LOWER BOUNDS FOR THE VARIANCE OF UNIFORMLY MINIMUM VARIANCE UNBIASED ESTIMATORS

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Thesis submitted to the Graduate Faculty of the

Virginia Polytechnic Institute

in candidacy for the degree of

MASTER OF SCIENCE

in

Statistics

September 1965

Blacksburg, Virginia

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I. INTRODUCTION

1.1 Introduction

One of the most fundamental problems in statistical theory is the problem of estimation. Let $X_1, X_2, \ldots X_n$ be independent and identically distributed random variables each with distribution function $F(x,\theta)$, where θ is a parameter. One problem in estimation is to obtain a function $t = t(X_1, X_2, \ldots X_n)$ of the n random variables such that t has expectation equal to a preassigned function $g(\theta)$. t is then said to be an unbiased estimator of $g(\theta)$. Without loss of generality we will consider the function $g(\theta) = \theta$ as the object of our estimation problem. Usually θ admits more than one unbiased estimator. In such a case we use the unbiased estimator which has uniformly minimum variance when it exists. It will be shown that if a minimum variance unbiased estimator for θ exists, then it is unique.

A very useful concept in the theory of statistical estimation is that of a lower bound for the variance of all unbiased estimators. The use made of this lower bound is the following. If we have an unbiased estimator whose variance is equal to this lower bound, then it has uniformly minimum variance among all unbiased estimators. A possible practical use of a lower bound might be as follows. Suppose that we have a relatively simple unbiased estimator whose variance is only slightly greater than our lower bound. In such a case, we might be willing to use this estimator.

Another method for finding uniformly minimum variance unbiased estimators without finding the lower bound involves the concept of Completeness. This method was introduced by Blackwell and developed further by Lehmann and Scheffe, but will not be considered here.

The problem considered here is to find lower bounds for the variance of unbiased estimators and to determine when they are attained.

We will derive the lower bounds due to Cramer and Rao, Bhattacharyya, Hammersley, Chapman and Robbins, and Kiefer; discuss each on its own merits and then compare each with the other.

Throughout this paper we will denote by $X_1, X_2, \ldots X_n$, n independent identically distributed random variables each with distribution function $F(x,\theta)$. We will also use the following conventions:

$$F'(x,\theta) = p(x,\theta) \text{ if } X \text{ is discrete}$$
$$= f(x,\theta) \text{ if } X \text{ is continuous,}$$

$F'(x_1,\theta)\ldots F'(x_n,\theta) = L(\underline{x},\theta)$ (the likelihood function),

and $\int dx$ is interpreted as $\sum_x$ if X is a discrete random variable.

1.2 Uniqueness of the Minimum Variance Unbiased Estimator

In order to assure that an unbiased estimator whose variance attains a lower bound is "the" uniformly minimum variance unbiased estimator for θ we must prove the following theorem (Kendall and Stuart [8]).

<u>Theorem 1.1</u> If a minimum variance unbiased estimator for θ exists, then it is unique.

Proof. Let t_1 and t_2 be any two distinct unbiased estimators for θ and suppose that both have minimum variance V. Define a new estimator

$$t = \frac{1}{2}(t_1 + t_2). \tag{1.1}$$

Now

$$E(t) = \frac{1}{2}[E(t_1) + E(t_2)] = \theta,$$

and

$$\text{Var}(t) = \frac{1}{4}[\text{Var}(t_1) + \text{Var}(t_2) + 2\,\text{Cov}(t_1,t_2)].$$

We therefore see that t is also an unbiased estimator of θ. Using the Cauchy-Schwarz's inequality we have

$$\begin{aligned} \text{Cov}(t_1,t_2) &= \int\cdots\int(t_1-\theta)(t_2-\theta)dF(x_1,\theta)\ldots dF(x_n,\theta) \\ &\le \{\int\cdots\int(t_1-\theta)^2 dF(x_1,\theta)\ldots dF(x_n,\theta) \\ &\qquad \int\cdots\int(t_2-\theta)^2 dF(x_1,\theta)\ldots dF(x_n,\theta)\}^{1/2} \\ &\le \{\text{Var}(t_1)\cdot\text{Var}(t_2)\}^{1/2} = \{V.V\}^{1/2} \\ &\le V \end{aligned} \tag{1.2}$$

and hence

$$\begin{aligned} \text{Var}(t) &\le \frac{1}{4}(\text{Var}(t_1) + \text{Var}(t_2) + 2V) \\ &\le V. \end{aligned}$$

By hypothesis the estimator with minimum variance has variance V which implies that $\text{Cov}(t_1,t_2) = V$. Parzen [12] showed that this can only be so if $(t_1-\theta) = K(\theta)(t_2-\theta)$. Multiplying both sides of this equation by $(t_2-\theta)$ and taking expectations we obtain

$$E[(t_1-\theta)(t_2-\theta)] = K(\theta)[E(t_2-\theta)^2].$$

Therefore

$$\begin{aligned} \text{Cov }(t_1,t_2) &= K(\theta) \text{ Var }(t_2) \\ &= K(\theta)\, V. \end{aligned}$$

Since Cov $(t_1,t_2) = V$, we obtain that $K(\theta) = 1$ and $(t_1-\theta) = (t_2-\theta)$ or $t_1 = t_2$ identically. Thus if there exists a minimum variance unbiased estimator for θ, it is unique.

II. CRAMER-RAO LOWER BOUND

2.1 Derivation

Let $X_1, X_2, \ldots X_n$ be n independent identically distributed random variables each with distribution function $F(x,\theta)$.

Regularity Conditions

(i) θ may be any value in (a,b) $-\infty \leq a < b \leq \infty$.

(ii) $\frac{\partial}{\partial\theta} F'(x,\theta) < \infty$ for all x and all θ in (a,b).

(iii) $\frac{\partial}{\partial\theta} \int\cdots\int dF(x_1,\theta)\ldots dF(x_n,\theta) = \int\cdots\int \frac{\partial}{\partial\theta} dF(x_1,\theta)\ldots dF(x_n,\theta)$ for all θ in (a,b).

(iv) For any statistic $t = t(X_1, X_2, \ldots X_n)$

$\frac{\partial}{\partial\theta} \int\cdots\int t\, dF(x_1,\theta)\ldots dF(x_n,\theta) = \int\cdots\int t\, \frac{\partial}{\partial\theta} dF(x_1,\theta)\ldots dF(x_n,\theta)$ for all θ in (a,b).

(v) $E[\frac{\partial}{\partial\theta} \ln F'(x,\theta)]^2 > 0$ for all θ in (a,b).

The following theorem is due to Cramer and Rao[4],[14].

Theorem 2.1 If $E(t) = \theta$, i.e., $t = t(X_1, X_2, \ldots X_n)$ is an unbiased estimator for the parameter θ, and regularity conditions (i),...(v) are satisfied, then

$$\text{Var}(t) \geq \frac{1}{n\, E[\frac{\partial}{\partial\theta} \ln F'(x,\theta)]^2} \qquad (2.1)$$

Proof. $E(t) = \theta = \int\cdots\int t\, dF(x_1,\theta)\ldots dF(x_n,\theta)$. Taking a derivative with respect to θ we obtain

$$1 = \frac{\partial}{\partial\theta} \int\cdots\int t\, dF(x_1,\theta)\ldots dF(x_n,\theta)$$

$$= \int\cdots\int t \frac{\partial}{\partial\theta} dF(x_1,\theta)\ldots dF(x_n,\theta)$$

$$= \int\cdots\int t \Big[\frac{\frac{\partial}{\partial\theta} dF(x_1,\theta) \prod_{i=1}^{n} dF(x_i,\theta)}{dF(x_1,\theta)} + \ldots + \frac{\frac{\partial}{\partial\theta} dF(x_n,\theta) \prod_{i=1}^{n} dF(x_i,\theta)}{dF(x_n,\theta)} \Big]$$

$$= \int\cdots\int (t) \Big(\sum_{i=1}^{n} \frac{\partial}{\partial\theta} \ln F'(x_i,\theta) \Big) dF(x_1,\theta)\ldots dF(x_n,\theta) \tag{2.2}$$

since

$$\frac{\frac{\partial}{\partial\theta} F'(x_i,\theta)}{F'(x_i,\theta)} = \frac{\partial}{\partial\theta} \ln F'(x_i,\theta).$$

Define $Z = \sum_{i=1}^{n} \frac{\partial}{\partial\theta} \ln F'(x_i,\theta)$. We therefore have from (2.2)

$$1 = \int\cdots\int t\, Z\, dF(x_1,\theta)\ldots dF(x_n,\theta) \tag{2.3}$$

or $E(tZ) = 1$.

