalent.

Chapter 6

Fractal Squares and their Graph Representations

We are finally ready to learn how to form a graph to represent a fractal square. First, we are going to give conditions that implies that a fractal square does not have an infinite chain, and hence has a finite graph. We will then learn how to form the graph.

Theorem 8 *If for some $n \in \mathbb{N}$ there is a complete path in the $n \times n$ block, then no infinite chain exists.*

Proof:

Case 1 Consider the case where there is a complete path in the 1×1 block.

Choose a filled in square in an arbitrary set of adjacent squares in an arbitrary level in an arbitrary array of blocks. The chosen square will be in one "quadrant", defined by the given complete path, of the 1×1 block that contains it. Fill in around the 1×1 block and form a 2×2 block. Now, the chosen filled in square will be trapped by a closed path. This closed path will guarantee a bound on the number of squares in a chain because all possible connections of copies of the fractal square will be a subset of the inside of the closed path. By lemma 4, we know that this is the bound on the number of squares in a chain in any scaled 2×2 block. Note that this bound persists at every level of construction because every possible chain will occur in some scaled 2×2.

Case 2 Let there be such an $n \in \mathbb{N}$ so that the $n \times n$ block has a complete path.

Let $n \in \mathbb{N}$ be such that the $n \times n$ block has a complete path.

Choose a filled in square in a set of adjacent squares at an arbitrary level in some array of 1×1 blocks. The relevant $n \times n$ block is not obvious, i.e. our filled in square is a subset of a 1×1 block. Now, this block could sit in the upper left corner of an $n \times n$ block, in the middle, or at the bottom right or anywhere else. To determine the necessary positioning in the $n \times n$ block one must look at all n^2 possibilities.

Consider all n^2 possibilities separately. In each case the filled in square in question lies in a quadrant of the $n \times n$ block's complete path. Now, fill in $n \times n$ blocks to form a $2n \times 2n$ block as in the 1×1 case. By Lemma 4 we know that this is also a bound on our original array of blocks. As in case one, note that this bound also persists are every level of construction.

In every case, the filled in square is inside the closed path that is formed by the $n \times n$ complete path. As in case one, this gives a bound on the total number squares in a chain. Let this bound be denoted by M.

In an infinite chain there is an arbitrarily large number of adjacencies. We will show later on in lemma 15 that we can find $M + 1$ adjacencies in any infinite chain. Since we have a maximum of M squares in a chain, we cannot have an infinite chain.

Thus, a complete path in any $n \times n$ block will guarantee that no infinite chain exists. □

As in example 7 the complete path is in the 1×1 block. Thus, a 2×2 block would be formed to find the maximal number of adjacencies. In the picture below the 2×2 block is shown twice. On the left the complete path is in gray and on the right the closed path is in gray. This shows that the maximal number of squares in a chain is 8.

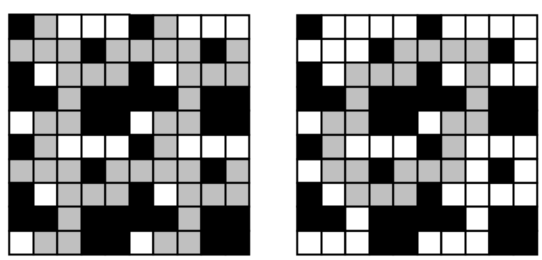


Figure 6.1: 2×2 Fractal Square with Complete Path and Closed Path

Now, in the above proof the complete path existed in the first stage of the construction of a fractal square. However, a complete path may not be found in the first stage of construction, but rather the second stage, third stage, fourth stage, etc. In this case, the fractal square will not have an infinite chain.

Example 10

In this example, in stage 1 of construction it looks like there is going to be an infinite chain.

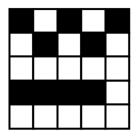


Figure 6.2: First Stage Fractal Square with No Complete Path

The path of 4 filled in blocks towards the bottom is no problem, we can get around those using the blank square to the right. However, the chain at the top of the square is an issue. There is no way to break through this chain with a blank path. We cannot go from the top of this square to the bottom with a path.

However, in the second stage of construction of the 1×1 block, it is clear that a complete path exists:

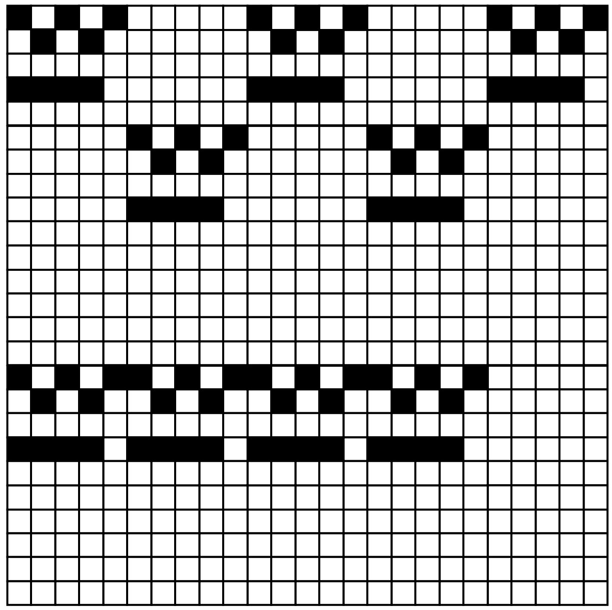


Figure 6.3: Second Stage Fractal Square with Complete Path

The complete path is in gray:

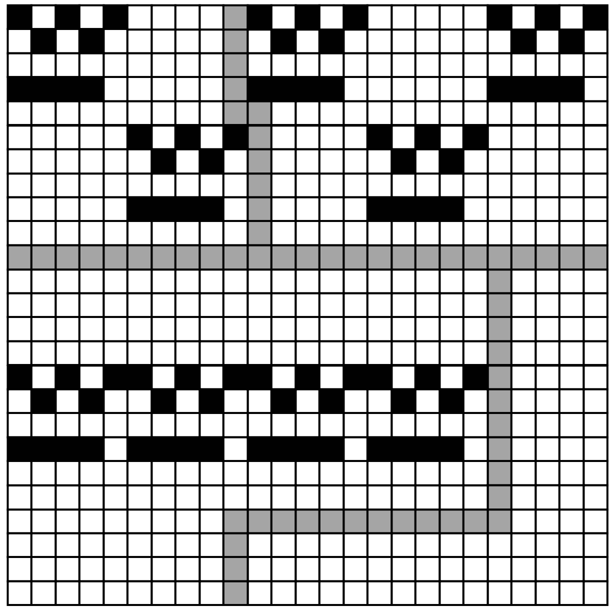


Figure 6.4: Complete Path of Second Stage Fractal Square

Theorem 9 *If a complete path exists in some $n \times n$ block of 1×1 blocks, each in the k^{th} stage of construction for some k, then no infinite chain exists.*

Proof:

Combining Lemma 2 with Theorem 8 gives the desired result □

Definition 35 *Let F be a fractal square with a complete path at level k. Let C be a subchain of the interior of the closed path formed in the 2×2 block of copies of F at level k. Form the collection, call it B, of $[0,1] \times [0,1]$ blocks formed using the same edge and corner connections as are used in C. A* **shape** *of F is the collection of copies of F formed by replacing the $[0,1] \times [0,1]$ blocks of B with copies of F.*

Note, in arguments that involve shapes we often need to replace the copies of F with copies of F at a particular level of construction. By lemma 2 we can consider the first level of construction. However a shape is a collection of copies of the fractal square, not a collection of copies of the fractal square at a particular level of construction. These shapes are critical in the formation of the graph that represents F since the vertices of the graph will represents all the different shapes of F.

Now, we showed that a complete path gives a bound on the maximum length of a chain in a fractal square. This is not the only benefit of a complete path. It guarantees that there are indeed a finite number of vertices for the graph representing the fractal square.

Lemma 15 *Let F be a fractal square with a complete path at some level of construction. Then, the graph representing F will have a finite number of vertices.*

Proof:

Let F be a fractal square. By Lemma 2 let's assume that F has a complete path in its first stage of construction. Then we know the 2×2 block has a closed path. Inside of this closed path lies a subset that represents every possible connection of F via corner to corner or edge to edge. For example, two copies of F connected by bottom left corner to top right corner is represented, as well as two copies connected top to bottom. This gives every shape is a subset of the filled in blocks inside the closed path. Since every vertex is represented by a shape, we know that each of these blocks is a subset of the filled in blocks inside of the closed path. Since there are only a finite number of filled in blocks inside the closed path there is only a finite number of shapes and hence a finite number of vertices for the graph representing F. □

Now, we can discuss how to form a finite directed graph to represent a fractal square F with a complete path.

Theorem 10 *Assume that F is a fractal square with a complete path at some stage $k \in \mathbb{N}$. Then there exists a finite directed graph G^* that represents the fractal F.*

Proof:

Let F be a fractal square with a complete path at some stage $k \in \mathbb{N}$. Without loss of generality we can assume that $k = 1$ by lemma 2. Since F has a complete path, we know that there are a finite number of shapes. Look at the first stage construction of F. There are a finite number of shapes formed by adjacent filled in blocks. Each of their corresponding shapes will be represented by a vertex in our graph. For vertex v_i let us call its respective shape F_i. For each F_i choose a point in the shape to represent the origin.

Now that we have some of our vertices we can begin filling in some directed edges. There will be a directed edge connecting each vertex v, representing F, to the vertex v_i representing F_i if the chain corresponding to the shape F_i is a chain in the first level of construction of F. Note that the arrow should be tending toward the vertex v_i, not towards the vertex v representing F.

Form each shape F_i. For the purposes of this proof we will consider the copies of F that form F_i to be in the first level of construction. Again, a certain number of chains will appear. Form the shape associated with each new chain. To each new shape F_j form a new vertex v_j. For each of these new shapes F_j that appears, draw a directed edge connecting the vertex v_i with the vertex v_j representing that shape and with the arrow facing towards v_j. Note that when a shape created by a chain is repeated we need not form a new vertex. Continue this process. Since there are a finite number of chains we know that this process will end in a finite number of steps. We now have our graph, all that is left is to form the similitudes that are paired with each directed edge.

