Spectral and Superpotential Effects in Heterotic Compactifications

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(ABSTRACT)

In this dissertation we study several topics related to the geometry and physics of heterotic string compactification. After an introduction to some of the basic ideas of this field, we review the heterotic line bundle standard model construction and a complex structure moduli stabilization mechanism associated to certain hidden sector gauge bundles. Once this foundational material has been presented, we move on to the original research of this dissertation. We present a scan over all known heterotic line bundle standard models to examine the frequency with which the particle spectrum is forced to change, or "jump," by the hidden sector moduli stabilization mechanism just mentioned. We find a significant percentage of forced spectrum jumping in those models where such a change of particle content is possible. This result suggests that one should consider moduli stabilization concurrently with model building, and that failing to do so could lead to misleading results. We also use state of the art techniques to study Yukawa couplings in these models. We find that a large portion of Yukawa couplings which naively would be expected to be non-zero actually vanish due to certain topological selection rules. There is no known symmetry which is responsible for this vanishing. In the final part of this dissertation, we study the Chern-Simons contribution to the superpotential of heterotic theories. This quantity is very important in determining the vacuum stability of these models. By explicitly building real bundle morphisms between vector bundles over Calabi-Yau manifolds, we show that this contribution to the superpotential vanishes in many cases. However, by working with more complicated, and realistic geometries, we also present examples where the Chern-Simons contribution to the superpotential is non-zero, and indeed fractional.

Spectral and Superpotential Effects in Heterotic Compactifications

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(GENERAL AUDIENCE ABSTRACT)

String theory is a candidate for a unified theory of all of the known interactions of nature. To be consistent, the theory needs to be formulated in 9 spatial dimensions, rather than the 3 of everyday experience. To connect string theory with reality, we need to reproduce the known physics of 3 dimensions from the 9 dimensional theory by hiding, or "compactifying," 6 directions on a compact internal space. The most common choice for such an internal space is called a Calabi-Yau manifold. In this dissertation, we study how the geometry of the Calabi-Yau manifold determines physical quantities seen in 3 dimensions such as the number of particle families, particle interactions and potential energy. The first project in this dissertation studies to what extent the process of making the Calabi-Yau manifold rigid, something which is required observationally, affects the particle spectrum seen in 3 dimensions. By scanning over a large model set, we conclude that computation of the particle spectrum and such "moduli stabilization" issues should be considered in concert, and not in isolation. We also showed that a large portion of the interactions that one would naively expect between the particles in such string models are actually absent. There is no known symmetry of the theory that accounts for this structure, which is linked to the topology of the extra spatial dimensions. In the final part of the dissertation, we show how to calculate previously unknown contributions to the potential energy of these string theory models. By linking to results from the mathematics literature, we show that these contributions vanish in many cases. However, we present examples where it is non-zero, a fact of crucial importance in understanding the vacua of heterotic string theories.

Dedication

I dedicate this dissertation to my parents

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Chapter 1

Introduction

In this chapter we will introduce some basic concepts of particle physics, and then talk about why supersymmetry is desirable. After this we will discuss how the unification of the Standard Model with gravity motivates the development of string theory. Finally, we will give an overview of the structure of this dissertation.

1.1 Particle Physics and String Theory

1.1.1 The Standard Model of Particle Physics

To understand what the basic building blocks of matter are and how they interact with each other has been a long standing problem in human history. Throughout the development of science and technology, people have been trying to come up with different ideas and techniques to answer this question. Huge breakthroughs started to come into play from the beginning of 20th century. Experimentally, people developed new strategies and techniques to go into the micro-world. To mention a famous example among many important experiments, Rutherford and his students, Geiger and Marsden, used α particles to hit a plate of gold and studied what came out [1]. Through this experiment people first learnt for certain that the mass of an atom is actually sitting in a nucleus at its center. Even today, almost all the important particle experiments are still using this strategy to see the sub-structures of certain particles: scattering two particles with one another and seeing what comes out. Based on experimental findings, pioneer physicists gradually built the framework of quantum mechanics to understand the physics of the microscopic world during the first 3 or 4 decades in the 20th century. Since then, our understanding of particle physics has accelerated, culminating in the development of the Standard Model of particle physics, which is based on a huge amount of experimental facts and innovative physics ideas to understand them. A good reference combining physics and historical introduction to the above ideas is [2].