Using $1 = \int\cdots\int dF(x_1,\theta)\ldots dF(x_n,\theta)$ and taking a derivative with respect to θ we obtain

$$0 = \frac{\partial}{\partial\theta} \int\cdots\int dF(x_1,\theta)\ldots dF(x_n,\theta)$$

$$= \int\cdots\int \Big[\sum_{i=1}^{n} \frac{\partial}{\partial\theta} \ln F'(x_i,\theta) \Big] dF(x_1,\theta)\ldots dF(x_n,\theta)$$

$$= \int\cdots\int Z\, dF(x_1,\theta)\ldots dF(x_n,\theta). \tag{2.4}$$

or $E(Z) = 0$. We now have that $E(tZ) = 1$ and $E(Z) = 0$, which imply that

$$\text{Cov}(t,Z) = E(tZ) - E(t)\,E(Z) = E(tZ) = 1.$$

It also follows that

$$\mathrm{Var}\ (Z) = \mathrm{Var}[\sum_{i=1}^{n} \frac{\partial}{\partial\theta}\ln F'(x_i,\theta)]$$

$$= E[\sum_{i=1}^{n} \frac{\partial}{\partial\theta}\ln F'(x_i,\theta) - E(Z)]^2$$

$$= n\ E[\frac{\partial}{\partial\theta}\ln F'(x,\theta)]^2. \qquad (2.5)$$

Recalling that for any two variables X and Y

$$[\mathrm{Cov}(X,Y)]^2 \leq \mathrm{Var}(X)\ \mathrm{Var}(Y),$$

we obtain

$$\mathrm{Var}(t) \geq \frac{[\mathrm{Cov}(t,Z)]^2}{\mathrm{Var}(Z)}$$

$$\geq \frac{1}{n\ E[\frac{\partial}{\partial\theta}\ln F'(x,\theta)]^2}. \qquad (2.6)$$

This inequality (2.6) is the Cramer-Rao lower bound for the variance of any unbiased estimator for the parameter θ in the distribution $F(x,\theta)$.

If we add the regularity condition

(vi) $\frac{\partial^2}{\partial\theta^2}\int\cdots\int dF(x_1,\theta)\ldots dF(x_n\theta) = \int\cdots\int\frac{\partial^2}{\partial\theta^2}dF(x_1,\theta)\ldots dF(x_n,\theta)$,

then

$$\frac{\partial}{\partial\theta}\int\cdots\int dF(x_1,\theta)\ldots dF(x_n,\theta) = \int\cdots\int Z dF(x,\theta)\ldots dF(x_n,\theta) = 0$$

becomes by taking a derivative with respect to θ

$$0 = \int\cdots\int\frac{\partial}{\partial\theta}[Z\ dF(x_1,\theta)\ldots dF(x_n,\theta)]$$

$$= \int\cdots\int[\frac{\partial}{\partial\theta}Z + Z^2]\ dF(x_1,\theta)\ldots dF(x_n,\theta). \qquad (2.7)$$

Therefore

$$E[\frac{\partial}{\partial\theta}Z + Z^2] = E[\frac{\partial}{\partial\theta}Z] + E[Z^2] = 0$$

or

$$E[Z^2] = -E[\frac{\partial}{\partial\theta}Z]$$

$$= -E[\sum_{i=1}^{n} \frac{\partial^2}{\partial\theta^2}\ln F'(x_i,\theta)]$$

$$= -n\ E[\frac{\partial^2}{\partial\theta^2}\ln F'(x,\theta)]. \qquad (2.8)$$

The Cramer-Rao inequality may now be equivalently stated as

$$\mathrm{Var}(t) \geq \frac{1}{-n\ E[\frac{\partial^2}{\partial\theta^2}\ln F'(x,\theta)]} \qquad (2.9)$$

where t is any unbiased estimator for θ. The inequality (2.9) is due to Rao [14].

In most cases this later inequality adds ease in computing the Cramer-Rao lower bound.

2.2 Conditions Under Which the Cramer-Rao Lower Bound is Attained

Theorem 2.2 Assuming that the regularity conditions are satisfied, then

$$\mathrm{Var}(t) = \frac{1}{n\ E[\frac{\partial}{\partial\theta}\ln F'(x,\theta)]^2}$$

if and only if $Z = A(\theta)(t-\theta)$ where $t = t(X_1,X_2,\ldots X_n)$ is an unbiased estimator for θ, and $A(\theta)$ is some function of θ.

Lemma (Parzen[12]) For any two variables X_1 and X_2,

$[E(X_1X_2)]^2 = E[X_1^2]\ E[X_2^2]$ if and only if, for some constant A, $X_2 = AX_1$.

The above lemma applied to the random variables Z and $t-\theta$ states that

$$[\text{Cov}(Z,t)]^2 = \text{Var}(Z)\ \text{Var}(t) \tag{2.10}$$

if and only if $Z = A(\theta)(t-\theta)$ since $E(Z(t-\theta)) = \text{Cov}(Z,t)$. Since in our case Cov $(Z,t) = 1$, equation (2.10) becomes

$$\text{Var}(t) = \frac{1}{\text{Var}(Z)}$$

$$= \frac{1}{n\ E[\frac{\partial}{\partial\theta}\ln F'(x,\theta)]^2} \tag{2.11}$$

if and only if $Z = A(\theta)(t-\theta)$. In particular $A(\theta) = \frac{1}{\text{Var}(t)}$ since $\text{Var}(Z) = A(\theta)^2\ \text{Var}(t)$ and $\text{Var}(t) = \frac{1}{\text{Var}(Z)}$.

Therefore if the Cramer-Rao lower bound exists, a necessary and sufficient condition that it be attained is that Z can be written in the form

$$Z = \frac{1}{\text{Var}(t)}(t-\theta), \tag{2.12}$$

or equivelently that t may be written in the form

$$t = \frac{Z}{\text{Var}(Z)} + \theta. \tag{2.13}$$

2.3 Sufficient Statistics and the Cramer-Rao Lower Bound

Blackwell [2] showed that if a sufficient statistic exists for estimating θ, then the unbiased estimator with uniformly minimum variance is a statistic based on a sufficient statistic.

One may show Blackwell's result as follows. Let S be a sufficient statistic for θ and t be any unbiased estimator for θ. Define $T = E(t|S)$. Now

$$E(T) = E[E(t|S)] = E(t) = \theta \tag{2.14}$$

or T is an unbiased estimator of θ. It is easily seen that for any two random variables t and S (Parzen [13]) that

$$Var(t) = Var[E(t|S)] + E[Var(t|S)], \tag{2.15}$$

which implies that

$$Var(T) \leq Var(t). \tag{2.16}$$

Therefore we obtain Blackwell's result.

Fend [6] proved the following. If the regularity conditions are satisfied and if $Var(t) > 0$ for all θ, then a necessary and sufficient condition that the variance of t, where t is unbiased, achieve the Cramer-Rao lower bound is that

$$F'(x_1,\theta)\ldots F'(x_n,\theta) = \exp[tg(\theta) + g_o(\theta) + f(x_1\ldots x_n)] \tag{2.17}$$

where $\frac{\partial}{\partial\theta}g(\theta) \neq 0$ for all θ in (a,b).

2.4 Examples

Example 2.1 The Poisson Distribution

$$F'(x,\theta) = \frac{e^{-\theta}\theta^x}{x!} \qquad x = 0,1,2\ldots$$

It is easily seen that the regularity conditions are satisfied for this mass function. Continuing we see that

$$\ln F'(x,\theta) = -\theta + x \ln \theta - \ln(x!)$$

$$\frac{\partial}{\partial\theta}\ln F'(x,\theta) = -1 + \frac{x}{\theta}$$

$$\frac{\partial^2}{\partial\theta^2}\ln F'(x,\theta) = -\frac{x}{\theta^2}.$$

Therefore

$$\mathrm{Var}(t) \geq \frac{1}{-n\, E(\frac{-x}{\theta^2})}$$

$$\geq \frac{\theta}{n}$$

where t is any unbiased estimator for θ. Now $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is an unbiased estimator of θ, and has variance $\frac{\theta}{n}$. Hence $\bar{X}$ is the uniformly minimum variance unbiased estimator for θ.

Example 2.2 The Normal Distribution

$$F'(x,\theta) = (2\pi\sigma^2)^{-1/2}\exp[-\frac{(x-\theta)^2}{2\sigma^2}] \qquad -\infty < x < \infty.$$

It is easily seen that the regularity conditions are satisfied for this density function. Continuing we see that

$$\ln F'(x,\theta) = -\frac{1}{2}\ln 2\pi\sigma^2 - \frac{(x-\theta)^2}{2\sigma^2}$$

$$\frac{\partial}{\partial\theta}\ln F'(x,\theta) = \frac{x-\theta}{\sigma^2}$$

$$\frac{\partial^2}{\partial\theta^2}\ln F'(x,\theta) = -\frac{1}{\sigma^2}.$$

Therefore for any unbiased estimator t

$$\text{Var}(t) \geq \frac{1}{-n\,E(-\frac{1}{\sigma^2})}$$

$$\geq \frac{\sigma^2}{n}.$$

This lower bound is attained by the variance of the unbiased estimator $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Thus $\bar{X}$ is the uniformly minimum variance unbiased estimator for θ.

Example 2.3 The Uniform Distribution

$$F'(x,\theta) = \frac{1}{\theta} \quad 0 < x < \theta$$

$$\frac{\partial}{\partial\theta}\int\cdots\int\frac{1}{\theta^n}\,dx_1\ldots dx_n = 0,$$

whereas

$$\int\cdots\int\frac{\partial}{\partial\theta}\,\frac{1}{\theta^n}\,dx_1\ldots dx_n = \int\cdots\int -n\,\frac{1}{\theta^{n+1}}\,dx_1\ldots dx_n$$

$$= -\frac{n}{\theta}.$$

Hence regularity condition (iii) is not siatisfied, and the Cramer-Rao lower bound does not exist for this distribution.

2.5 Discussion of the Examples

In example 2.1 $Z = \frac{n}{\theta}(\bar{X}-\theta)$, which indicates that $\bar{X}$ is the uniformly minimum variance unbiased estimator for θ with variance $\frac{\theta}{n}$. Here $\bar{X}$ is sufficient for θ and

$$F'(x_1,\theta)\ldots F'(x_n,\theta) = \exp[\bar{x}n\,\ln\theta - n\theta - \ln\prod_{i=1}^{n} x_i!]$$

which is in agreement with Fend's result (2.1.7).