For each distinct shape represented by a vertex we have designated a point to represent the origin. Choose any edge e. Suppose that e connects vertex v_i to v_j, i.e. the arrow is pointed toward v_j. Now we know that a scaled version of F_j appears in F_i. The map will be $c(x, y) + (a, b)$ where (a, b) value needed to add from the origin of F_i to the origin of the scaled version of F_j and c is the value of which F_j is scaled in F_i. Now we have the directed graph G^* for the fractal F with underlying graph G. $\square$

This finite directed graph will form unique decimal representations for each point in the fractal square. By definition of shape, we know each shape that represents a vertex is disjoint from every other shape. If not, the two would have been connected by either an edge, corner or square. If this were the case the two shapes would have formed one single shape. Now, choose a point $p = (x, y)$ in the fractal square F. At level one p is in exactly one shape, F_{i_1}. In level 2 the point p will be in exactly one shape, F_{i_2}. Continue this process so in level k, the point p is in exactly one shape F_{i_k}. This gives that p will have a unique decimal representation. The graph was formed to give us exactly that!

Corollary 2 *Assume F is a fractal square with a complete path at some level of construction. Assume that all shapes in F and all shapes inside the closed path arising from the complete path are rectangular. The shape-forming process in theorem 10 results in all shapes at all levels being rectangular.*

Remark: Recall, when we say all shapes inside the closed path, we mean that the closed curve arising from the closed path has winding number one with respect to the midpoint of all filled in squares in that shape.

Proof:

Without loss of generality, assume that F has a complete path at the first level of construction by lemma 2. All shapes in F are rectangular and all shapes inside the closed path are rectangular. Also, recall that a shape formed at any level of construction is a subset of the closed path at that level in the 2×2 block. Thus, any shape is a subset of an appropriately scaled 2×2 block.

Choose a shape in the first level of construction of F. Look at the associated 2×2 block. To form this shape we use a horizontal slice, vertical slice, neither or both of the 2×2. Since all shapes inside the closed path of F are rectangular, any vertical slice, horizontal slice, or a vertical and horizontal slice, will give a rectangular shape. Thus, the shapes formed using the first level of construction are all rectangular.

Assume that all shapes in the $k-1^{st}$ level of construction are rectangular as well as all subsets of the closed path. Look at one of the shapes, F_s formed in the k^{th} level of construction of F. Form the 2×2 block of copies of F in the $k-1^{st}$ level of construction. All shapes that are a subset of F_s are subsets of this 2×2 block. Since F_s is rectangular, all subshapes are formed by slicing vertically, horizontally, both or neither. Since each of those actions leads to rectangular shapes, we have that all shapes in F_s are rectangular. By induction on the level of F we have that all shapes are rectangular. □

Now that we know when fractal squares can be represented by a graph. Next we will look at different ways of viewing our fractal squares by breaking them up into four quadrants. This representation will help us prove bilipschitz equivalence of fractal squares.

Chapter 7

Fractal Squares and Cantor Sets

Now, we will expand the technique we used to show Cantor sets are equivalent to Cantor sets to show that Cantor sets and fractal squares can, in fact, be equivalent.

Theorem 11 *Let C be a Cantor set of dimension $\frac{\ln(a)}{\ln(b)}$ with a subintervals with scale ratio $\frac{1}{b}$. Let F a fractal square of the same dimension with a filled in squares with scale ratio $\frac{1}{b}$. Then $C \equiv F$.*

Proof:

As in Theorem 5 we will create a Cantor set S and show that $C \equiv S$ and $F \equiv S$. As before let S be a Cantor set of dimension $\frac{ln(a)}{ln(b)}$. (Note that $\frac{ln(a)}{ln(b)} < 1$.) In the first stage of construction subdivide the unit interval into b equal length subintervals and keep the first a. By looking at the second stage one will see that we actually have separated subintervals.

Again, following the technique given by Theorem 4 construct a graph for C and show that this graph can be imposed on S.

Now, we need to construct a graph for F and show that this graph can be imposed on S. Since $\frac{ln(a)}{ln(b)} < 1$ we know that $a < \frac{1}{b}$. This gives that the number of filled in squares is strictly less than the number of squares in any row or column. Thus we have a complete path in the first stage of construction of F. Now, using the techniques given in Theorem 10 we can construct a graph, G, associated with F. Now, we wish to impose this graph on S. Now, each vertex of G represents a collection of adjacent squares. For each vertex v_i let k_i be the number of adjacent squares represented by v_i. Let S_i be k_i adjacent subintervals. Letting S_i be the vertices of G we can see that S does impose on the graph associated with F. Also note that all other hypotheses of Theorem 3 are satisfied in both cases. This gives us, by transitivity of equivalence relations that $F \equiv C$ as desired. □

Example 11 *Let I be the Cantor set and S be the fractal square with first stage of construction below. Then $I \equiv S$.*

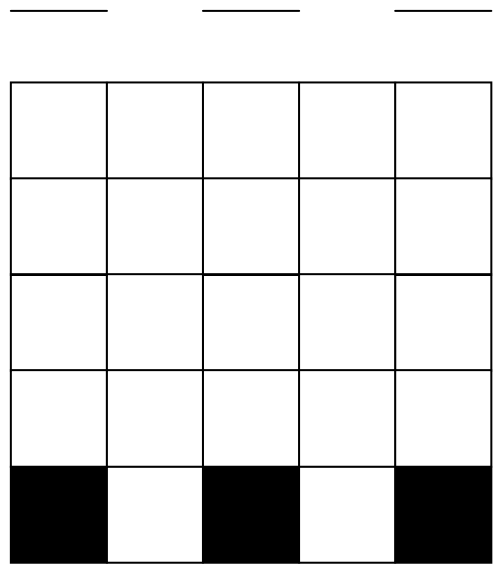


Figure 7.1: Cantor set and Fractal Square

Both S and I have dimension $\frac{\ln(3)}{\ln(5)}$ and has graph G with one vertex and three edges.

For S the edges have similitudes: $S_1(x) = \frac{1}{5}x, S_2(x) = \frac{1}{5}x + (\frac{2}{5}, 0)$ and $S_3(x) = \frac{1}{5}x + (\frac{4}{5}, 0)$.

For I the edges have similitudes: $I_1(x) = \frac{1}{5}x, I_2(x) = \frac{1}{5}x + \frac{2}{5}$ and $I_3(x) = \frac{1}{5}x + \frac{4}{5}$.

Now $\{S_j\}_1^3$ and $\{I_j\}_1^3$ are contractive similitudes. Each S_j and I_j have similitude ratio $\frac{1}{5}$. There is only one invariant set S for $\{S_j\}_1^3$ and similarly just I for $\{I_j\}_1^3$. S and I are both dust-like since, clearly, $S = \cup_1^3 S_j(S)$ and $I = \cup_1^3 I_j(I)$ are disjoint unions.

Since S is dust-like for every $x \in S$ there is a unique coding $e_1e_2e_3\ldots$ such that $\{x\} = \cap_1^\infty S_{e_1e_2\ldots e_k}(S)$. Let $f : S \to I$ be defined as $\{f(x)\} = \cap_1^\infty I_{e_1e_2\ldots e_k}(I)$

Is this a bijection? By the uniqueness of the codings f is one-to-one. Now if $y \in I$ has the coding $\cap_1^\infty I_{e_1 e_2 \dots e_k}(I)$ then $x = \cap_1^\infty S_{e_1 e_2 \dots e_k}(S)$ maps to y. So f is indeed a bijection.

Is f bilipschitz?

Let $x, x' \in S$ have codings $e_1 e_2 \dots$ and $e'_1 e'_2 \dots$ respectively. Set m to be the largest integer such that $e_1 e_2 \dots e_m = e'_1 e'_2 \dots e'_m$. This gives that $x, x' \in S_{e_1 e_2 \dots e_m}(S)$.

So we know $|x - x'| \leq diam(S_{e_1 e_2 \dots e_m}(S)) = \frac{1}{5^m} diam(S)$.

By maximality of m,

$|x - x'| \geq mind(S_{e_1 e_2 \dots e_{m+1}}(S), S_{e'_1 e'_2 \dots e'_{m+1}}(S)) = \frac{1}{5^m} diam(S)$

Thus, $diam(S)\frac{1}{5^m} \leq |x - x'| \leq \frac{1}{5^m} diam S$.

This gives that there exists c_1 so that $c_1^{-1} \frac{1}{5^{m+1}} \leq |x - x'| \leq \frac{1}{5^m} c_1$

Similarly, there exists c_2 such that $c_2^{-1} \frac{1}{5^{m+1}} \leq |f(x) - f(x')| \leq \frac{1}{5^m} c_2$

This gives $c_1^{-1} c_2^{-1} |x - x'| \leq |f(x) - f(x')| \leq c_1 c_2 |x - x'|$. Hence f is bilipschitz and $S \equiv I$.

Theorem 12 *Let F be either a fractal square or a Cantor set. Let M be either a fractal square or a Cantor set. Let F have scaling factor b and M scaling factor b^n. If M and F admit the same graph then $M \equiv F$.*

Proof:

Let $dim(F) = \frac{\ln(a)}{\ln(b)}$ and $dim(M) = \frac{\ln(c)}{\ln(b^n)}$. Suppose that $c \neq a^n$ then $dim(F) \neq \dim(M)$ and the two graphs would not be equivalent. So we know that $c = a^n$. Now, let S be a separated

Cantor set or fractal square with the same dimension as F and M. If S is a Cantor set we already know that G imposes on S by proof of Theorem 4. If S is a fractal square we will show in chapter 8 that G does impose on S. Thus we know that $M \equiv S \equiv F$ and we have our result. $\square$

Chapter 8

Bilipschitz Equivalence of Fractal Squares

Now, let's return to studying only fractal squares. Let's look at the following example:

Example 12

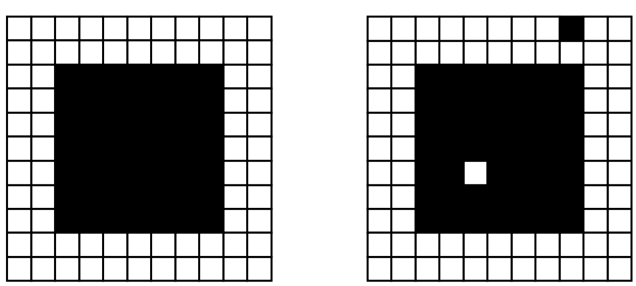


Figure 8.1: F and M

Examine at the following first stages constructions of fractal square F and fractal square M shown above, respectively. It turns out that each fractal square has a directed graph with one vertex and forty-nine loops. The graph for each fractal square is formed as follows.

