Superconvergence and *A posteriori* Error Estimation for the Discontinuous Galerkin Method Applied to Hyperbolic Problems on Triangular Meshes

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ABSTRACT

In this thesis, we present new superconvergence properties of discontinuous Galerkin (DG) methods for two-dimensional hyperbolic problems. We investigate the superconvergence properties of the DG method applied to scalar first-order hyperbolic partial differential equations on triangular meshes. We study the effect of finite element spaces on the superconvergence properties of DG solutions on three types of triangular elements. Superconvergence is described for structured and unstructured meshes. We show that the DG solution is $O(h^{p+2})$ superconvergent at Legendre points on the $outflow$ edge on triangles having one $outflow$ edge using three $p-$ degree polynomial spaces. For triangles having two $outflow$ edges the finite element error is $O(h^{p+2})$ superconvergent at the end points of the $inflow$ edge for an augmented space of degree p. Furthermore, we discovered additional mesh-orientation dependent superconvergence points in the interior of triangles. The dependence of these points on orientation is explicitly given. We also established a global superconvergence result on meshes consisting of triangles having one $inflow$ and one $outflow$ edges.

Applying a local error analysis, we construct simple, efficient and asymptotically correct *a posteriori* error estimates for discontinuous finite element solutions of hyperbolic problems on triangular meshes. *A posteriori* error estimates are needed to guide adaptive enrichment and to provide a measure of solution accuracy for any numerical method. We develop an inexpensive superconvergence-based *a posteriori* error estimation technique for the DG solutions of conservation laws. We explicitly write the basis functions for the error spaces corresponding to several finite element solution spaces. The leading term of the discretization error on each triangle is estimated by solving a local problem where no boundary conditions are needed. The computed error estimates are shown to converge to the true error under mesh refinement in smooth solution regions. We further present a numerical study of superconvergence properties for the DG method applied to time-dependent convection problems. We also construct asymptotically correct *a posteriori* error estimates by solving local hyperbolic problems with no boundary conditions on general unstructured meshes. The global superconvergence results are numerically confirmed. Finally, the *a posteriori* error estimates are tested on several linear and nonlinear problems to show their efficiency and accuracy under mesh refinement.