In example 2.2 $Z = \frac{n}{\sigma^2}(\bar{X}-\theta)$ which indicated that $\bar{X}$ is the uniformly minimum variance unbiased estimator for θ with variance $\frac{\sigma^2}{n}$. Here also $\bar{X}$ is sufficient for θ and

$$F'(x_1,\theta)\ldots F'(x_n,\theta) = \exp[\bar{x}\,\frac{n\theta}{\sigma^2} - \frac{n\theta^2}{2\sigma^2} - \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2-n\bar{x}}{2\sigma^2} - \frac{n}{2}\ln(2\pi\sigma^2)]$$

which is in agreement with Fend's result (2.17).

III. BHATTACHARYYA LOWER BOUND

3.1 Derivation

Let $X_1, X_2, \ldots X_n$ be n independent identically distributed random variables each with distribution function $F(x,\theta)$.

Regularity Conditions

(i) θ is in (a,b) $-\infty \leq a \leq b\ \infty$.

(ii) $\frac{\partial}{\partial\theta} F'(x,\theta) < \infty$ for all x and all θ in (a,b).

(iii) $\frac{\partial}{\partial\theta} \int\cdots\int dF(x_1,\theta)\ldots dF(x_n,\theta) = \int\cdots\int \frac{\partial}{\partial\theta} dF(x_1,\theta)\ldots dF(x_n,\theta)$

for all θ in (a,b).

(iv) For any statistic $t = t(X_1, X_2, \ldots X_n)$

$\frac{\partial}{\partial\theta} \int\cdots\int t\ dF(x_1,\theta)\ldots dF(x_n,\theta) = \int\cdots\int t\ \frac{\partial}{\partial\theta} dF(x_1,\theta)\ldots dF(x_n,\theta)$

for all θ in (a,b).

(v) $E[\frac{\partial}{\partial\theta} \ln F'(x,\theta)]^2 > 0$ for all θ in (a,b).

(vi) $\frac{\partial^i F'(x,\theta)}{\partial\theta^i}$ $i = 1,2,\ldots K$ are all linearly independent functions of the variables $X_1, X_2, \ldots X_n$ for some fixed K.

Theorem 3.1 Under the above regularity conditions we have the following inequality for the variance of any unbiased estimator t for the parameter θ.

$$\mathrm{Var}(t) \geq \frac{1}{J_{11} - \sum_{i=2}^{K} \sum_{j=2}^{K} J_{1i} J_{1j} (J_{ij.1})^{-1}} \qquad (3.1)$$

where $J_{ij} = E[Z_i Z_j]$ with $Z_i = \dfrac{\frac{\partial^i}{\partial\theta^i}[F'(x_1,\theta)\ldots F'(x_n,\theta)]}{F'(x_1,\theta)\ldots F'(x_n,\theta)}$

and

$$J_{ij.1} = \begin{bmatrix} J_{22} & J_{23} & \cdots & J_{2K} \\ J_{32} & J_{33} & \cdots & J_{3K} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ J_{K2} & J_{K3} & \cdots & J_{KK} \end{bmatrix}$$

The proof as given here is a special case of that of Bhattacharyya [1] but is much more detailed and complete.

Proof. Consider the function $R_K = R_K(X_1, X_2, \ldots X_n, \theta)$ defined by

$$R_K = t - \theta - \sum_{i=1}^{K} \lambda_i Z_i \tag{3.2}$$

where $t = t(X_1, X_2, \ldots X_n)$ is an unbiased estimator for θ, and λ_i, $i = 1, 2, \ldots K$, are constants to be determined.

$$E(R_K) = \int \cdots \int (t-\theta) dF(x_1,\theta) \ldots dF(x_n,\theta) - \sum_{i=1}^{K} \lambda_i \int \cdots \int Z_i dF(x_1,\theta) \ldots dF(x_n,\theta)$$

$$= 0$$

since

$$E(Z_i) = \frac{\partial^i}{\partial \theta^i} \int \cdots \int dF(x_1,\theta) \ldots dF(x_n,\theta)$$

$$= \frac{\partial^i}{\partial \theta^i} (1)$$

$$= 0.$$

Now we choose the λ_i in such a way that the variance of R_K,

$$\mathrm{Var}(R_K) = \int \cdots \int [t - \theta - \sum_{i=1}^{K} \lambda_i Z_i]^2 dF(x_1,\theta) \ldots dF(x_n,\theta), \tag{3.3}$$

is a minimum.

We differentiate (3.3) with respect to λ_j, set it equal to zero, and obtain

$$0 = \int\cdots\int [t-\theta-\sum_{i=1}^{K}\lambda_i Z_i] Z_j dF(x_1,\theta)\ldots dF(x_n,\theta)$$

$$= \int\cdots\int tZ_j dF(x_1,\theta)\ldots dF(x_n,\theta) - \sum_{i=1}^{K}\lambda_i Z_i Z_j dF(x_1,\theta)\ldots dF(x_n,\theta) \tag{3.4}$$

for J = 1,2,...K. Since $\int\cdots\int tZ_j dF(x_1,\theta)\ldots dF(x_n,\theta) = \frac{\partial^i}{\partial\theta^j}\theta$, the system of equations (3.4) can be written as

$$\begin{aligned} \sum_{i=1}^{K}\lambda_i J_{i1} &= 1 \\ \sum_{i=1}^{K}\lambda_i J_{i2} &= 0 \\ &\vdots \\ \sum_{i=1}^{K}\lambda_i J_{iK} &= 0. \end{aligned} \tag{3.5}$$

The system of equations (3.5) can be written in matrix form as

$$\begin{bmatrix} J_{11} & J_{21} & \cdots & J_{K1} \\ J_{12} & J_{22} & \cdots & J_{K2} \\ \vdots & & & \\ J_{1K} & J_{2K} & \cdots & J_{KK} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_K \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3.6}$$

Since regularity condition (vi) holds (3.6) becomes

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_K \end{bmatrix} = \begin{bmatrix} J^{11} & J^{12} & \cdots & J^{1K} \\ J^{21} & J^{22} & \cdots & J^{2K} \\ \vdots & & & \\ J^{K1} & J^{K2} & \cdots & J^{KK} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} J^{11} \\ J^{21} \\ \cdot \\ \cdot \\ \cdot \\ J^{K1} \end{bmatrix}, \tag{3.7}$$

where J^{ij} represents the (i,j) element of the inverse of the matrix J_{ij}. Therefore $\lambda_i = J^{i1}$.

Hence equation (3.3) becomes

$$\begin{aligned} Var(R_K) &= \int \cdots \int [t-\theta-\sum_{i=1}^{K} J^{i1} Z_i] dF(x_1,\theta) \ldots dF(x_n,\theta) \\ &= Var(t) - 2\sum_{i=1}^{K} J^{i1} E[(t-\theta)Z_i] + E[\sum_{i=1}^{K} J^{i1} Z_i]^2 \\ &= Var(t) - 2\sum_{i=1}^{K} J^{i1} E(tZ_i) + \sum_{i=1}^{K}\sum_{j=1}^{K} J^{i1} J^{ji} E[Z_i Z_j] \\ &= Var(t) - 2J^{11} + \sum_{i=1}^{K}\sum_{j=1}^{K} J^{i1} J^{j1} J_{ij} \\ &= Var(t) - J^{11} \end{aligned} \tag{3.8}$$

since

$$\begin{aligned} \sum_{i=1}^{K} J^{i1} \sum_{j=1}^{K} J^{j1} J_{ij} &= \sum_{i=1}^{K} J^{i1} E(t-\theta) Z_i \text{ by (3.4)} \\ &= \sum_{i=1}^{K} J^{i1} E(tZ_i) \\ &= J^{11} \end{aligned}$$

where

$$\begin{aligned} E(tZ_i) &= 1 \text{ if } i = 1 \\ &= 0 \text{ if } i = 2,3,\ldots K. \end{aligned}$$

From equation (3.8) we obtain the inequality

$$Var(t) \geq J^{11}. \tag{3.9}$$

The equality sign holds when $Var(R_K) = 0$.

By definition

$$\begin{bmatrix} J_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} J^{11} & K^{12} \\ K^{21} & K^{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \cdots\cdots & 0 \\ 0 & 1 \cdots\cdots & 0 \\ \vdots & & \\ 0 & 0 \cdots\cdots & 1 \end{bmatrix} \tag{3.10}$$

where $K_{12} = (J_{12} \ldots J_{1K})$, $K_{21} = (J_{21} \ldots J_{K1})'$, $K_{22} = J_{ij.1}$, $K^{12} = (J^{12} \ldots J^{1K})$, $K^{21} = (J^{21} \ldots J^{K1})'$, and

$$K^{22} = \begin{bmatrix} J^{22} \cdots & J^{2K} \\ \vdots & \\ J^{K2} \cdots & J^{KK} \end{bmatrix} .$$

From equation (3.10) we obtain

$$J_{11}J^{11} + K_{12}K^{21} = 1 \tag{3.11}$$

and

$$K_{21}J^{11} + K_{22}K^{21} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} . \tag{3.12}$$

Solving (3.12) for K^{21} we obtain

$$K^{21} = -(K_{22})^{-1}K_{21}J^{11}$$

and hence

$$K_{12}K^{21} = -K_{12}(K_{22})^{-1}K_{21}J^{11}. \tag{3.13}$$

Substituting equation (3.13) into equation (3.11) we obtain

$$J_{11}J^{11} - K_{12}(K_{22})^{-1}K_{21}J^{11} = 1.$$

Therefore

$$J^{11} = \frac{1}{J_{11} - K_{12}(K_{22})^{-1}K_{21}}$$

$$= \frac{1}{J_{11} - \sum_{i=2}^{K}\sum_{j=2}^{K} J_{1i} J_{1j} (J_{ij.1})^{-1}} \tag{3.14}$$

where $(J_{ij.1})^{-1}$ is as defined before. Hence (3.9) can be written

$$\text{Var}(t) \geq \frac{1}{J_{11} - \sum_{i=2}^{K}\sum_{j=2}^{K} J_{1i} J_{1j} (J_{ij.1})^{-1}} .$$

This lower bound is known as the K-th Bhattacharyya lower bound. It gives us a whole sequence of non-decreasing lower bounds as K = 1,2,... for the variance of any unbiased estimator for θ (Lehmann [10]). This set of bounds may not increase at all or they may increase toward a limiting value which may or may not be attained by the variance of any unbiased estimator for θ.