Both F and M have no new adjacencies in the 2×2 construction. So, if you look at both F and M at the next level of construction each of those filled in squares are actually disjoint, not adjacent as they originally appeared to be! The subsets of the fractal square contained in these filled in squares are, in fact, disjoint closed sets. Recall from chapter 4 that vertices represent the different shapes that appear in a fractal square. Since we only have one shape we only need one vertex representing a single square.

There are forty-nine loops which come from the forty-nine filled in squares in the stage one construction per fractal square. Clearly, each similitude has ratio $\frac{1}{11}$ for each fractal square. It is also clear that we are dealing with dust-like sets since our union of the forty-nine sets is disjoint. Thus, we have everything needed to implement Theorem 3. Therefore $F \equiv M$, as desired. $\square$

These fractal squares have one important property in common. These fractal squares are both what will be called separated. Separated fractal squares play an important role in showing equivalences.

Definition 36 *A fractal square is called* **separated** *if there is a level at which every chain in the* 2×2 *block is contained in one and only one of the four* 1×1 *blocks used to form the* 2×2 *block.*

There are many properties that are associated with separated fractal squares. Two filled in squares at any stage of construction of a separated fractal square represent two disjoint scaled copies of the separated fractal square. More specifically, each filled in square of the separated fractal square can be viewed as a scaled copy of the stage one construction of the fractal square. Supposed two filled in squares are adjacent. By definition, any chain in one of these two filled in squares does not intersect with the other filled in square. The distance from any filled in square at the next stage of construction is at least $\frac{1}{n^2}$ to another filled in square in that next stage of construction. This can be similarly discussed for any stage of construction of the fractal square. Thus, even if it looks like there are two adjacent filled in squares in a separated fractal square, the subsets of the fractal square they represent are in fact disjoint. In all actuality, in every stage, each filled in block has no adjacencies. We can consider the squares as adjacent or disjoint.

Example 13

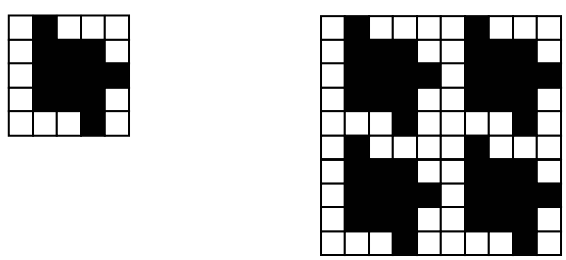


Figure 8.2: Separated Fractal Square and its 2x2 block

A separated fractal square is given as well as its 2×2 block. Now, you can see that each of the four copies of S_F in the 2×2 block are disjoint from each other. We can consider this block of 4 copies of S_F either as four disjoint copies, two horizontal rectangles, two vertical rectangles, and "L" shape and a single copy of S_F or as a single larger square. Thus, we can consider this collection of copies of a separated fractal square in many different ways. The same is true for any collection of copies of any separated fractal square. This is going to be key in showing the equivalence of two general fractal squares.

Theorem 13 *Let F be a separated fractal square. Let F have k filled in blocks in the possible n^2 in the first stage of construction, i.e. F has dimension $\frac{\ln(k)}{\ln(n)}$. Then F can be represented by a graph with a single vertex and k edges.*

Proof:

Let v be a vertex representing F, i.e. the shape consisting of a single square. Recall, each filled in square of F represents a scaled copy of F. These scaled copies, by definition of separated fractal square, are disjoint. Hence, we can map a $\frac{1}{n}$ scaled version of F to each of these filled in squares. This gives the k edges of the graph and each of the k similitudes corresponding to each of the edges with similitude ratios $\frac{1}{n}$. More specifically, the similitudes are $\frac{1}{n}x + (a, b)$ where (a, b) is the location of the origin of the filled in square. Thus F can be represented by a single vertex with k edges. □

Example 14 *Let S and T, respectively, be fractal squares with first stage depicted below. Note that these are separated fractal squares.*

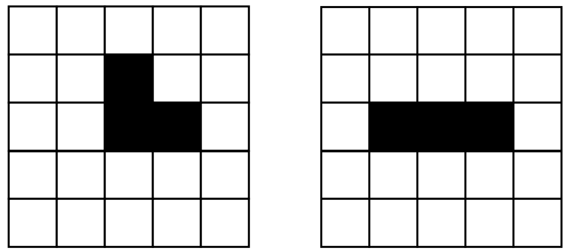


Figure 8.3: S and T

Now, following the proof given in [2] we will show that these two sets are equivalent.

These two fractal squares are of the same dimension $\frac{\ln(3)}{\ln(5)}$ and each clearly has a graph with one vertex and three edges.

For S the edges have similitudes: $S_1(x) = \frac{1}{5}x + (\frac{2}{5}, \frac{2}{5}), S_2(x) = \frac{1}{5}x + (\frac{3}{5}, \frac{2}{5})$ and $S_3(x) = \frac{1}{5}x + (\frac{2}{5}, \frac{3}{5})$.

For T the edges have similitudes: $T_1(x) = \frac{1}{5}x + (\frac{1}{5}, \frac{2}{5}), T_2(x) = \frac{1}{5}x + (\frac{2}{5}, \frac{2}{5})$ and $T_3(x) = \frac{1}{5}x + (\frac{3}{5}, \frac{2}{5})$.

Clearly, we have that $\{S_j\}_1^3$ and $\{T_j\}_1^3$ are contractive similitudes. Each S_j and T_j have the same similitude ratio $\frac{1}{5}$. Each has only one invariant set, namely S for $\{S_j\}_1^3$ and T for $\{T_j\}_1^3$. Note that S and T are both dust-like since $S = \cup_1^3 S_j(S)$ and $T = \cup_1^3 T_j(T)$ are disjoint unions.

Since S is dust-like we have for every $x \in S$ there is a unique coding $e_1e_2e_3\ldots$ where $\{x\} = \cap_1^\infty S_{e_1e_2\ldots e_k}(S)$. Let $f : S \to T$ be defined as $\{f(x)\} = \cap_1^\infty T_{e_1e_2\ldots e_k}(T)$

Since the codings are unique we know that f is one-to-one. Let $t \in T$ has the coding $\cap_1^\infty T_{e_1 e_2 \ldots e_k}(T)$ then $x = \cap_1^\infty S_{e_1 e_2 \ldots e_k}(S)$ maps to t. So f is onto and hence a bijection.

Lastly, we need to know if f is bilipschitz. Let $x, x' \in S$ have codings $e_1 e_2 \ldots$ and $e_1' e_2' \ldots$ respectively. Let m be the largest integer such that $e_1 e_2 \ldots e_m = e_1' e_2' \ldots e_m'$. Thus $x, x' \in S_{e_1 e_2 \ldots e_m}(S)$.

This tells us that $|x - x'| \leq diam(S_{e_1 e_2 \ldots e_m}(S)) = \frac{1}{5^m} diam(S)$.

Now, the diameter of S is at most $\frac{\sqrt{8}}{5}$, which is the distance from $(\frac{2}{5}, \frac{4}{5})$ to $(\frac{4}{5} . \frac{2}{5})$. So we actually have that $|x - x'| \leq diam(S_{e_1 e_2 \ldots e_m}(S)) = \frac{1}{5^m} diam(S) \leq \frac{2\sqrt{2}}{5^{m+1}}$

Since m is maximal we know that

$|x - x'| \geq mind(S_{e_1 e_2 \ldots e_{m+1}}(S), S_{e_1' e_2' \ldots e_{m+1}'}(S)) = \frac{1}{5^m} \frac{2}{25} = \frac{2}{5^{m+2}}$

Therefore, $\frac{2}{5^{m+2}} \leq |x - x'| \leq \frac{2\sqrt{2}}{5^{m+1}}$.

Similarly, we can show that $\frac{3}{5^{m+2}} \leq |f(x) - f(x')| \leq \frac{3}{5^{m+1}}$.

This gives that $\frac{1}{5}|x - x'| \leq |f(x) - f(x')| \leq 5|x - x'|$. Hence f is bilipschitz and $S \equiv T$.

Now, we will expand this idea to show that separated fractal squares with certain properties are equivalent.

Definition 37 *Let F be a fractal square with dimension $\frac{\ln(k)}{\ln(n)}$. Let F have a complete path at the m^{th} stage of construction. In the 2×2 block of the m^{th} stage of construction the complete path forms a closed path. Suppose the chains inside the closed path can be translated, so that no changes in shape or orientation occur, to a subset of $[0, 1] \times [0, 1]$ without breaking any* *adjacencies. This forms the first stage of construction for a new fractal square. If this new fractal square is separated, we call it the* **separated fractal square** *associated with F. We denote this separated fractal square by S_F.*

There are a couple of items to note. First, a separated fractal square arises from a process that forms a fractal square with the separated property at some level of construction. Second, there can be more than one separated fractal square associated with a particular fractal square, call it F. However, each separated fractal square associated with F can be represented by the same graph, a graph with a single vertex, where each edge vertex has the same similitude ratio by lemma 13. We also have that each separated fractal square is dust-like. This means that all the separated fractal squares associated with F are bilipschitz equivalent. This means we can make a choice of a particular separated fractal square for F and call it the separated fractal square associated with F.

We will show that if a fractal square has a complete path at some level of construction then it has an associated separated fractal square.

Theorem 14 *A fractal square with a complete path at some level of construction has an associated separated fractal square.*

Remark: The separated property reveals itself at the level following the level on which the complete path first exists.

Proof:

Assume F is a fractal square with n rows and n columns. By lemma 2 assume that F has a complete path at the first level of construction. Form the second level of construction of F. Now, in the right-most column of F, which consists of scaled copies of F or blank squares, a vertical portion of a complete path can be formed using each of the vertical portions of the complete path from the n squares that form the column. (Note, if a square is blank one can just use a vertical path.) Similarly form the horizontal portion of a complete path in the bottom row. Thus, we have formed a complete path in F at the second level of construction.

