1 Supplemental Material

1.1 Sketch of Solution to Z_1

The case where $\ell = 0$ must be handled specially, however, it is not challenging to show that $\zeta_{1,0} \equiv 0$. This is required for the correct representation of the far field conditions. The solution for the remaining can be found using the method of variation of parameters.

The solution to the homogeneous equation can be expressed in terms of the functions

$$y_1 = \sqrt{\gamma} e^{-\gamma} I_{\ell+1/2}(\gamma) \qquad \text{(Supp. 1)}$$
$$y_2 = \sqrt{\gamma} e^{-\gamma} K_{\ell+1/2}(\gamma) \qquad \text{(Supp. 2)}$$

where $\gamma = -f_0/(2r)$. I and K are the modified Bessel functions of first and second kind, respectively [1]. First noting that $\partial_r Z_0$ is known, and apply variation of parameters the general solution can then be expressed

$$\zeta_{1,\ell} = (c_{1,\ell} + a_{1,\ell}(r)) I_{\ell+1/2}\left(-\frac{1}{2r}\right) + (d_{1,\ell} + b_{1,\ell}(r)) K_{\ell+1/2}\left(-\frac{1}{2r}\right) \qquad \text{(Supp. 3)}$$
$$a(r) = -\int_1^r f_{0,0} f_{1,\ell} g(\ell+1) \sqrt{\frac{-f_0}{2}} \exp\left[f_0\left(\frac{1}{2\chi} - 1\right)\right] \frac{K_{\ell+1/2}\left(-\frac{f_0}{2\chi}\right)}{\chi^{\ell+1/2+2}} d\chi \qquad \text{(Supp. 4)}$$
$$b(r) = \int_1^r f_{0,0} f_{1,\ell} g(\ell+1) \sqrt{\frac{-f_0}{2}} \exp\left[f_0\left(\frac{1}{2\chi} - 1\right)\right] \frac{I_{\ell+1/2}\left(-\frac{f_0}{2\chi}\right)}{\chi^{\ell+1/2+2}} d\chi \qquad \text{(Supp. 5)}$$

where χ is a dummy variable introduced for the integration. To get this form it was used that $f_{1,\ell} = h_{1,\ell} f_0$. In both cases the integrand is smooth. To ensure the integral converges at infinity, the integrand must decay rapidly in this limit. It can be demonstrated [1] that as $r \to \infty$

$$I_{\ell+1/2} \to \left(-\frac{1}{2}\frac{f_0}{2r}\right)^{\ell+1/2} / \Gamma(\ell+3/2) \qquad \text{(Supp. 6)}$$
$$K_{\ell+1/2} \to \frac{1}{2}\Gamma(\ell+3/2)\left(-\frac{1}{2}\frac{f_0}{2r}\right)^{-(\ell+1/2)} \qquad \text{(Supp. 7)}$$

In either case, substitution of this into the integrals yields an integrand that decays faster than r^{-2} which implies that the integrals are finite. The functions $K_{\ell+1/2}$ diverge at large r, therefore the solution must have $d_{1,\ell} = -b(\infty)$. The remaining unknown coefficient, $c_{1,\ell}$ is solved from the boundary conditions, Eq. (40). We use the notation, $\gamma_s = -f_0/2$,

$$q = \sqrt{\gamma} e^{-\gamma} \qquad \text{(Supp. 8)}$$

and use $'$ to denote differentiation with respect to gamma. It is recovered,

$$c_{1,\ell} = -f_{0,0} h_{1,\ell} \frac{g f_0 - g(\ell+1) + b(\infty) \partial_r \gamma \left[q' K_{\ell+1/2} + 1/2\, q \left(K_{\ell-1/2} + K_{\ell+1/2}\right)\right]}{\partial_r \gamma \left[q' I_{\ell+1/2} + 1/2\, q \left(I_{\ell-1/2} + I_{\ell+1/2}\right)\right]} \qquad \text{(Supp. 9)}$$

where it is understood that the argument for the Bessel functions is γ_s. It is noted that the term $f_{0,0} h_{1,\ell}$ factors out, so the scalar perturbation is zero whenever the geometric perturbation is zero.