Seth [16] showed necessary and sufficient conditions under which the Bhattacharyya lower bound is attained. The conditions are that there exist an unbiased estimator t such that (1) the regularity conditions are satisfied, and (2) the probability density $g(t,\theta)$ of t satisfies the equation

$$t = \sum_{i=1}^{K} \frac{P_i}{g(t,\theta)} \frac{\partial^i}{\partial\theta^i} g(t,\theta)$$

where the P_i are constants.

He also showed that the Mth, M>1, Bhattacharyya lower bound is higher than the first Bhattacharyya lower bound if and only if $R_{1\cdot 23\ldots M}$ is not zero where $R_{1\cdot 23\ldots M}$ is the multiple correlation between $\phi_1(x)$ and $\phi_2(x)\ldots\phi_M(x)$ and where $\phi_i(x) = \frac{\partial^i}{\partial\theta^i}F'(x,\theta) \div F'(x,\theta)$. This is equivalent to the condition that for at least one i $(i\geq 2)$, the correlation coefficient between $\phi_1(x)$ and $\phi_i(x)$ is different from zero.

Fend [6] gave an important theorem which helps in establishing which Bhattacharyya lower bound is achieved by the variance of the uniformly minimum variance unbiased estimator. He showed that if the regularity conditions for the Bhattacharyya lower bound are satisfied and if for some unbiased estimator t, Var(t) achieves the K-th Bhattacharyya lower bound but not the (K-1)-th Bhattacharyya lower bound, then the density may be expressed in the form

$$F'(x_1,\theta)\ldots F'(x_n,\theta) = \exp[p(x_1,\ldots x_n)\ g(\theta) + g_o(\theta) + f(x_1,\ldots x_n)]$$

where t is a polynomial in $p(X_1,\ldots X_n)$ of degree K. Further, the variance of any polynomial in $p(X_1,\ldots X_n)$ of degree K will achieve the K-th Bhattacharyya lower bound.

Hence if

$$F'(x_1,\theta)\ldots F'(x_n,\theta) = \exp[p(x_1,\ldots x_n)\ g(\theta) + g_o(\theta) + f(x_1,\ldots x_n)]$$

we should try to find an unbiased polynomial in $p(X_1,\ldots X_n)$ for θ. If this polynomial is of degree K, then its variance will achieve the K-th Bhattacharyya lower bound and is therefore the uniformly minimum variance unbiased estimator for θ.

We can also see that if the K-th lower bound is achieved by the variance of t, then t is necessarily a polynomial of degree K in $p(X_1,\ldots X_n)$.

3.2 General Comparison with the Cramer-Rao Lower Bound

1. The existence of the Bhattacharyya lower bound requires an added regularity condition not required for the existence of the Cramer-Rao lower bound, namely that the $\frac{\partial^i}{\partial\theta^i}F'(x,\theta)$ $i = 1,2,\ldots K$ are all linearly independent functions of the variables $X_1,X_2,\ldots X_n$ for some fixed K.

2. For the case K = 1 the Bhattacharyya inequality becomes

$$\text{Var}(t) \geq \frac{1}{J_{11}} \tag{3.15}$$

where t is any unbiased estimator for θ, and where

$$J_{11} = \int\cdots\int\frac{\partial}{\partial\theta}\ln F'(x_1,\theta)\ldots F'(x_n,\theta)\cdot\frac{\partial}{\partial\theta}\ln F'(x_1,\theta)\ldots F'(x_n,\theta)\, dF(x_1,\theta)\ldots dF(x_n,\theta)$$

$$= E[\frac{\partial}{\partial\theta}\ln F'(x_1,\theta)\ldots F'(x_n,\theta)]^2$$

$$= n\ E[\frac{\partial}{\partial\theta}\ln F'(x,\theta)]^2.$$

Therefore the inequality (3.15) becomes

$$\text{Var}(t) \geq \frac{1}{n\ E[\frac{\partial}{\partial\theta}\ln F'(x,\theta)]^2} \tag{3.16}$$

which is exactly the Cramer-Rao lower bound.

3. The K-th, K>1, Bhattacharyya lower bound is higher than that obtained by employing the Cramer-Rao lower bound

if and only if for at least one i, i = 2,3,...K, the correlation coefficient between $\phi_1(x)$ and $\phi_i(x)$ is different from zero where $\phi_i(x) = \frac{\partial^i}{\partial\theta^i} F'(x,\theta) \div F'(x,\theta)$.

4. The K-th Bhattacharyya lower bound becomes increasingly difficult to compute as K increases.

5. Both the Bhattacharyya and the Cramer-Rao lower bounds are non-existant when the range of the distribution depends on the parameter θ.

3.3 Examples

Example 3.1 The Poisson Distribution

$$F'(x,\theta) = \frac{e^{-\theta}\,\theta^x}{x!} \quad x = 0,1,2,....$$

Since the Cramer-Rao lower bound is attained, the first Bhattacharyya lower bound is also attained and the calculations are identical.

Example 3.2 The Normal Distribution

$$F'(x,\theta) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp[-\frac{(x-\theta)^2}{2\sigma^2}] \quad -\infty < x < \infty.$$

Since the Cramer-Rao lower bound is attained, the first Bhattacharyya lower bound is also attained and the calculations are identical.

Example 3.3 The Uniform Distribution

$$F'(x,\theta) = \frac{1}{\theta} \quad 0 < x < \theta.$$

Regularity condition (iii) does not hold as was shown in the Cramer-Rao example. Hence none of the Bhattacharyya lower bounds exist.

Example 3.4 A Variation on the Gamma Distribution

Consider one observation from the distribution with density

$$F'(x,\theta) = (\theta)^{-1/2} \exp[-x\theta^{-1/2}] \quad 0<x,\ 0<\theta.$$

It is easily seen that for this distribution the regularity conditions for the Bhattacharyya lower bound are satisfied. Continuing we see that

$$\frac{\partial}{\partial\theta}F'(x,\theta) = \frac{-\exp(-x\theta^{-1/2})}{2\theta^{3/2}} + \frac{x\exp(-x\theta^{-1/2})}{2\theta^{2}}$$

$$\frac{\frac{\partial}{\partial\theta}F'(x,\theta)}{F'(x,\theta)} = -\frac{1}{2\theta} + \frac{x}{2\theta^{3/2}}$$

$$\frac{\partial^2}{\partial\theta^2}F'(x,\theta) = \frac{3\exp(-x\theta^{-1/2})}{4\theta^{5/2}} - \frac{5x\exp(-x\theta^{-1/2})}{4\theta^{3}} + \frac{x^2\exp(-x\theta^{-1/2})}{4\theta^{5/2}}$$

$$\frac{\frac{\partial^2}{\partial\theta^2}F'(x,\theta)}{F'(x,\theta)} = \frac{3}{4\theta^2} - \frac{5x}{4\theta^{5/2}} + \frac{x^2}{4\theta^3}.$$

It can be shown that $E(x) = \theta^{1/2}$, $E(x^2) = 2\theta$, $E(x^3) = 6\theta^{3/2}$. $E(x^4) = 24\theta^2$. Now

$$J_{11} = E[-\frac{1}{2\theta} + \frac{x}{2\theta^{3/2}}]^2 = E[\frac{1}{4\theta^2} - \frac{2x}{4\theta^{5/2}} + \frac{x^2}{4\theta^3}]$$

$$= \frac{1}{4\theta^2} - \frac{2}{4\theta^2} + \frac{2}{4\theta^2}$$

$$= \frac{1}{4\theta^2},$$

$$J_{12} = E[(\frac{3}{4\theta^2} - \frac{5x}{4\theta^{5/2}} + \frac{x^2}{4\theta^3})(-\frac{1}{2\theta} + \frac{x}{2\theta^{3/2}})]$$

$$= -\frac{1}{8\theta^3},$$

and

$$J_{22} = E[\frac{3}{4\theta^2} - \frac{5x}{4\theta^{5/2}} + \frac{x^2}{4\theta^3}]^2$$

$$= E[\frac{9}{16\theta^4} - \frac{30x}{16\theta^{9/2}} + \frac{31x^2}{16\theta^5} - \frac{10x^3}{16\theta^{11/2}} + \frac{x^4}{16\theta^6}]$$

$$= \frac{5}{16\theta^4}.$$

The first Bhattacharyya (Cramer-Rao) lower bound is given by

$$Var(t) \geq \frac{1}{J_{11}} = 4\theta^2.$$

The second Bhattacharyya lower bound is given by

$$Var(t) \geq \frac{1}{J_{11} - J_{12}J_{12}(J_{22})^{-1}}$$

$$\geq \frac{1}{\frac{1}{4\theta^2} - \frac{1}{64\theta^6}\frac{16\theta^4}{5}}$$

$$\geq 5\theta^2.$$

The estimator $X^2/2$ is unbiased for θ and has variance $5\theta^2$. Therefore $X^2/2$ is the uniformly minimum variance unbiased estimator for θ.