Form the 2×2 block using the second stage of construction. Consider the complete path from the second level of construction using the right-most column and bottom row as formed above. The complete path forms a closed path in the 2×2 block. All possible shapes of F's second level of construction appear as a subset of this closed path. This closed path lies in a square with $n(n+1)$ rows and $n(n+1)$ columns, i.e. $n+1$ first level rows and $n+1$ first level columns. (This is a copy of F in its second level of construction with an extra n rows on top and an extra n columns on the left.) We will consider this from now on as $(n+1)n$ rows and $(n+1)n$ columns since we are looking at the second stage of construction and for the rest of the proof we will only be considering this square.

Look at the collection squares in the $n(n+1)$ rows and $n(n+1)$ columns such that the closed curve formed from the closed path has winding number zero with respect to the midpoints of the squares. These squares are "outside" the closed path. Since we are trying to form a separated fractal square using only the shapes that are "inside" the closed path we can consider the "outside" squares as blank squares. Note that, by forcing the "outside" squares to be blank, we are guaranteed that in the first row and leftmost column are, in fact, blank. They were either blank to begin with or they were filled in and, as discussed, we consider them as blank squares. We will need this later in the proof.

Consider the collection of filled in squares that are "inside" the closed path that lies in the $(n+1)n$ rows and $(n+1)n$ columns. Specifically, the collection of filled in squares such that the closed curve formed from the closed path has winding number one with respect to the midpoints of the filled in squares. Call this collection C. We will show that the shapes in C to the right and/or below the stage one complete path can be translated left n columns and/or up n rows so that all the shapes inside C are now in an $n^2 \times n^2$ block and no adjacencies will be broken. This will be the needed S_F since we will not translate any shapes into the top row or leftmost column and as discussed in the previous paragraph, the top row and the leftmost column are blank.

To form S_F we will first consider the level one complete path in the lower right square of the level one 2×2 block of F in the second level of construction. This is an $n(n+1) \times n(n+1)$ block. The path is extended horizontally to include one level one square in the left column and vertically to include one level one square in the top row. Thus, returning to consider everything in the second level of construction, the vertical portion has a width of n blocks and the horizontal portion has a height of n blocks. Note that this is a complete path in the $n(n+1) \times n(n+1)$ block by the definition of complete path since the path both enters and exits in the same row and same column. Now, the complete path from the first stage of construction breaks the $n(n+1) \times n(n+1)$ block into four quadrants. As usual, label these as $1, 2, 3, 4$ starting in the upper right and continuing in a clockwise manner. We will shift each filled in square in quadrant 1 left n steps. We will shift each filled in square in quadrant 3 up n steps. Lastly, we will shift each filled in square in quadrant 2 left n steps and up n steps. These are clearly translations. We need to show that no two filled in squares intersect in more than a line. (More specifically, two squares may either be disjoint, share a corner or share an edge.) If so, then we will have a new square that can be show to be a separated fractal square associated with F.

To show that the only non-empty intersections that occur are edge or corner intersections, we will look at all of the possible overlapping squares. Now, first look to see if any filled in squares from quadrant 1 overlap with a filled in square from quadrant 4. Now, we shifted each filled in square n squares to the left. Now, since there are n adjacent empty squares from the level one complete path separating quadrant 4 from quadrant 1, we know that at most two squares from the stationary quadrant 4 and the shifted quadrant 1 share an edge. Similarly, two squares from shifted quadrant 3 and stationary quadrant 4 will at most share an edge since there are n empty adjacent squares from the horizontal portion of the level one complete path. Now, if two squares, one from shifted quadrant 1 and one from shifted quadrant 3, are to intersect at more than an edge or a point, then one square from quadrant 3 and one square from quadrant 1 must have had a corner-to-corner adjacency. However, since no chain can cross a complete path, by propositions 4 and 5, we know this is not possible. Now, if one square from quadrant 2 and one square from quadrant 4 were to intersect at more than a line or a point then those two squares must have had at least a corner-to-corner adjacency. Since we have a complete that we know that no such adjacency cannot occur. Two squares from quadrant 2 and 1 cannot intersect since both started with a separation greater than or equal to n from the level one complete path and both were moved left by n columns and thus retain that no filled in squares overlap. Similarly, filled in squares from quadrant 2 and 3 cannot overlap. Once these translations have occurred, the first row and the leftmost column remain empty. We did not shift any squares in quadrants 1 and 4 up, nor did we shift any squares in quadrants 4 and 3 to the left. This gives that the top row and leftmost column remain blank. Now, shifting squares in quadrants 2 and 3 up gives that the bottom n rows are blank. This is from shifting all shapes up by n rows. Now, shifting shapes in quadrant 1 and 2 to the left gives the the rightmost n columns are blank. This is due to the fact that we shifted all shapes to the left n columns.

Thus, all filled in squares fit inside an $n \times n$ block with a blank top row and a blank leftmost column. Call this new fractal square S_F This is a separated fractal square that is formed with all the shapes from C without breaking any of the adjacencies from C. Clearly, all shapes of the original C are each subshapes of S_F such that their union forms all the shapes of C and none of the shapes overlap and no squares are left over. Thus, S_F is a separated fractal square associated with the second level of construction of F. $\square$

Example 15

Let F be the following fractal square.

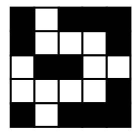


Figure 8.4: F

We will use the above theorem to find an associated separated fractal square.

Here is the second level of construction:

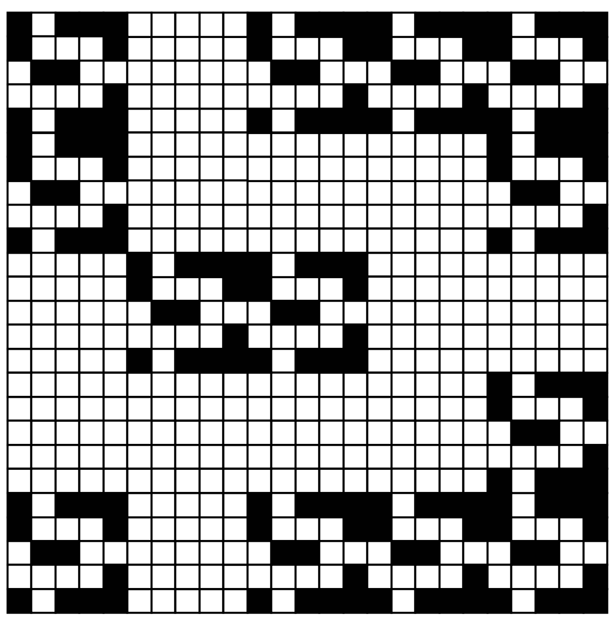


Figure 8.5: Stage 2 Construction of F

Here is the 2×2 construction of the second level with complete path:

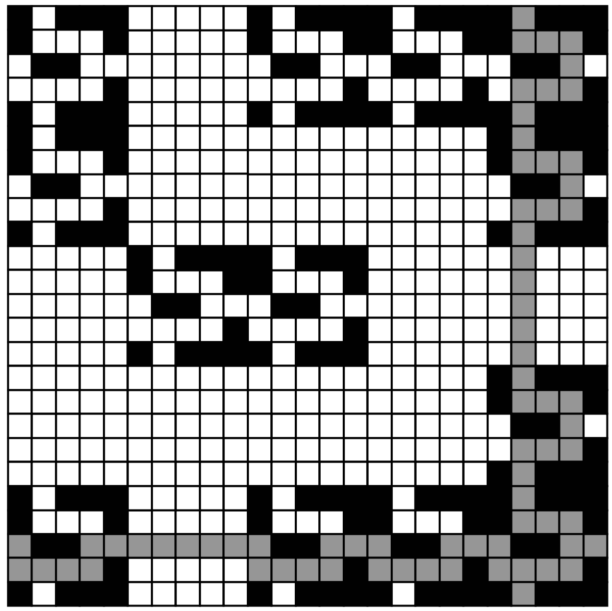


Figure 8.6: Complete Path

Below is the 6×6 block that contains the closed path.

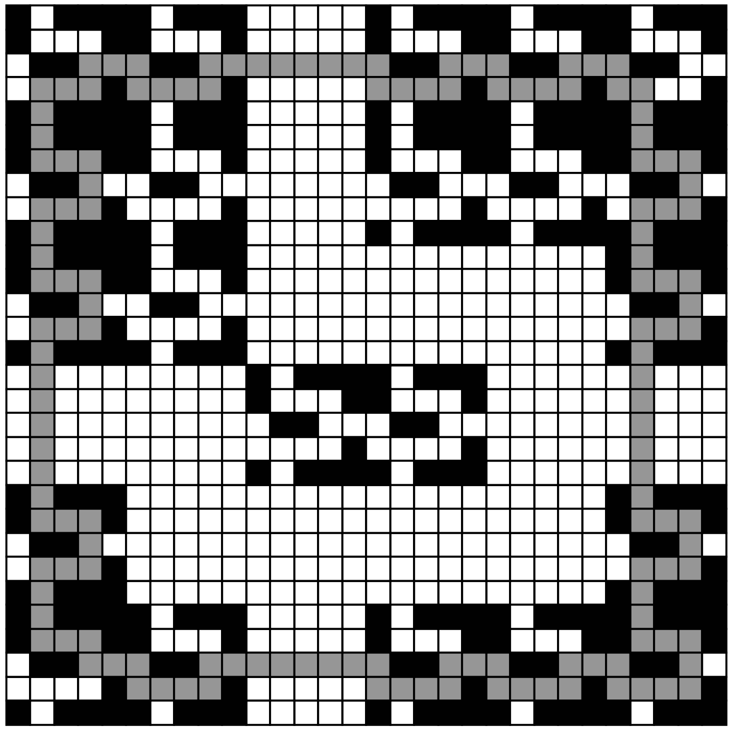
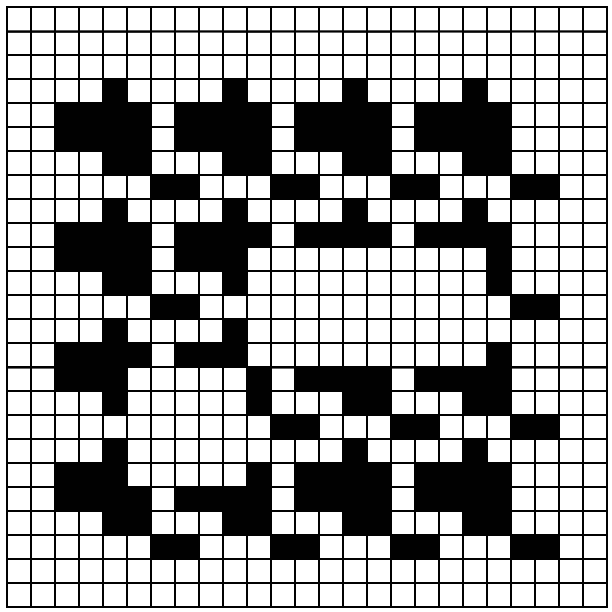


Figure 8.7: 6x6 Containing Closed Path

Now, shift all squares in quadrant 1 left 5 columns. Shift a squares in quadrant 3 up 5 rows and shift all squares in quadrant 4 to the left 5 columns and up 5 rows. The following fractal square is formed.