1.2 Sketch of solution to Φ_2 and Z_2

Since Φ_2 satisfies Laplace's equation, its solution is specified by simply solving the boundary conditions. Expansion of Eq. (35) under the appropriate conditions and rearranging yields the expression

$$\partial_\theta \Phi_2 = -4 h_{1,2}^2 f_{0,0} P_2 P_2' \sin\theta - h_{2,3} P_3' \sin\theta \qquad \text{(Supp. 10)}$$

where $'$ denotes differentiation with respect to $\cos\theta$. Some manipulation shows

$$P_2 P_2' = \frac{9}{5} P_3 + \frac{6}{5} P_1 \tag{Supp. 11}$$

which may be substituted. Since the new expression is a linear expression of Legendre functions and their derivatives, this expression may now be integrated directly using the result of [2]. However this result specifies the integral with the addition of an arbitrary constant factor. Now note that $f_{2,0}$ must be zero because Φ_2 must satisfy a far field condition similar to Eq. (62). Using this information, the constant of integration can be determined, and the resulting expressions are

$$f_{2,0} = 0 \tag{Supp. 12}$$

$$f_{2,2} = \frac{4}{7} h_{1,2}^2 f_{0,0} \tag{Supp. 13}$$

$$f_{2,3} = h_{2,3} f_{0,0} \tag{Supp. 14}$$

$$f_{2,4} = \frac{36}{35} h_{1,2}^2 f_{0,0} \tag{Supp. 15}$$

and all other coefficients $f_{2,\ell}$ are null.

The solution for Z_2 proceeds analogously to Z_1. In fact, the homogeneous part of the equation governing Z_2 is the same as that which govern Z_1. Accordingly, the homogeneous solution for the result is equivalent and the particular solution can be computed from variation of parameters strategy. One major difference between the Z_1 and Z_2 solution is that $\zeta_{2,0}$ is not zero. The function $\zeta_{2,0}$ satisfies a qualitatively different governing equation than the other $\zeta_{2,\ell}$, so it is best solved separately. Its solution is

$$\zeta_{2,0} = c_{0,0} + d_{0,0} \exp\Phi_0 \tag{Supp. 16}$$

With appropriate substitutions and simplifications, the boundary condition (Eq. (41)) becomes

$$\zeta_{2,0}' = -h_{2,0} g f_{0,0}^2 + \frac{1}{5}\left[\frac{1}{2} h_{1,2}^2 g\left(f_{0,0}^3 + 6 f_{0,0}^2\right) + h_{1,2} g\left(12 f_{1,2} - \zeta_{1,2}''\right)\right] + \frac{6}{5}\left(h_{1,2}\zeta_{1,2} - f_{0,0} h_{1,2}^2\right) \tag{Supp. 17}$$

$$\zeta_{2,2}' = \frac{2}{7}\left[\frac{1}{2} h_{1,2}^2 g\left(f_{0,0}^3 + 6 f_{0,0}^2\right) + h_{1,2} g\left(12 f_{1,2} - \zeta_{1,2}''\right)\right] - \frac{4}{7} g h_{1,2}^2 f_{0,0} + \frac{6}{7}\left(h_{1,2}\zeta_{1,2} - f_{0,0} h_{1,2}^2\right) \tag{Supp. 18}$$

$$\zeta_{2,3}' = -h_{2,3} g\left(f_{0,0}^2 + f_{0,0}\right) P_3 \tag{Supp. 19}$$

$$\zeta_{2,4}' = \frac{18}{35}\left[\frac{1}{2} h_{1,2}^2 g\left(f_{0,0}^3 + 6 f_{0,0}^2\right) + h_{1,2} g\left(12 f_{1,2} - \zeta_{1,2}''\right)\right] - \frac{36}{35} g h_{1,2}^2 f_{0,0} - \frac{52}{35}\left(h_{1,2}\zeta_{1,2} - f_{0,0} h_{1,2}^2\right) \tag{Supp. 20}$$

For all other ℓ, $\zeta_{2,\ell} = 0$.