3.4 Discussion of the Examples

Examples 3.1 and 3.2 were discussed in section 2.5.

In example 3.4 the Cramer-Rao lower bound was $4\theta^2$ whereas the second Bhattacharyya lower bound was $5\theta^2$ which is attained by the variance of $X^2/2$. This is in accordance with Fend's [6] results, since $F'(x,\theta) = \exp[-x\theta^{-1/2} - 1/2 \ln\theta]$ and $X^2/2$ is a polynomial in X of degree two. Hence the second Bhattacharyya lower bound should be attained by the variance of the uniformly minimum variance unbiased estimator.

IV. HAMMERSLEY LOWER BOUND

4.1 Derivation

Let $X_1, X_2, \ldots X_n$ be n independent identically distributed random variables each with distribution function $F(x,\theta)$.

Regularity Conditions

(i) θ is in (a,b) $-\infty \leq a < b \leq \infty$.

(ii) $\frac{\partial}{\partial\theta} F'(x,\theta) < \infty$ for all θ in (a,b) and all x.

(iii) $E[\frac{F'(x,\theta_1)}{F'(x,\theta_2)}]^2 \neq 1$ for all θ_1, θ_2 in (a,b) where $\theta_1 \neq \theta_2$.

With the aid of the above regularity conditions, Hammersley [7] derived an inequality which gives a lower bound for the variance of any unbiased estimator for the parameter θ in the distribution $F(x,\theta)$.

Hammersley's lower bound is

$$\mathrm{Var}(t) \geq \sup_{d} \frac{d^2}{[E\{\frac{F'(x,\theta+d)}{F'(x,\theta)}\}^2]^n - 1} \qquad (4.1)$$

where $t = (X_1, X_2, \ldots X_n)$ is any unbiased estimator for θ and the sup is taken over all d $(d \neq 0)$ such that $\theta+d$ is in (a,b) (Mitra [11]).

This lower bound is derived as follows (Hammersley [7]). Let us suppose that θ_o is the true (unknown) value of the parameter θ, and that θ_1 and θ_2 are any two distinct members of the set of values of θ in (a,b). Let $L(\underline{x},\theta) = F'(x_1,\theta) \ldots F'(x_n,\theta)$ be the likelihood function. Since

$\int\cdots\int L(\underline{x},\theta)dx_1\ldots dx_n = 1$ and t is unbiased for all θ in (a,b), we have that

$$\int\cdots\int\{L(\underline{x},\theta_1)-L(\underline{x},\theta_2)\}dx_1\ldots dx_n = 0 \qquad (4.2)$$

and

$$\int\cdots\int t\{L(\underline{x},\theta_1)-L(\underline{x},\theta_2)\}dx_1\ldots dx_n = \theta_1 - \theta_2. \qquad (4.3)$$

By multiplying (4.2) by θ_2 and subtracting from (4.3) we obtain

$$\theta_1-\theta_2 = \int\cdots\int(t-\theta_2)\{L(\underline{x},\theta_1)-L(\underline{x},\theta_2)\}dx_1\ldots dx_n$$

$$= \int\cdots\int\{(t-\theta_2)[L(\underline{x},\theta_2)]^{1/2}\}\{\frac{L(\underline{x},\theta_1)-L(\underline{x},\theta_2)}{[L(\underline{x},\theta_2)]^{1/2}}\}dx_1\ldots dx_n. \qquad (4.4)$$

Applying the Cauchy-Schwarz inequality to (4.4) and rearranging we obtain

$$\mathrm{Var}(t) \geq \frac{(\theta_1 - \theta_2)^2}{\int\cdots\int\{\frac{[L(\underline{x},\theta_1)-L(\underline{x},\theta_2)]^2}{L(\underline{x},\theta_2)}\}dx_1\ldots dx_n}. \qquad (4.5)$$

Now

$$\frac{[L(\underline{x},\theta_1)-L(\underline{x},\theta_2)]^2}{L(\underline{x},\theta_2)} = \frac{[L(\underline{x},\theta_1)]^2}{L(\underline{x},\theta_2)} - 2\,L(\underline{x},\theta_1) + L(\underline{x},\theta_2). \qquad (4.6)$$

If we integrate the expression (4.6) over all x, we see that the last two terms on the right yield -2 and +1 respectfully. Therefore our inequality (4.5) becomes, since the variables X_i are independent,

$$\text{Var}(t) \geq \frac{(\theta_1 - \theta_2)^2}{\int \cdots \int \frac{[L(\underline{x},\theta_1)]^2}{L(\underline{x},\theta_2)}\, dx_1 \ldots dx_n - 1}$$

$$\geq \frac{(\theta_1 - \theta_2)^2}{\prod_{i=1}^{n} \int \frac{[F'(x_i,\theta_1)]^2}{F'(x_i,\theta_2)}\, dx_i - 1}$$

$$\geq \frac{(\theta_1 - \theta_2)^2}{\{\int \frac{[F'(x,\theta_1)]^2}{F'(x,\theta_2)}\, dx\}^n - 1} . \tag{4.7}$$

The inequality (4.7) holds for all values of θ_1 and θ_2 in (a,b). We may put $\theta_1 = \theta + d$ and $\theta_2 = \theta$ and allow $\theta+d$ ($d \neq 0$) to vary over the whole set (a,b). We therefore find that (4.7) becomes

$$\text{Var}(t) \geq \sup_d \frac{d^2}{\{E[\frac{F'(x,\theta+d)}{F'(x,\theta)}]^2\}^n - 1} , \tag{4.8}$$

where the sup is taken over all d ($d \neq 0$) such that $\theta+d$ is in (a,b).

4.2 Comparison with the Bhattacharyya Lower Bound

1. The existence of the Hammersley lower bound requires less restrictive regularity conditions than those required for the existence of the Bhattacharyya lower bound.

2. The author has not found a general inequality that exists between the Hammersley and the K-th Bhattacharyya lower bound. However given that both the first Bhattacharyya

(Cramer-Rao) and the Hammersley lower bounds exist, then (Chapman and Robbins [3])

$$
\begin{aligned}
n\,E[\tfrac{\partial}{\partial\theta}\ln F'(x,\theta)]^2 &= E[\lim_{d\to 0}\{\frac{L(\underline{x},\theta+d)-L(\underline{x},\theta)}{d\,L(\underline{x},\theta)}\}^2] \\
&= \lim_{d\to 0}\frac{1}{d^2}\,E[\{\frac{L(\underline{x},\theta+d)-L(\underline{x},\theta)}{L(\underline{x},\theta)}\}^2] \\
&\geq \inf_{d}\frac{1}{d^2}\,E[\{\frac{L(\underline{x},\theta+d)-L(\underline{x},\theta)}{L(\underline{x},\theta)}\}^2] \\
&\geq \inf_{d}\frac{1}{d^2}\,[E\{\frac{L(\underline{x},\theta+d)}{L(\underline{x},\theta)}\}^2-1] \\
&\geq \inf_{d}\frac{1}{d^2}\,\{[E\{\frac{F'(x,\theta+d)}{F'(x,\theta)}\}^2]^n-1\}
\end{aligned}
$$

which is the denominator of the Hammersley lower bound. Therefore we may conclude that under fewer regularity conditions, the Hammersley lower bound will be greater than or equal to the first Bhattacharyya (Cramer-Rao) lower bound. We can now be assured that in those distributions in which the first Bhattacharyya (Cramer-Rao) lower bound exists and is attained by the variance of the uniformly minimum variance unbiased estimator that the Hammersley lower bound will exist and be attained by the variance of this same estimator.

4.3 Examples

Example 4.1 The Poisson Distribution

Consider n independent observations from the Poisson distribution with mass function

$$F'(x,\theta) = \frac{e^{-\theta}\,\theta^{x}}{x!} \qquad x = 0,1,2\ldots$$

It is easily seen that the regularity conditions of the Hammersley lower bound are satisfied for this mass function. Continuing we see that

$$E\left[\frac{F'(x,\theta+d)}{F'(x,\theta)}\right]^2 = \sum_{x=0}^{\infty} \frac{e^{-2(\theta+d)}\,(\theta+d)^{2x}}{e^{-\theta}\,\theta^{x}\,x!}$$

$$= e^{-\theta}e^{-2d}\sum_{x=0}^{\infty}\frac{\left[\frac{(\theta+d)^2}{\theta}\right]^x}{x!}$$

$$= e^{-\theta}e^{-2d}e^{\frac{(\theta+d)^2}{\theta}} .$$

Therefore

$$\sup_d \frac{d^2}{\left[E\left\{\frac{F'(x,\theta+d)}{F'(x,\theta)}\right\}^2\right]^n - 1} = \sup_d \frac{d^2}{e^{\frac{nd^2}{\theta}} - 1}$$

$$= \sup_d \frac{d^2}{\left(1 + \frac{nd^2}{\theta} + \frac{n^2d^4}{2\theta^2} + \ldots\right) - 1}$$

$$= \lim_{d\to 0} \frac{1}{\frac{n}{\theta} + \frac{n^2d^2}{2\theta^2} + \ldots}$$

$$= \frac{\theta}{n}$$

which is also the Cramer-Rao lower bound.