This is an associated separated fractal square S_F associated with F. Thus we have used our theorem to find an associated separated fractal square. □

Now, we wish to show that if F is a fractal square with a complete path at some level of construction and each shape associated with the graph representing F is rectangular, then $F \equiv S_F$. To show this we need a few more definitions and lemmas.

Theorem 15 *Assume F is a fractal square with complete path at some level. If each F_i is rectangular then $F \equiv S_F$.*

Proof:

By theorem 14 S_F can be formed. If S_F and F can be represented by the same graph such that per edge the similitude ratios are equal and all shapes associated with vertices are dust-like then by theorem 3 we will have that $F \equiv S_F$. First, we will show that S_F can be represented by the same graph as F. Form S_F as previously discussed in Theorem 14.

F has a complete path and by lemma 2 we can assume the complete path is at level 1. However, by theorem 14 we form S_F using the second stage of construction. Thus, we will consider F at its second stage of construction. Recall from theorem 10 that, for a fractal square F, the shapes required by chains are represented by a vertices in its associated graph. We called the vertex v_i and we call its respective shape F_i. For the remainder of this proof we will be looking at the copies of F in F_i in the second level of construction. Form the graph G representing F. Assume G has m vertices. Let S_i be the shape formed by replacing the copies of F in F_i with copies of S_F. The copies of S_F will have the same scale ratio and number of filled in squares as F in the second level of construction. We will group subsets of F_i, each subset a chain of maximum length, i.e. shapes, and associate the grouped subsets with copies of S_F without breaking any corner or edge connections. We will then use these associations to impose the graph G associated with F onto S_F.

To form the needed associations of subsets of F_i with copies of S_F we must begin by studying the graph representing F. Choose a vertex v_i where $1 \leq i \leq m$. Form F_i considering copies of F at the second level of construction, as discussed above. F_i has a collection of chains, call these chains F_{ij}. Thus, we have that the second stage of construction of F_i can be written as $F_i = \bigcup_{j=1}^{m} \bigcup_{k=1}^{n(j)} F_{ij}^{k}$, where $n(j)$ is the number of times F_{ij} appears in the second stage of construction of F_i. More specifically, if the complete path is of blocks in the decomposition in n^2 into $[0,1] \times [0,1]$ then the F_{ij}'s are the unions of squares of side length $\frac{1}{n}$ and F_i and F_j are unions of copies of $[0,1] \times [0,1]$ each decomposed into the union of n^2 squares. Note that once we consider the fractal in its final form, not the second stage of construction, the F_{ij} are actually scaled F_j for $1 \leq j \leq m$. Also, we will denote the maps associated with G as $T_{ij}^k(F_j) = F_{ij}^k$. Where T_{ij}^k is a scaling and a translation. More specifically, $T_{ij}^k(x,y) = \frac{1}{n}(x,y) + (a,b)$ where $\frac{1}{n}$ is the scaling factor of F_i and (a,b) is the displacement from the origin of F_i to the origin of the scaled F_j, or F_{ij}, in F_i. Also, form S_i, as discussed above, and choose a point to be the origin. The S_i will represent the same vertex as F_i.

Recall from corollary 1, the complete path of F breaks the fractal square up into quadrants. Note, these quadrants persist at all further levels of construction. As before, we label them $1, 2, 3$ and 4 clockwise from top right. The first level of construction of S_F, which corresponds to the second level of F, is a collection of filled in squares that can be decomposed (without overlap, although possibly with edge or corner intersection at the first level of construction) into a union of translations of the chains in the $3, 4, 1, 2$ arrangement of quadrants. Recall, this is the order in which the quadrants appeared in the component surrounded by the closed path in lemma 7. These quadrants will give us a way to group the F_{ij} of F_i and associate them with chains in copies of S_F, that form S_i. S_{ij}^k will refer to the chains associated with F_{ij}^k.

Form F_i and note the quadrants. Since F_i is rectangular, there is one quadrant in each of the "corners" of the rectangle F_i that is disjoint from all other quadrants. More specifically, the fourth quadrant of F in the top left "corner" of F_i, the first quadrant of F in the top right "corner" of F_i, the second quadrant of F in the bottom right "corner" of F_i and the third quadrant of F in the bottom left "corner" of F_i. By the construction of S_F, the chains in these four quadrants can be associated, without breaking any adjacencies, with subchains of a copy of S_F. Under the natural identifications of the plane containing F_i with the plane containing S_i, the associations can be regarded as translations. The associations create a one-to-one correspondence between the squares appearing in chains in these four corners of F_i and the squares appearing in chains in the chosen copy of S_F in S_i.

Now, choose a component that consists of two adjacent first and fourth quadrants on the top edge of F_i. Specifically, the quadrant on the left is the first quadrant and the quadrant on the right is the fourth quadrant. Directly below these components, on the bottom edge of F_i, lie adjacent second and third quadrants that are also components. Due to the complete path, the chains in this pair of pairs of adjacent quadrants, without breaking any edge or corner connections, can be associated with a copy of S_F, as discussed above. Do this for all first and fourth quadrants on the top edge and their corresponding second and third quadrants on the bottom edge.

Similarly to the top/bottom edge case, group the adjacent third and fourth quadrants on the left edge of F_i with their corresponding first and second adjacent quadrants on the right edge. Again, if the third quadrant is on top of the fourth quadrant and the second quadrant is on top of the first quadrant, then the complete path guarantees that the chains, without breaking any edge or corner connections, in this grouping of quadrants can be associated with a copy of S_F, as discussed above. Do this for all third and fourth quadrants on the left edge and their corresponding first and second quadrants on the right edge.

The remaining quadrants left to discuss are all in fact copies of the four original quadrants in the order $3, 4, 1, 2$. Due to the complete path we can associate the chains of these groups of quadrants each to a copy of S_F without breaking any edge or corner connections of the chains.

Now, consider S_i in its first stage of construction. We formed associations sending the chains that appear in F_i to k copies of S_F, where k is the number of copies of S_F that form S_i. Choose a $F_{ij} \in \bigcup_{j=1}^{m} \bigcup_{k=1}^{n(j)} F_{ij}^k$. Assume F_{ij} is a part of grouping of quadrants Q, as discussed above. We have an association between the collection of F_{il} that form Q and a copy of S_F. Now that we have an association with the chains forming Q and the chains in a copy of S_F we can choose a location for this S_F inside S_i. Then, for each l where a scaled F_{il} is a chain in Q, consider a subchain of the associated S_F to be a scaled copy of S_l, more specifically S_{il} instead of F_{il}, as given by the known association. Thus, as with F_i, $S_i = \bigcup_{j=1}^{m} \bigcup_{j=1}^{n(j)} S_{ij}^k$.

We have that, due to the complete path and the rectangular nature of F_i, a one to one association of the F_{ij} in F_i with S_{ij} in copies of S_F. Thus, we have maps sending scaled S_j to S_{ij} if and only if there are maps sending scaled F_j to F_{ij}. More specifically, $T_{ij}^k(F_j) = F_{ij}$ if and only if $\tau_{ij}^k(S_j) = S_{ij}^k$. The maps τ_{ij}^k sending scaled S_j into S_i will be of the form $\frac{1}{n}(x, y) + (a, b)$, a translation and a scaling. The scale factor for S_i is $\frac{1}{n}$ and hence the maps will have the same scale factor. The maps shift by (a, b) where (a, b) is the distance from the origin of S_i to the origin of the scaled S_j, or S_{ij}.

Do this for all F_i for $1 \leq i \leq m$.

Now, impose the graph G onto S_F. Let v_i be a vertex in G. If v_i is connected by and edge to v_j we have that a scaled copy of F_j, namely F_{ij} appears in F_i. We have already shown that this implies a scaled copy of S_j, namely S_{ij} appears in S_i. Thus, we need the edge connecting v_i to v_j. Do this for all edges e from v_i and for all vertices v_i for $1 \leq i \leq m$.

The only topic left to note is which group of quadrants is associated with which S_F in S_i. At the first stage of construction one must make a choice where each copy of S_F is placed for each particular S_i. If we had chosen a different location for a particular S_F the scale factor would remain the same, however, the distance from the origin of a scaled S_j, or an S_{ij}, to the origin of S_i will differ. Different choices will give different similitudes, however, the ratios will all still remain $\frac{1}{n}$. We need only have the same similitude ratios, not the same similitudes. These are the only differences that can occur depending on the choice of placements of the copies of S_F inside S_i.

Now, we have already shown that F and S_F can be represented by the same graph. For each edge on the graph the similitude paired with the edge, either associated with F or S_F, has similitude ratio $\frac{1}{n}$, as previously noted. Also note, we are given that F_i and S_i are dust-like, once considering the copies of F and S_F as the fractal squares themselves and not levels of construction, for each $1 \leq i \leq m$. S_i is dust-like due to the fact that S_F is separated. Each filled in square of any level of construction of S_i is disjoint from all other filled in square when one looks at the next level of construction. Thus, we know that F and S_F are represented by the same graph, all similitude ratios are the same, and that each shape represented by a vertex are dust-like. Therefore, by theorem 3 we have that $F \equiv S_F$ as desired. $\square$

Now, in many examples it may be easier to form an associated S_F in the stage one construction. We do this by translating shapes inside the stage one closed path to sit in an $n \times n$ grid which forms a separated fractal square. However, the previous theorem guarantees a separated fractal square exists and gives a process to follow to form such a fractal square.