1.3 Curvature Definitions and Expansion to Second Order

Following [3], the expression for the mean and Gaussian curvature in this work can be expressed as

$$H = \frac{-\eta + \partial_\theta\eta\cot\theta}{2\eta\sqrt{\eta^2 + (\partial_\theta\eta)^2}} + \frac{\eta\partial_\theta\partial_\theta\eta - \eta^2 - 2(\partial_\theta\eta)^2}{2\left[\eta^2 + (\partial_\theta\eta)^2\right]^{3/2}} \tag{Supp. 21}$$

$$K = \frac{\left[\eta\left(\partial_\theta\partial_\theta\eta - \eta\right) - 2(\partial_\theta\eta)^2\right]\left[-\eta^2\sin^2\theta + \eta\partial_\theta\eta\cos\theta\sin\theta\right]}{\left[\eta^2 + (\partial_\theta\eta)^2\right]^2 \eta^2\sin^2\theta} \tag{Supp. 22}$$

These definitions can be used to generate the following expressions, to the order of the solution accuracy.

$$H = -1 + \varepsilon\frac{1}{2}\left(\partial_\theta\partial_\theta\eta_1 + 2\eta_1 + \partial_\theta\eta_1\cot\theta\right) + \frac{1}{2}\varepsilon^2\left(\partial_\theta^2\eta_2 - 2\partial_\theta^2\eta_1\eta_1 - 2\eta_1^2 + 2\eta_2 + \partial_\theta\eta_2\cot\theta - 2\partial_\theta\eta_1\eta_1\cot\theta\right) \tag{Supp. 23}$$

$$\begin{aligned}\sqrt{K} = {} & 1 - \varepsilon\frac{1}{2}\left(\partial_\theta\partial_\theta\eta_1 + 2\eta_1 + \partial_\theta\eta_1\cot\theta\right) \\ & + \frac{1}{8}\varepsilon^2\left[-\left(\partial_\theta\eta_1\right)^2 - 4\partial_\theta^2\eta_2 + 8\partial_\theta^2\eta_1\eta_1 + 8\eta_1^2 + 2\partial_\theta^2\eta_1\partial_\theta\eta_1\cot\theta - 4\partial_\theta\eta_2\cot\theta + 8\partial_\theta\eta_1\eta_1\cot\theta - \left(\partial_\theta\eta_1\right)^2\cot^2\theta\right]\end{aligned} \tag{Supp. 24}$$

And using the relevant geometric perturbation function for this work it simplifies to,

$$H = -1 - \varepsilon 2\eta_1 + \varepsilon^2\left(5h_{1,2}^2P_2^2 + h_{2,0} - 2h_{2,2}P_2 - 5h_{2,3}P_3 - 9h_{2,4}P_4\right) \tag{Supp. 25}$$

$$\sqrt{K} = 1 + \varepsilon 2\eta_1 + \varepsilon^2\left\{-5h_{1,2}^2P_2^2 + 3h_{2,2}P_2 + 6h_{2,3}P_3 + 10h_{2,4}P_4 - \frac{1}{8}(P_2'')^2h_{1,2}^2\sin^4\theta\right\} \tag{Supp. 26}$$

1.4 Second Order Expansion of Vaporization Flux and Comparison to Tonini & Cossali Curvature Law [4]

$$\boldsymbol{n}\cdot\nabla\Phi = -f_{0,0}\left\{1 + \varepsilon h_{1,2}P_2 + \varepsilon^2\left[-h_{2,0} + \frac{3}{5}h_{1,2}^2 - \frac{64}{7}h_{1,2}^2P_2 - 3h_{2,3}P_3 - \frac{72}{35}h_{1,2}P_4\right]\right\} \tag{Supp. 27}$$

$$\sqrt[4]{K}\frac{\dot{m}}{4\pi} = -f_{0,0}\left\{1 + \varepsilon h_{1,2}P_2 + \varepsilon^2\left[\frac{14}{5}h_{1,2}^2 - 16h_{2,0} + \left(32h_{2,2} - 88h_{1,2}^2\right)P_2 + 80h_{2,3}P_3 + 144h_{2,4}P_4\right]\right\} \tag{Supp. 28}$$

References

[1] NIST's Digital Library of Mathematical Functions.

[2] DiDonato, A. R., 1982, "Recurrence Relations for the Indefinite Integrals of the Associated Legendre Functions," *Mathematics of Computation,* **38**(158), p. 547.

[3] Korn, G. A., and Korn, T. M., 2000, *Mathematical Handbook for Scientists and Engineers*. Dover.

[4] Tonini, S., and Cossali, G. E., 2013, "An exact solution of the mass transport equations for spheroidal evaporating drops," *International Journal of Heat and Mass Transfer,* **60**(1), pp. 236–240.