Example 4.2 The Normal Distribution

Consider n independent observations from the Normal distribution with density function

$$F'(x,\theta) = (2\pi\sigma^2)^{-1/2}\exp\left[-\frac{(x-\theta)^2}{2\sigma^2}\right] \qquad -\infty < x < \infty.$$

It is easily seen that the regularity conditions of the Hammersley lower bound are satisfied for this density function. Continuing we see that

$$\frac{[F'(x,\theta+d)]^2}{F'(x,\theta)} = e^{(d/\sigma)^2}\,(2\pi\sigma^2)^{-1/2}\exp\left[-\frac{(x-\theta-2d)^2}{2\sigma^2}\right]$$

and

$$\sup_d \frac{d^2}{[E\{\frac{F'(x,\theta+d)}{F'(x,\theta)}\}^2]^n-1} = \sup_d \frac{d^2}{e^{n(d/\sigma)^2}}\cdot$$

$$\cdot\frac{1}{\int_{-\infty}^{\infty}(2\pi\sigma^2)^{-1/2}\exp\left[-\frac{(x-\theta-2d)^2}{2\sigma^2}\right]dx-1}$$

$$= \sup_d \frac{d^2}{e^{n(d/\sigma)^2}-1}$$

$$= \sup_d \frac{d^2}{[1+\frac{nd^2}{\sigma^2}+\frac{n^2d^4}{\sigma^4}+\ldots]-1}$$

$$= \sup_d \frac{1}{\frac{n}{\sigma^2}+\frac{n^2d^2}{\sigma^4}+\frac{n^3d^4}{\sigma^6}+\ldots}$$

$$= \sigma^2/n$$

which is also the Cramer-Rao lower bound.

Example 4.3 The Uniform Distribution

Consider one observation from the distribution $F(x,\theta)$ with density function

$$F'(x,\theta) = \frac{1}{\theta} \quad 0 < x < \theta.$$

It is easily seen that for this distribution the regularity conditions of the Hammersley lower bound are satisfied. Continuing we see that

$$E[\frac{F'(x,\theta+d)}{F'(x,\theta)}]^2 = \frac{\theta}{\theta+d} .$$

The inequality (4.1) becomes

$$\text{Var}(t) \geq \sup_d \frac{d^2}{\frac{\theta}{\theta+d} -1} = -d(\theta+d).$$

Since $\frac{\partial}{\partial d}(-\theta d-d^2) = 0$ when $d = -\frac{\theta}{2}$, the inequality (4.1) finally becomes

$$\text{Var}(t) \geq \frac{\theta^2}{4}.$$

The statistic 2x is unbiased for θ. Davis [5] showed that the statistic 2x is the uniformly minimum variance unbiased estimator for θ with variance $\frac{\theta^2}{3}$. The Hammersley lower bound therefore exists but is not attained.

Example 4.4 The Gamma Distribution (Mitra [11])

Consider n independent observations from the Gamma distribution with density function

$$F'(x,\theta) = \frac{\theta^p}{\Gamma(p)} x^{p-1} e^{-\theta x} \quad 0 \leq x,\ \theta > 0.$$

It is easily seen that for this distribution the regularity conditions of the Hammersley lower bound are satisfied. Continuing we see that

$$E[\frac{F'(x,\theta+d)}{F(x,\theta)}]^2 = \int_0^\infty \frac{(\theta+d)^{2p} x^{p-1} e^{-2(\theta+d)x}}{\theta^p \Gamma(p) e^{-\theta x}} dx$$

$$= (\frac{\theta+d}{\theta})^{2p} (\frac{\theta}{\theta+2d})^p \int_0^\infty \frac{(\theta+2d)^p}{\Gamma(p)} x^{p-1} e^{-(\theta+2d)x} dx$$

$$= (\frac{\theta+d}{\theta})^{2p} (\frac{\theta}{\theta+2d})^p.$$

Hence for any unbiased statistic t

$$\mathrm{Var}(t) \geq \sup_{d} \frac{d^2}{\dfrac{(1+\frac{d}{\theta})^{2pn}}{(1+\frac{2d}{\theta})^{pn}} - 1} \quad . \tag{4.9}$$

Let $d = \frac{\theta\alpha}{n}$ where α is a constant not depending on n. Now (4.9) becomes

$$\mathrm{Var}(t) \geq \sup_{\alpha} \frac{\alpha^2}{\dfrac{(1+\frac{\alpha}{n})^{2pn}}{(1+\frac{2\alpha}{n})^{pn}} - 1} \left(\frac{\theta^2}{n^2}\right) \quad . \tag{4.10}$$

Now

$$\ln\left[\frac{(1+\frac{\alpha}{n})^{2pn}}{(1+\frac{2\alpha}{n})^{np}}\right] = 2pn \ln\left(1+\frac{\alpha}{n}\right) - pn \ln\left(1+\frac{2\alpha}{n}\right)$$

$$= 2pn\left[\frac{\alpha}{n} - \frac{1}{2}\left(\frac{\alpha}{n}\right)^2 + \frac{1}{3}\left(\frac{\alpha}{n}\right)^3 - \ldots\right]$$

$$- pn\left[\frac{2\alpha}{n} - \frac{1}{2}\left(\frac{2\alpha}{n}\right)^2 + \frac{1}{3}\left(\frac{2\alpha}{n}\right)^3 - \ldots\right]$$

$$= \frac{p\alpha^2}{n} - \frac{2p\alpha^3}{n^2} + 0\left(\frac{1}{n^3}\right).$$

Hence (4.10) becomes

$$\mathrm{Var}(t) \geq \sup_{\alpha} \frac{\theta^2}{n^2} \frac{\alpha^2}{\exp\left[\frac{p\alpha^2}{n} - \frac{2p\alpha^3}{n^2} + 0\left(\frac{1}{n^3}\right)\right] - 1}$$

$$\geq \sup_{\alpha} \frac{\theta^2}{np - \frac{(p^2\alpha^2 - 4p\alpha)}{2} + 0\left(\frac{1}{n}\right)}$$

$$\geq \frac{\theta^2}{np - 2 + 0\left(\frac{1}{n}\right)} \tag{4.11}$$

since $\frac{\partial}{\partial\alpha}\,(p^2\alpha^2 - 4p\alpha) = 0$ when $\alpha = \frac{2}{p}$. The estimator $\frac{np-1}{\bar{X}}$ is unbiased for θ and Rao [15] showed that it is the uniformly minimum variance unbiased estimator for θ with variance $\frac{\theta^2}{np-2}$.

4.4 Discussion of the Examples

In examples 4.1 and 4.2 the same lower bounds are obtained as were obtained by the techniques of Cramer-Rao and Bhattacharyya.

In example 4.3 the Cramer-Rao and Bhattacharyya lower bounds do not exist. The Hammersley lower bound exists, but is not attained by the variance of the uniformly minimum variance unbiased estimator for θ.

In example 4.4 Hammersley's technique gives a lower bound which is only slightly smaller than the variance of the uniformly minimum variance unbiased estimator for θ (the difference being of order $\frac{1}{n^3}$).

V. CHAPMAN AND ROBBINS LOWER BOUND

5.1 Derivation (Chapman and Robbins [3])

Let $X_1, X_2, \ldots X_n$ be n independent identically distributed random variables each with distribution function $F(x,\theta)$. Let θ be a real parameter belonging to some set Ω. Let χ be the whole space and define $S(\theta)$ a subset of χ as follows:

$$F'(x,\theta) > 0 \text{ for all } x \text{ in } S(\theta)$$

$$F'(x,\theta) = 0 \text{ for all } x \text{ in } \chi - S(\theta).$$

Let $t = t(X_1, X_2, \ldots X_n)$ be any unbiased estimator for θ, such that for all θ in Ω

$$\int \cdots \int_{\chi} t\, L(\underline{x},\theta)\, dx_1 \ldots dx_n = \theta$$

where $L(\underline{x},\theta) = F'(x_1,\theta)F'(x_2,\theta)\ldots F'(x_n,\theta)$ is the likelihood function.

Now if θ and $\theta+h$ are any two distinct values belonging to Ω such that $S(\theta+h)$ is a subset of $S(\theta)$, then

$$\int \cdots \int_{S(\theta)} L(\underline{x},\theta)\, dx_1 \ldots dx_n = 1,$$

$$\int \cdots \int_{S(\theta+h)} L(\underline{x},\theta+h)\, dx_1 \ldots dx_n = \int \cdots \int_{S(\theta)} L(\underline{x},\theta+h)\, dx_1 \ldots dx_n = 1,$$

$$\int \cdots \int_{S(\theta)} t\, L(\underline{x},\theta)\, dx_1 \ldots dx_n = \theta, \text{ and}$$

$$\int \cdots \int_{S(\theta)} t\, L(\underline{x},\theta+h)\, dx_1 \ldots dx_n = \theta+h.$$

Therefore

$$\int \cdots \int_{S(\theta)} (t-\theta)\,[L(\underline{x},\theta)]^{1/2}\, \frac{L(\underline{x},\theta+h)-L(\underline{x},\theta)}{h\, L(\underline{x},\theta)} \cdot [L(\underline{x},\theta)]^{1/2}\, dx_1 \ldots dx_n = 1. \tag{5.1}$$

Applying the Cauchy-Schwarz inequality to (5.1) we obtain the relation

$$1 \le \int\cdots\int_{S(\theta)} (t-\theta)^2 \, L(\underline{x},\theta)dx_1\ldots dx_n \cdot$$

$$\cdot \int\cdots\int_{S(\theta)} \left[\frac{L(\underline{x},\theta+h)-L(\underline{x},\theta)}{h\, L(\underline{x},\theta)}\right]^2 L(\underline{x},\theta)dx_1\ldots dx_n$$

$$\le \text{Var}(t) \cdot \frac{1}{h^2}\left\{\int\cdots\int_{S(\theta)}\left[\frac{L(\underline{x},\theta+h)}{L(\underline{x},\theta)}\right]^2 L(\underline{x},\theta)dx_1\ldots dx_n - 1\right\}, \tag{5.2}$$

since

$$\frac{[L(\underline{x},\theta+h)-L(\underline{x},\theta)]^2}{L(\underline{x},\theta)} = \frac{[L(\underline{x},\theta+h)]^2}{L(\underline{x},\theta)} - 2L(\underline{x},\theta+h) + L(\underline{x},\theta).$$

Therefore the inequality (5.2) can be written as

$$\text{Var}(t) \ge \frac{1}{E\left[\frac{1}{h^2}\left\{\left[\frac{L(\underline{x},\theta+h)}{L(\underline{x},\theta)}\right]^2 - 1\right\}\right]}. \tag{5.3}$$

Since the inequality (5.3) holds whenever θ and $\theta+h$ are any two distinct elements of Ω such that $S(\theta+h)$ is a subset of $S(\theta)$ we obtain the fundamental inequality

$$\text{Var}(t) \ge \frac{1}{\inf_h E\left[\frac{1}{h^2}\left\{\left[\frac{L(\underline{x},\theta+h)}{L(\underline{x},\theta)}\right]^2 - 1\right\}\right]}, \tag{5.4}$$

where the infimum is taken over all h ($h \neq 0$) such that $S(\theta+h)$ is a subset of $S(\theta)$.