Example 16

The following fractal square, F, is equivalent to S_F.

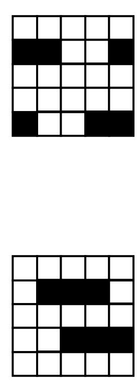


Figure 8.8: Fractal Square F and its Separated Fractal Square S_F

Proof:

First, we will form the graph G associated with F. The graph is pictured below.

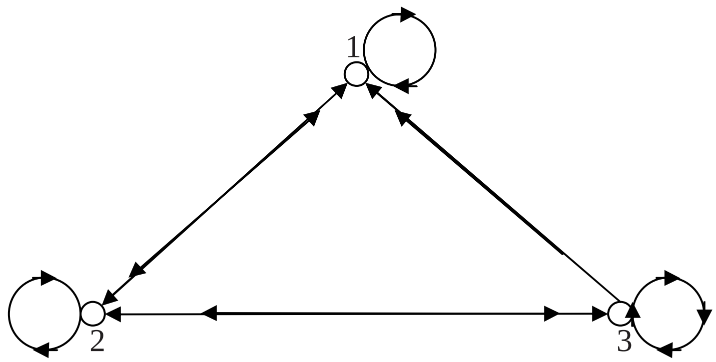


Figure 8.9: The Directed Graph representing F and S_F

The shape associated with vertex 1 is F, pictured below.

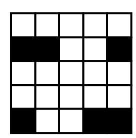


Figure 8.10: Fractal Square F

Let F_2 be the shape associated with vertex 2, pictured below.

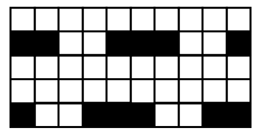


Figure 8.11: Shape F_2 Representing Vertex 2

Finally, let F_3 be the shape associated with vertex 3, pictured below.

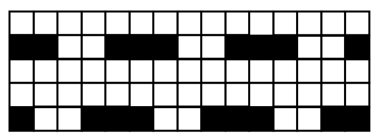


Figure 8.12: Shape F_3 representing vertex 3

The following functions are the similitudes associated with the graph, G.

From vertex 1:

There are two maps heading from vertex 1 to vertex 1. They are $\frac{1}{5}x$ and $\frac{1}{5}x + (\frac{4}{5}, \frac{3}{5})$.

There are two maps heading from vertex 1 to vertex 2. They are $\frac{1}{5}x + (\frac{3}{5}, 0)$ and $\frac{1}{5}x + (0, \frac{3}{5})$.

From vertex 2:

There are two maps heading from vertex 2 to vertex 1. They are $\frac{1}{5}x$ and $\frac{1}{5}x + (\frac{9}{5}, \frac{3}{5})$.

There are two maps heading from vertex 2 to vertex 2. They are $\frac{1}{5}x + (0, \frac{3}{5})$ and $\frac{1}{5}x + (\frac{8}{5}, 0)$.

There are two maps heading from vertex 2 to vertex 3. They are $\frac{1}{5}x + (\frac{3}{5}, 0)$ and $\frac{1}{5}x + (\frac{4}{5}.\frac{3}{5})$.

From vertex 3:

There are two maps heading from vertex 3 to vertex 1. They are $\frac{1}{5}x$ and $\frac{1}{5}x + (\frac{14}{5}, \frac{3}{5})$.

There are two maps heading from vertex 3 to vertex 2. They are $\frac{1}{5}x + (0, \frac{3}{5})$ and $\frac{1}{5}x + (\frac{13}{5}, 0)$.

There are four maps heading from vertex 3 to vertex 3. They are $\frac{1}{5}x + (\frac{3}{5}, 0)$, $\frac{1}{5}x + (\frac{8}{5}, 0)$, $\frac{1}{5}x + (\frac{4}{5}, \frac{3}{5})$ and $\frac{1}{5}x + (\frac{9}{5}, \frac{3}{5})$.

Now, we wish to impose S_F on the graph G. To do this, we must first form shapes S_i to be represented by vertex 1, 2, and 3 of G.

Let S_F, the separated fractal square associated with F, be the shape associated with vertex 1 pictured below.

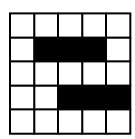


Figure 8.13: Separated Fractal Square S_F

Let S_2 be the shape associated with vertex 2. Form S_2 by replacing the copies of F in F_2 with copies of S_F, pictured below.

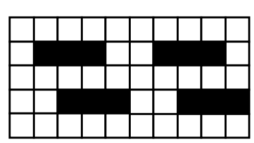


Figure 8.14: Shape S_2 Representing Vertex 2

Let S_3 be the shape associated with vertex 3. Form S_3 by replacing the copies of F in F_3 with copies of S_F, pictured below.

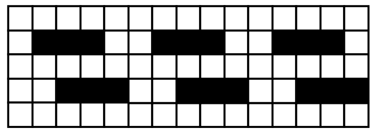


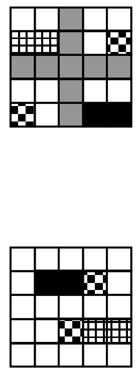
Figure 8.15: Shape S_3 representing vertex 3

Now, we wish to impose S_F on G using S_2 and S_3. To do this we will use the complete path of F and the rectangular nature of F, F_2 and F_3. Note that the complete path breaks F into quadrants. We will group subsets of F_i into collections of quadrants. Each collection contains exactly one of each of the four quadrants. These collections will be associated with a copy of S_F in S_i.

The collections are formed as follows. The corners of F_i are associated with one copy of S_F. Two adjacent first and fourth quadrants on the top edge of F_i and the two adjacent second and third quadrants on the bottom edge of F_i can be grouped into one collection. Group adjacent third and fourth quadrants on the left edge of F_i with their corresponding first and second adjacent quadrants on the right edge. The remaining quadrants left to discuss are all in fact copies of the four original quadrants.

For our example, the groupings are shown as follows. The scaled shape shaded in F_i is sent to a subset of S_i with the same shading. Remember that since S_F is separated, two subsquares that seem adjacent can be considered non-adjacent.

For S_F, the grouping is straightforward and pictured below:



For S_2 the groupings are:

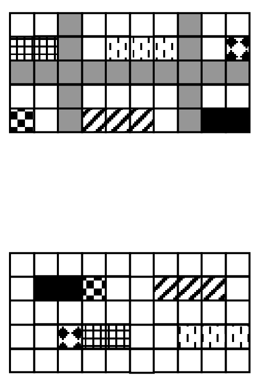


Figure 8.16: Fractal Square F and its Separated Fractal Square S_F

The "corners" are sent to the copy of S_F on the left, and the remaining quadrants are sent to the copy of S_F on the right.

For S_3 the groupings are:

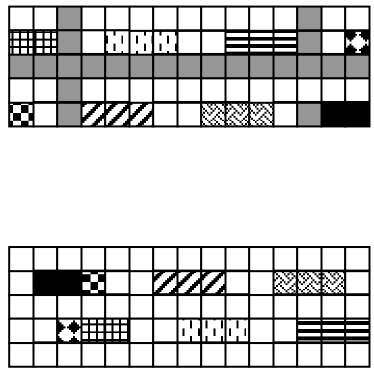


Figure 8.17: Fractal Square F and its Separated Fractal Square S_F

The "corners" are sent to the copy of S_F on the left, the left-middle group of quadrants is sent to the copy of S_F in the middle of S_3 and the right-middle group of quadrants is sent to the copy of S_F in the right of S_F.

These groupings show how to impose S_F on G. The following are the mappings associated with S_F on G.

From vertex 1:

There are two maps heading from vertex 1 to vertex 1. They are $\frac{1}{5}x + (\frac{2}{5}, \frac{1}{5})$ and $\frac{1}{5}x + (\frac{3}{5}, \frac{3}{5})$.

There are two maps heading from vertex 1 to vertex 2. They are $\frac{1}{5}x + (\frac{3}{5}, \frac{1}{5})$ and $\frac{1}{5}x + (\frac{1}{5}, \frac{3}{5})$.

From vertex 2:

There are two maps heading from vertex 2 to vertex 1. They are $\frac{1}{5}x + (\frac{2}{5}, \frac{1}{5})$ and $\frac{1}{5}x + (\frac{3}{5}, \frac{3}{5})$.

There are two maps heading from vertex 2 to vertex 2. They are $\frac{1}{5}x + (\frac{3}{5}, \frac{1}{5})$ and $\frac{1}{5}x + (\frac{1}{5}, \frac{3}{5})$.

There are two maps heading from vertex 2 to vertex 3. They are $\frac{1}{5}x+(\frac{7}{5},\frac{1}{5})$ and $\frac{1}{5}x+(\frac{6}{5},\frac{3}{5})$.

From vertex 3:

There are two maps heading from vertex 3 to vertex 1. They are $\frac{1}{5}x+(\frac{2}{5},\frac{1}{5})$ and $\frac{1}{5}x+(\frac{3}{5},\frac{3}{5})$.

There are two maps heading from vertex 3 to vertex 2. They are $\frac{1}{5}x+(\frac{3}{5},\frac{1}{5})$ and $\frac{1}{5}x+(\frac{1}{5},\frac{3}{5})$.

There are four maps heading from vertex 3 to vertex 3. They are $\frac{1}{5}x+(\frac{7}{5},\frac{1}{5})$, $\frac{1}{5}x+(\frac{6}{5},\frac{3}{5})$, $\frac{1}{5}x+(\frac{12}{5},\frac{1}{5})$ and $\frac{1}{5}x+(\frac{11}{5},\frac{3}{5})$.

Now, F and S_F can be represented by the same graph, each shape is dust-like and all the similitude ratios are $\frac{1}{5}$. By theorem 3 we have that $F \equiv S_F$. □

Example 17

The following F and its S_F are equivalent.