The inequality (5.4) gives a lower bound for the variance of any unbiased estimator $t = t(X_1, X_2, \ldots X_n)$ with the only restrictions (i) θ belongs to some set Ω, (ii) for each θ there exists at least one h ($h \neq 0$) such that θ and $\theta+h$ belong to Ω and such that $S(\theta+h)$ is a subset of $S(\theta)$.

5.2 General Comparison With Other Lower Bounds

Comparison with the Hammersley Lower Bound

1. The Hammersley lower bound and the Chapman and Robbins lower bound are similar in that neither requires differentiation under the intergal sign.

2. They were both derived in a similar manner using the concept of differences and applying the Cauchy-Schwarz inequality. In fact, it is easy to show that they are identical when $\Omega = (a,b)$ and χ is the real number line. In this case the Chapman and Robbins lower bound,

$$\frac{1}{\inf\limits_h E[\frac{1}{h^2}\{[\frac{L(\underline{x},\theta+h)}{L(\underline{x},\theta)}]^2-1\}]} = \sup_h \frac{h^2}{E[\frac{L(\underline{x},\theta+h)}{L(\underline{x},\theta)}]^2-1}$$

$$= \sup_h \frac{h^2}{\{E[\frac{F'(x,\theta+h)}{F'(x,\theta)}]^2\}^n-1}$$

which is Hammersley's lower bound.

Comparison with the Bhattacharyya and Cramer-Rao Lower Bounds

The regularity conditions for the Bhattacharyya and Cramer-Rao lower bounds include the assumptions that $\Omega = (a,b)$ and χ is the real line. In this case the Chapman and Robbins lower bound is identical to the Hammersley lower bound. The comparisons are therefore identical to those of section 4.2.

5.3 Examples

The only examples this author has been able to find assume that $\Omega = (a,b)$ and χ is the real line. Examples of this type have already been given in section (4.3).

VI. KIEFER LOWER BOUND

6.1 Derivation (Kiefer [9])

Let $X_1, X_2, \ldots X_n$ be n independent identically distributed random variables each with distribution function $F(x,\theta)$, where θ is a parameter belonging to some set Ω and where x belongs to some set χ. For each θ, let $\Omega_\theta = \{h | (\theta+h)$ is in $\Omega\}$. For fixed θ, let $\lambda_1(h)$ and $\lambda_2(h)$ be two completely arbitary distribution functions such that $E_i(h) = \int_{\Omega_\theta} h \, d\lambda_i(h)$ exists for i = 1,2. Let $L(\underline{x},\theta)$ be the likelihood function.

Now for any estimator $t = t(X_1, X_2, \ldots X_n)$ which is unbiased for θ, θ in Ω, we have for i = 1,2

$$\int_{\Omega_\theta} [\int\cdots\int (t-\theta) L(\underline{x},\theta+h) dx_1 \ldots dx_n] d\lambda_i(h) = E_i(h). \quad (6.1)$$

Assuming we can interchange the order of integration we can write

$$\begin{aligned} E_1(h)-E_2(h) &= \int_{\Omega_\theta} [\int\cdots\int (t-\theta) L(\underline{x},\theta+h) dx_1 \ldots dx_n] d(\lambda_1-\lambda_2) \\ &= \int\cdots\int (t-\theta) [\int_{\Omega_\theta} L(\underline{x},\theta+h) d(\lambda_1-\lambda_2)] dx_1 \ldots dx_n \\ &= \int\cdots\int (t-\theta) [L(\underline{x},\theta)]^{1/2} [\frac{\int_{\Omega_\theta} L(\underline{x},\theta+h) d(\lambda_1-\lambda_2)}{[L(\underline{x},\theta)]^{1/2}}] \\ &\qquad dx_1 \ldots dx_n. \end{aligned} \quad (6.2)$$

Using the Cauchy-Schwarz inequality we obtain from (6.2)

$$[E_1(h)-E_2(h)]^2 \leq \int\cdots\int (t-\theta)^2 L(\underline{x},\theta) dx_1 \ldots dx_n. \cdot \int\cdots\int [\frac{\int_{\Omega_\theta} L(\underline{x},\theta+h) d(\lambda_1-\lambda_2)}{(L(\underline{x},\theta))^{1/2}}]^2 dx_1 \ldots dx_n \quad (6.3)$$

which on rearrangement becomes

$$\mathrm{Var}(t) \geq \frac{[E_1(h) - E_2(h)]^2}{\int\cdots\int\{\frac{[\int_{\Omega_\theta} L(\underline{x},\theta+h)\,d(\lambda_1-\lambda_2)]^2}{L(\underline{x},\theta)}\,dx_1\ldots dx_n} .$$

The inequality (6.4) is true for every λ_1 and λ_2 and therefore we obtain

$$\mathrm{Var}(t) \geq \sup_{\lambda_1,\lambda_2} \frac{[E_1(h) - E_2(h)]^2}{E[\{\frac{\int_{\Omega_\theta} L(\underline{x},\theta+h)\,d(\lambda_1-\lambda_2)}{L(\underline{x},\theta)}\}^2]} \tag{6.5}$$

where the supremum is taken over all λ_1 and λ_2 for which $\lambda_1 \neq \lambda_2$ and for which the expectation in the denominator is defined.

This lower bound is subject to the difficulty of locating appropriate forms of $\lambda_1(h)$ and $\lambda_2(h)$ in each case of interest.

6.2 General Comparison With Other Lower Bounds

Comparison with the Chapman and Robbins Lower Bound

1. Neither the Kiefer lower bound nor the Chapman and Robbins lower bound require differentiation under the intergal sign to ensure their existence.

2. If in the Kiefer lower bound $\lambda_2(h)$ is allowed to degenerate to a point at h = 0, (6.5) becomes (Kiefer [9])

$$\mathrm{Var}(t) \geq \sup_{\lambda_1} \frac{[E_1(h)]^2}{E[\{\frac{\int_{\Omega_\theta} L(\underline{x},\theta+h)\,d\lambda_1}{L(\underline{x},\theta)}\}^2]-1} . \tag{6.6}$$

If we further allow $\lambda_1(h)$ to degenerate to a point $h \neq 0$, then the Kiefer lower bound becomes

$$\mathrm{Var}(t) \geq \frac{1}{\inf_h \frac{1}{h^2} \{E[\frac{L(\underline{x},\theta+h)}{L(\underline{x},\theta)}]^2 - 1\}} \tag{6.7}$$

which is precisely the Chapman and Robbins lower bound for the variance of any unbiased estimator t.

Therefore it follows that in general Kiefer's lower bound is at least as good as that of Chapman and Robbins. In fact if the Chapman and Robbins lower bound is attained by the variance of some unbiased estimator, then Kiefer's lower bound is attained by the variance of this same unbiased estimator.

3. Chapman and Robbins lower bound may be obtained directly whereas Kiefer's lower bound is subject to the difficulty of locating the appropriate forms of $\lambda_1(h)$ and $\lambda_2(h)$ for each problem.

Comparison with the Hammersley Lower Bound

The comparison between the Kiefer and Hammersley lower bounds can only be made when Ω is an interval (a,b) and χ is the real line. In this case the Hammersley lower bound is identical to the Chapman and Robbins lower bound and these comparisons have already been made.

Comparison with the Bhattacharyya and Cramer-Rao Lower Bounds

1. The existance of the Bhattacharyya and Cramer-Rao lower bounds requires more restrictive regularity conditions than those required for the existence of the Kiefer lower bound.

2. The author has not found a general inequality that exists between the K-th Bhattacharyya lower bound and the Kiefer lower bound. However as was shown, the Kiefer lower bound is always greater than or equal to the Chapman and Robbins lower bound, and the Chapman and Robbins lower bound is always greater than or equal to the first Bhattacharyya (Cramer-Rao) lower bound. Hence the Kiefer lower bound is always greater than or equal to the first Bhattacharyya (Cramer-Rao) lower bound.