Figure 8.18: Fractal Square F and its Separated Fractal Square S_F

Proof:

Form an associated fractal square associated with F. Note, we did not use theorem 14 to form S_F, however we could have done so.

The graph associated with F is as follows

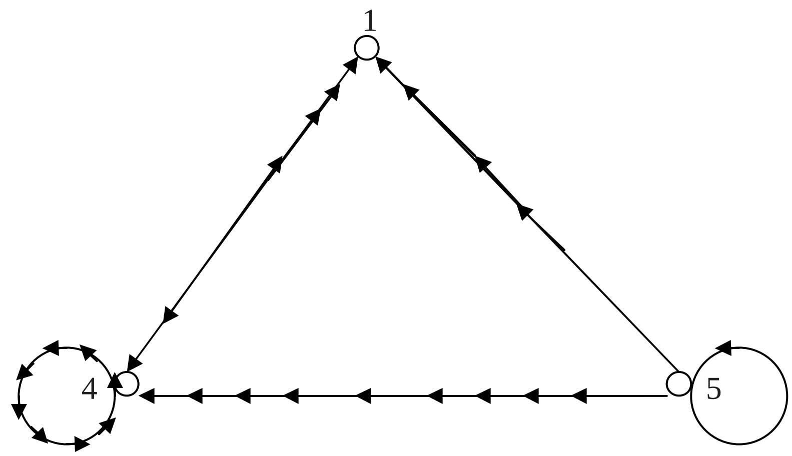


Figure 8.19: Graph of F

Let's look at the similitudes associated with each vertex:

For vertex 1:

There is one map heading to vertex 1. It is $\frac{1}{5}(x,y)+(0,\frac{4}{5})$.

There are two maps heading to vertex 4. They are $\frac{1}{5}(x,y)+(\frac{2}{5},0)$ and $\frac{1}{5}(x,y)+(\frac{4}{5},\frac{1}{5})$.

For vertex 4:

There are four maps heading toward vertex 1. They are $\frac{1}{5}(x,y)+(0,\frac{4}{5})$, $\frac{1}{5}(x,y)+(0,\frac{9}{5})$, $\frac{1}{5}(x,y)+(0,\frac{14}{5})$ and $\frac{1}{5}(x,y)+(0,\frac{19}{5})$.

There are eight maps heading toward vertex 4. They are $\frac{1}{5}(x,y)+(\frac{2}{5},0)$, $\frac{1}{5}(x,y)+(\frac{2}{5},1)$, $\frac{1}{5}(x,y)+(\frac{2}{5},2)$, $\frac{1}{5}(x,y)+(\frac{2}{5},3)$, $\frac{1}{5}(x,y)+(\frac{4}{5},\frac{1}{5})$, $\frac{1}{5}(x,y)+(\frac{4}{5},\frac{6}{5})$, $\frac{1}{5}(x,y)+(\frac{4}{5},\frac{11}{5})$ and $\frac{1}{5}(x,y)+$

$(\frac{4}{5}, \frac{16}{5})$.

For vertex 5:

There are four maps heading to vertex 1. They are $\frac{1}{5}(x, y) + (0, \frac{4}{5})$, $\frac{1}{5}(x, y) + (0, \frac{9}{5})$, $\frac{1}{5}(x, y) + (0, \frac{14}{5})$ and $\frac{1}{5}(x, y) + (0, \frac{19}{5})$.

There are nine maps heading to vertex 4. They are $\frac{1}{5}(x, y) + (\frac{2}{5}, 0)$, $\frac{1}{5}(x, y) + (\frac{2}{5}, 1)$, $\frac{1}{5}(x, y) + (\frac{2}{5}, 2)$, $\frac{1}{5}(x, y) + (\frac{2}{5}, 3)$, $\frac{1}{5}(x, y) + (\frac{4}{5}, \frac{1}{5})$, $\frac{1}{5}(x, y) + (\frac{4}{5}, \frac{6}{5})$, $\frac{1}{5}(x, y) + (\frac{4}{5}, \frac{11}{5})$, $\frac{1}{5}(x, y) + (\frac{7}{5}, 3)$ and $\frac{1}{5}(x, y) + (\frac{9}{5}, \frac{16}{5})$.

There is one map heading to vertex 5. It is $\frac{1}{5}(x, y) + (\frac{4}{5}, \frac{16}{5})$

Now, we can impose the graph onto S_F with the following similitudes.

For vertex 1:

There is one map heading to vertex 1. It is $\frac{1}{5}(x, y) + (\frac{1}{5}, \frac{4}{5})$.

There are two maps heading to vertex 4. They are $\frac{1}{5}(x, y) + (0, \frac{1}{5})$ and $\frac{1}{5}(x, y) + (\frac{3}{5}, \frac{1}{5})$.

For vertex 4:

There are four maps heading toward vertex 1. They are $\frac{1}{5}(x, y) + (\frac{1}{5}, \frac{4}{5})$, $\frac{1}{5}(x, y) + (\frac{1}{5}, \frac{9}{5})$, $\frac{1}{5}(x, y) + (\frac{1}{5}, \frac{14}{5})$ and $\frac{1}{5}(x, y) + (\frac{1}{5}, \frac{19}{5})$.

There are eight maps heading toward vertex 4. They are $\frac{1}{5}(x, y) + (0, \frac{1}{5})$, $\frac{1}{5}(x, y) + (0, \frac{6}{5})$, $\frac{1}{5}(x, y) + (0, \frac{11}{5})$, $\frac{1}{5}(x, y) + (0, \frac{16}{5})$, $\frac{1}{5}(x, y) + (\frac{3}{5}, \frac{1}{5})$, $\frac{1}{5}(x, y) + (\frac{3}{5}, \frac{6}{5})$, $\frac{1}{5}(x, y) + (\frac{3}{5}, \frac{11}{5})$ and $\frac{1}{5}(x, y) + (\frac{3}{5} \frac{16}{5})$.

For vertex 5:

There are four maps heading to vertex 1. They are $\frac{1}{5}(x,y)+(\frac{1}{5},\frac{4}{5})$, $\frac{1}{5}(x,y)+(\frac{1}{5},\frac{9}{5})$, $\frac{1}{5}(x,y)+(\frac{1}{5},\frac{14}{5})$ and $\frac{1}{5}(x,y)+(\frac{1}{5},\frac{19}{5})$.

There are nine maps heading to vertex 4. They are $\frac{1}{5}(x,y)+(0,\frac{1}{5})$, $\frac{1}{5}(x,y)+(0,\frac{6}{5})$, $\frac{1}{5}(x,y)+(0,\frac{11}{5})$, $\frac{1}{5}(x,y)+(0,\frac{16}{5})$, $\frac{1}{5}(x,y)+(\frac{3}{5},\frac{1}{5})$, $\frac{1}{5}(x,y)+(\frac{3}{5},\frac{6}{5})$, $\frac{1}{5}(x,y)+(\frac{3}{5},\frac{11}{5})$, $\frac{1}{5}(x,y)+(\frac{3}{5}\frac{16}{5})$ and $\frac{1}{5}(x,y)+(\frac{8}{5},\frac{16}{5})$.

There is one map heading to vertex 5. It is $\frac{1}{5}(x,y)+(1,\frac{16}{5})$

Thus, even though we could not form S_F with rigid motions, we can still form S_F and we also have $F \equiv S_F$.

Now, we can also show fractal squares, who have non-rectangular shapes, are equivalent to their separated fractal square. We will begin with an example and then state a theorem.

Example 18

The following fractal square F is equivalent to its S_F.

Figure 8.20: Fractal Square F and its Separated Fractal Square S_F

The graph associated with F is as follows:

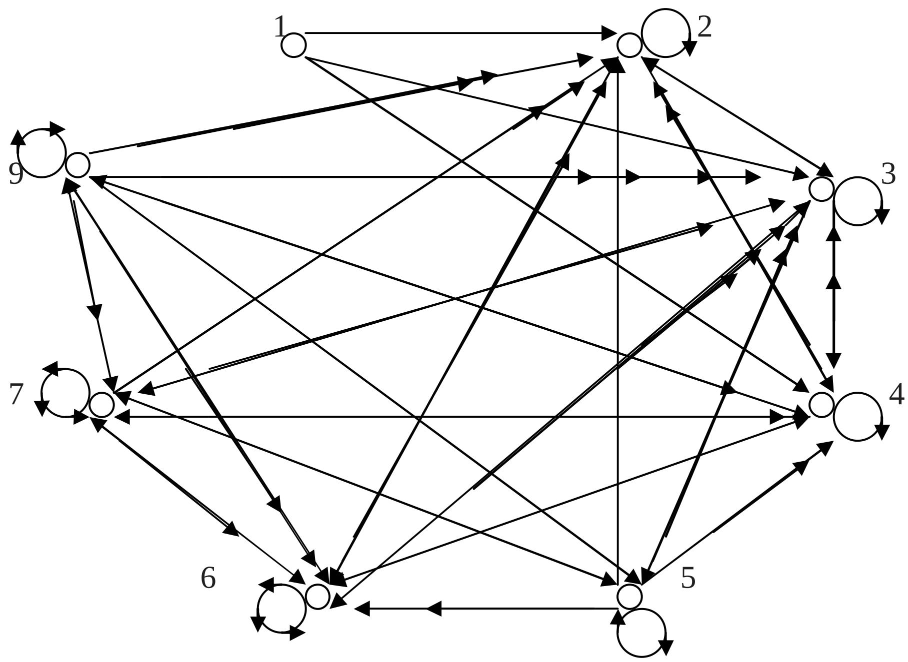


Figure 8.21: Graph representing F

It turns out F is equivalent to its separated fractal square S_F even though not all the shapes associated with F are rectangular because it satisfies the hypotheses of the following theorem.

Theorem 16 *Let F be a fractal square of dimension $\frac{\ln(k)}{\ln(n)}$. Assume that there exists $m \in \mathbb{N}$ such that F has a complete path at level m. Assume one can form S_F. Let G be the graph associated with F. Assume there exists a level of construction such that for all vertices v_i of G the following holds:*

Let R_{ij} be the shape formed using the same chain, i.e. begin with the same $[0,1] \times [0,1]$ adjacencies as F_{ij}, formed with copies of S_F in place of copies of F. For all directed edges *from vertices* v_{i_j} *for* $1 \leq i \leq p_i$ *that*

1. *The exists a map* $f_{i_j} = \frac{1}{n}x + b_j$ *for some* b_j
2. $\frac{1}{n}R_{ij} + b_j \subseteq S_{F_i}$ *for all* $i \leq j \leq p_i$
3. $\frac{1}{n}R_{ij} + b_j \cap \frac{1}{n}R_{il} + b_l = \emptyset$ *for* $j \neq l$
4. $\cup_1^{p_i} \dfrac{1}{n}R_{ij} + b_j = S_{F_i}$

   Then $F \equiv S_F$

Proof:

We would like to use Theorem 3 to show that $S_f \equiv F$. First we need to show that each R_{ij} are dust-like. Since S_F is separated, as previously discussed, we know that even two "adjacent" separated squares are disjoint. Also, given $\frac{1}{n}R_{ij} + b_j \cap \frac{1}{n}R_{il} + b_l = \emptyset$ for $j \neq l$ we have that each R_{ij} is dust-like.

Let v_i be a vertex in G. Form S_{F_i}. Then, by hypothesis, for all $1 \leq j \leq p_i$ we have $f_{i_j} : S_{F_{i_j}} \to S_{F_i}$ with similitude ratio $\frac{1}{n}$. Thus each vertex has at least the same incoming directed edges. There are no more since $\bigcup_1^{p_i} \left(\dfrac{1}{n} S_{F_{i_j}} + b_j = S_{F_i} \right)$. Thus G can be imposed on S_F. Now, each similitude ratio, no matter which directed edge we look at or which fractal square we impose G on, is $\frac{1}{n}$. This gives that, by theorem 3, $S_f \equiv F$. $\square$

Example 19

The fractal square F is equivalent to S_F, both shown below.

Figure 8.22: Fractal Square F and its Separated Fractal Square S_F

However, if one just studies the stage one construction it looks like this is not the case. For the shape below, call it F_9, there are no functions that satisfy the hypotheses of theorem 16. F_9 is a subset of F_3, the shape in the lower left corner of F.

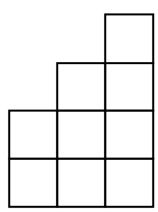


Figure 8.23: F_9

Look at F_9 in more detail:

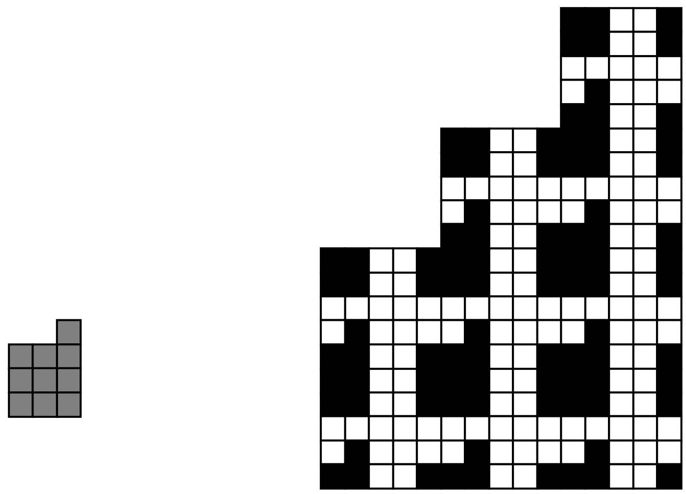


Figure 8.24: Single shape of S_F and the Second Stage of Construction for F_9

In theorem 16, we have the hypotheses that

1. The exists a map $f_{i_j} = \frac{1}{n}x + b_j$ for some b_j

2. $\frac{1}{n}R_{ij} + b_j \subseteq S_{F_i}$ for all $i \leq j \leq p_i$

3. $\frac{1}{n}R_{ij} + b_j \cap \frac{1}{n}R_{il} + b_l = \emptyset$ for $j \neq l$

4. $\cup_1^{p_i} \dfrac{1}{n} R_{ij} + b_j = S_{F_i}$

   Then $F \equiv S_F$

In this case, F_9 appears twice as scaled version of itself. Now, without breaking any edge connections, we need to take these scaled F_9's and join them with a scaled single filled in square to form S_F. However, we only have one single filled in square. By the hypotheses of our theorem we cannot break a corner or edge connection. This gives that, at stage one, we do not satisfy the hypotheses.

If one looks at the second stage of construction, there are functions that satisfy the needed hypotheses. Going down another level created more single filled in squares than F_9's! The hypotheses of theorem 16 can be satisfied at the second stage of construction. So $F \equiv S_F$ is true, however, one needed to study the second stage of construction instead of just the first stage.

Now, can we show even more. We can show that two fractal squares of the same dimension are equivalent if they are each equivalent to their own separated fractal square.

Theorem 17 *Let A and B be fractal squares with the same scale ratio and same number of filled in squares at the first stage of construction. Assume $A \equiv S_A$ and $B \equiv S_B$. Then $A \equiv B$.*

Proof:

Since A and B have the same scale ratio and same number of filled in squares we know that we can make a graph for S_A and S_B that has one vertex and k edges by theorem 13. Each edge with have similitude ratio $\frac{1}{n}$. This gives that $S_A \equiv S_B$. Thus, by transitivity of an equivalence relation, we have $A \equiv S_A \equiv S_B \equiv B$. Hence $A \equiv B$ as desired. □

Now, one of the first ways one thinks of to show that two fractal squares are equivalent is to show that they are each equivalent to an isolated fractal square, defined earlier in Chapter 4. The following corollary will show that an isolated fractal square is equivalent to a separated fractal square of the same dimension. This will give, by transitivity of equivalence relation, that an isolated fractal square is equivalent to fractal squares of the same dimension (assuming one can form a graph for the non-isolated fractal square.)

Corollary 3 *Let I be an isolated fractal square. Let S be a separated fractal square. Let S and I have the same scale ratio and same number of filled in squares at the first stage of construction, then $S \equiv I$.*

Proof:

We know by theorem 13 that we can impose on S the graph G with a single vertex and k edges each paired with a similitude with ratio $\frac{1}{n}$. We will now do the same for I.

To show that I can be represented by a graph with a single vertex we note, by definition of isolated, each filled in square is non-adjacent to every other filled in square. This means, not only are no new adjacencies created by going down another level, but each filled in square is just a contraction of the isolated fractal square and only the isolated fractal square. Thus, we need only the single vertex. We know that the contraction ratio is $\frac{1}{n}$ by the dimension of I. Since there are a total of k filled in squares we have a total of k edges from the single vertex. Thus, we have that we can impose a graph on I that is a single vertex with k edges where every edge is associated with a similitude with contraction ratio $\frac{1}{n}$.

Clearly, we have that both I and S are dust-like. This gives, by theorem 3, that $I \equiv S$, as desired. □

Corollary 4 *Let I be an isolated fractal square. Let S be a separated fractal square. Assume $dim(S) = \frac{\ln(k)}{\ln(n)} = dim(I)$ and at the first stage of construction either S has k filled in squares and I has k^m filled in squares or I has k filled in squares and S has k^m filled in squares. Then $S \equiv I$.*

Proof:

Case 1:

Assume $dim(S) = \frac{\ln(k)}{\ln(n)}$ and $dim(I) = \frac{\ln(k^m)}{\ln(n^m)}$. Now, by lemma 2 we know that we can iterate the m^{th} stage of construction for S and get the same fractal square, thus we can use the m^{th} stage of construction with k^m filled in squares and contraction ratio $\frac{1}{n^m}$. Thus, by theorem 13 to form a graph of S with a single vertex and k^m edges. Each edge has a similitude with corresponding ratio $\frac{1}{n^m}$.

As in corollary 3 we can represent I by a graph with k^m edges and one vertex. Each edge will have a similitude with ratio $\frac{1}{n^m}$.

By definition of isolated and separated, we have that both I and S are dust-like. Thus, by theorem 3, we have that $I \equiv S$, as desired.

Case 2:

This is very similar to case 1. Assume $dim(I) = \frac{\ln(k)}{\ln(n)}$ and $dim(S) = \frac{\ln(k^m)}{\ln(n^m)}$. Again, by lemma 2 we can iterate the m^{th} stage of construction for I and get the same fractal square, so we can use the m^{th} stage of construction with k^m filled in squares and contraction ratio $\frac{1}{n^m}$. As in corollary 3 we can represent I by a graph with a single vertex, k^m edges each with a similitude with ratio $\frac{1}{n^m}$.

By theorem 13 we can form a graph of S with a single vertex and k^m edges. Each edge has a similitude with corresponding ratio $\frac{1}{n^m}$.

By definition of isolated and separated, we have that both I and S are dust-like. Thus, by theorem 3, we have that $I \equiv S$, as desired. □

Thus, we know conditions in which fractal squares are equivalent as well as techniques to show the equivalence.

Chapter 9

A Look Ahead

There are many more areas in this subject that I wish to study. I would like to know when two general fractal squares are equivalent. More specifically, what happens if one of the fractal squares has non-rectangular subshapes? Recall that I know many examples in which a fractal square with non-rectangular shapes is equivalent to its separated fractal square. I wish to find more general sufficient conditions that guarantee this equivalence.

There are also many questions involving the graphs themselves. Is there anyway to categorize the fractals that belong to a particular graph when one begins with the graph itself. Can one look at two graphs and know that the fractal squares associated with them are equivalent? In essence, are there ways of proving two fractal squares are equivalent by proving their graphs are equivalent? Is it guaranteed that a finite graph will mean a complete path at some level of construction in some $n \times n$ block of copies of the fractal square?

I also wish to study if there are there any other methods for proving the equivalence of fractal squares. Another technique may prove useful when a finite graph is absent for a fractal square or if one can apply this technique with an infinite graph.

Once I complete the study of fractal squares another potential area of study is the equivalence of the three dimensional analogue of Cantor sets, or what I call fractal cubes. What is the extension of the complete path? the closed path? One point in question to consider is how to extend the winding number argument to the fractal cubes. I would be interested in determining if the "winding number of a cylinder about a line" could be an effective approach (and one that may evolve naturally using my two dimensional technique).

As described, there are many more avenues of research involved with this particular study.

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