6.3 Examples

Example 6.1 The Poisson Distribution

Consider n independent observations from the Poisson distribution with mass function

$$F'(x,\theta) = \frac{e^{-\theta}\,\theta^{x}}{x!} \qquad x = 0,1,2....$$

For this distribution the Kiefer lower bound gives the inequality

$$\mathrm{Var}(t) \geq \sup_{\lambda_1,\lambda_2} \frac{[E_1(h) - E_2(h)]^2}{E[\{\frac{\int_{-\theta}^{\infty} e^{-n(\theta+h)}(\theta+h)^{nx}\,d(\lambda_1-\lambda_2)}{e^{-n\theta}\theta^{nx}}\}^2]} . \qquad (6.8)$$

If in equation (6.8) $\lambda_2(h)$ is allowed to degenerate to a point at h=0 and then $\lambda_1(h)$ is allowed to degenerate to a point at $h \neq 0$, then equation (6.8) becomes

$$\mathrm{Var}(t) \geq \sup_h \frac{h^2}{[E\{\frac{e^{-(\theta+h)}(\theta+h)^x}{e^{-\theta}\theta^x}\}^2]^n - 1}$$

$$\geq \theta/n$$

as was shown in example 4.1.

Example 6.2 The Normal Distribution

Consider n independent observations from the Normal distribution with density function

$$F'(x,\theta) = (2\pi\sigma^2)^{-1/2} \exp[-\frac{(x-\theta)^2}{2\sigma^2}] \quad -\infty < x < \infty .$$

For this distribution the Kiefer lower bound gives the inequality

$$\mathrm{Var}(t) \geq \sup_{\lambda_1,\lambda_2} \frac{[E_1(h) - E_2(h)]^2}{E[\{\frac{\int_{-\infty}^{\infty} \exp[\frac{-n(x-\theta-h)^2}{2\sigma^2}] d(\lambda_1-\lambda_2)}{\exp[\frac{-n(x-\theta)^2}{2\sigma^2}]}\}^2]} . \quad (6.9)$$

If in equation (6.9) $\lambda_2(h)$ is allowed to degenerate to a point at h=0 and then $\lambda_1(h)$ is allowed to degenerate to a point at $h \neq 0$, then equation (6.9) becomes

$$\mathrm{Var}(t) \geq \sup_h \frac{h^2}{[E\{\frac{\exp[\frac{-(x-\theta-h)^2}{2\sigma^2}]}{\exp[\frac{-(x-\theta)^2}{2\sigma^2}]}\}^2]^n - 1}$$

$$\geq \sigma^2/n$$

as was shown in example 4.2.

It is interesting to note that two examples given by Kiefer [9] are incorrect. The examples are the following:

Example 6.3 The Uniform Distribution

Consider n independent observations from the Uniform distribution with density function

$$F'(x,\theta) = \frac{1}{\theta} \qquad 0 < x < \theta.$$

Kiefer lets $d\lambda_2(h)$ degenerate to a point at h=0 and lets

$$d\lambda_1(h) = \frac{n+1}{\theta^{n+1}}(h+\theta)^n dh \qquad -\theta < h < 0.$$

He claims that his lower bound gives $\frac{\theta^2}{n(n+2)}$, which is the variance of the uniformly minimum variance unbiased estimator $\frac{n+1}{n}Y$ where Y is the maximum of the n observations. In checking the rationale one finds the Kiefer lower bound to equal $\frac{\theta^2}{n(n+2)^3}$ which is not attained by the variance of the estimator $\frac{n+1}{n}Y$.

Example 6.4 The Exponential Distribution

Consider n independent observation for the Exponential distribution with density function

$$F'(x,\theta) = e^{-(x-\theta)} \qquad x \geq \theta.$$

Kiefer lets $d\lambda_2(h)$ degenerate to a point at h=0 and lets

$$d\lambda_1(h) = n\, e^{-nh}\, dh \qquad 0 < h < \infty.$$

He claims that his lower bound gives $\frac{1}{n^2}$, which is actually attained as the variance of the unbiased estimator $Z-\frac{1}{n}$, where Z is the minimum of the n observations.

In checking the rationale here one finds that

$$\int_{\Omega_\theta} L(\underline{x},\theta+h)\,d\lambda_1(h)$$

is infinite, which implies that the lower bound

$$\sup_{\lambda_1} \frac{[E_1(h)]^2}{E[\{\frac{\int_{\Omega_\theta} L(\underline{x},\theta+h)\,d\lambda_1}{L(\underline{x},\theta)}\}^2]-1}$$

is zero.

6.4 Discussion of the Examples

Example 6.1 and 6.2 follow by letting the Kiefer lower bound degenerate to a point and hence obtaining the Hammersley lower bound which is attained by the variance of the uniformly minimum variance unbiased estimator for θ.

The author has not been able to find an example in which the Kiefer lower bound is greater than the Chapman and Robbins lower bound, although one may exist. Kiefer [9] gives two examples in which he attempts to show that his lower bound is greater than that of Chapman and Robbins. These examples however do not seem to be correct.

VII. SUMMARY

The problem considered is that of finding a lower bound for the variance of the uniformly minimum variance unbiased estimator.

The lower bounds due to: Cramer and Rao; Bhattacharyya; Hammersley; Chapman and Robbins; and Kiefer; are derived and discussed. For the Cramer-Rao lower bound, necessary and sufficient conditions are given, which ensure that the variance of the uniformly minimum variance unbiased estimator attains this lower bound. Examples illustrating each lower bound are given.

The lower bounds are compared with each other. It was found that: the first Bhattacharyya lower bound is identical to the Cramer-Rao lower bound; the Chapman and Robbins lower bound is equal to the Hammersley lower bound whenever the latter exists; the Hammersley, Chapman and Robbins, and Bhattacharyya lower bounds are always greater than or equal to the Cramer-Rao lower bound. A summary of other comparisons of the various lower bounds is illustrated in Table 6.1, page 50.

Table 6.1 Comparison of Lower Bounds

Lower Bound	When lower bound is attained other lower bounds that are also attained.	Form of distribution function when lower bound is attained.
Cramer-Rao	Bhattacharyya Hammersley Chapman and Robbins Kiefer	$F'(x_1,\theta)\ldots F'(x_n,\theta) = \exp[t\, g(\theta) + g_o(\theta) + f(x_1\ldots x_n)]$.
Bhattacharyya	One considers the particular problem here.	$F'(x_1,\theta)\ldots F'(x_n,\theta) = \exp[p(x_1\ldots x_n)g(\theta) + g_o(\theta) + f(x_1\ldots x_n)]$ where t is a polynomial in $p(x_1\ldots x_n)$ of degree K.
Hammersley	Chapman and Robbins Kiefer	
Chapman and Robbins	Hammersley Kiefer	
Kiefer		

VIII. ACKNOWLEDGEMENTS

The author wishes to express his appreciation to the following:

Dr. R. G. Krutchkoff for serving as chairman of his thesis committee and whose assistance, guidance, and criticism made this paper possible.

Dr. J. Pickands, III for his reading and his corrections of the manuscript.

Dr. Boyd Harshbarger for his encouragement and general interest in the work.

His wife, Roie Ellen, for her conscientious typing of this paper.

The Department of Health, Education, and Welfare for their financial assistance.

IX. BIBLIOGRAPHY

1. Bhattacharyya, A. (1946). On some analogues of the amount of information and their use in statistical estimation. _Sankhya_ _8_, 1-14.

2. Blackwell, D. (1947). Conditional expectation and unbiased sequential estimation. _Annals of Mathematical Statistics_ _18_, 105-110.

3. Chapman, D. G. and Robbins, H. (1951). Minimum variance estimation without regularity assumptions. _Annals of Mathematical Statistics_ _22_, 581-585.

4. Cramer, H. (1946). _Mathematical Methods of Statistics_. Princeton University Press.

5. Davis, R. C. (1951). On minimum variance in non-regular estimation. _Annals of Mathematical Statistics_ _22_, 48.

6. Fend, A. V. (1959). On the attainment of Cramer-Rao and Bhattacharyya bounds for the variance of an estimate. _Annals of Mathematical Statistics_ _30_, 381-388.

7. Hammersley, J. M. (1950). On estimating restricted parameters. _Journal of the Royal Statistical Society Series B_ _12_, 192-229.

8. Kendall, M. G. and Stuart, A. (1961). _The Advanced Theory of Statistics II_. Hafner: New York.

9. Kiefer, J. (1952). On minimum variance estimators. _Annals of Mathematical Statistics_ _23_, 627-629.

10. Lehmann, E. L. (1950). _Notes on the Theory of Estimation_. University of California Press.

11. Mitra, S. K. (1954). A note on minimum variance in unbiased estimation. _Sankhya_ _14_, 53-60.

12. Parzen, E. (1960). _Modern Probability Theory and its Applications_. John Wiley and Sons: New York.

13. Parzen, E. (1962). _Stochastic Processes_. Holden-Day: San Francisco.

14. Rao, C. R. (1945). Information and the accuracy attainable in the estimation of statistical parameters. Calcutta Mathematical Society Bulletin 37, 81-91.

15. Rao, C. R. (1949). Sufficient statistics and minimum variance estimates. Proceedings of the Cambridge Philosophical Society 45, 213-218.

16. Seth, G. R. (1949). On the variance of estimates. Annals of Mathematical Statistics 20, 1-27.

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ABSTRACT

The object of this paper was to study lower bounds for the variance of uniformly minimum variance unbiased estimators.

The lower bounds of Cramer and Rao, Bhattacharyya, Hammersley, Chapman and Robbins, and Kiefer were derived and discussed. Each was compared with the other, showing their relative merits and shortcomings.

Of the lower bounds considered all are greater than or equal to the Cramer-Rao lower bound. The Kiefer lower bound is as good as any of the others, or better.

We were able to show that the Cramer-Rao lower bound is exactly the first Bhattacharyya lower bound. The Hammersley and the Chapman and Robbins lower bounds are identical when they both have the same parameter space, i.e., when $\Omega = (a,b)$.

The use of the various lower bounds is illustrated in examples throughout the paper.