The particles of the Standard Model fall into two categories, particles that constitute matter and particles that mediate interactions. (For a general introduction of Standard Model physics, please see [2, 3].) The most commonly known elementary particle in matter is the electron, e, which has one unit of negative charge and has spin $\frac{1}{2}$. Through experiments, people found that e also has two cousins the muon μ and the tau τ, with the same charge and spin but bigger mass. μ and τ can't form stable matter because they will decay into other particles through the so called weak interaction shortly after they are created. For example, μ will decay in the following way: $\mu \longrightarrow e + \bar{\nu}_e + \nu_\mu$. Here, $\bar{\nu}_e$ is anti-electron neutrino and ν_μ is muon neutrino. The electron e, muon μ and tau τ, together with their partner neutrinos, ν_e, ν_μ and ν_τ form 3 generations of leptons which we can summarize as:

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix} \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}. \tag{1.1}$$

Leptons only interact through electro-magnetism and the weak interaction. The protons and neutrons of the nucleus of atoms are not elementary and are formed by quarks, spin $\frac{1}{2}$ particles with fractional charge. There are 6 kinds of quarks and they are grouped in the following special way into 3 generations:

$$\begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \begin{pmatrix} t \\ b \end{pmatrix}. \tag{1.2}$$

For each generation of the quarks, the up type quarks, u, c and t, have $+\frac{2}{3}$ units of charge, the down type quarks, d, s and b, have $-\frac{1}{3}$ units of charge. So the type of quarks, up or down, is decided by the charge they take or in other words, how they interact through electro-magnetic interaction. The families of quarks are successively heavier, mirroring the features of the leptons. The heavier quarks will also decay into lighter quarks through various interactions. In fact, heavier quarks prefer to decay into the lighter quark of their own generation. u and d, as the lightest of all the quarks, have no other quarks to decay to and that's why they are relatively stable and are thus the constituents of the components of the nucleus. The force binding quarks together to form protons or neutrons is called the strong interaction. Quarks can interact through all 3 of the interactions we have mentioned thus far.

Apart from the above particles, which are the basic building blocks of matter, there are also particles mediating the fundamental forces called gauge bosons. The photon is the particle mediating the electro-magnetic force. For the weak interaction, there are 3 different particles, W^+, W^- and Z. Here W^+ takes a positive unit charge, W^- takes a negative unit charge and Z is neutral. The gluon g is the particle which mediates the strong force.

In the above two paragraphs we have summarized the particles in the Standard Model. One of the successes of modern particle physics is that all the above structure can be unified together in a frame work inspired by symmetry. The strong, weak and electro-magnetic interactions can be encompassed within a quantum gauge field theory with gauge group $G = SU(3) \times SU(2) \times U(1)$. All the matter fields, quarks and leptons, are in the fundamental or trivial representations of the factors of G. The gauge bosons are in the adjoint or trivial representations of the gauge group factors. The strong interaction is described by the $SU(3)$ and the electro-magnetic and weak forces are together described by the $SU(2) \times U(1)$ gauge group factors[1]. All the quarks and leptons experience electro-weak interactions and there is a very special chiral structure in this sector.

The left handed leptons, which we schematically write as l_L, include a lepton and a neutrino pairing with it to form a doublet under $SU(2)$ in the $(\mathbf{1}, \mathbf{2})_{-\frac{1}{2}}$ representation of the whole SM gauge group $SU(3) \times SU(2) \times U(1)$. l_R, which just includes a right handed lepton, on the other hand, is a singlet under $SU(2)$ and it is in the $(\mathbf{1}, \mathbf{1})_1$ representation of the SM gauge group. For quarks, as we have seen in the above paragraphs, each generation contains a quark doublet, which includes an up type quark U and a down type quark D. The left handed quark doublet is a doublet under $SU(2)$ and more generally is in the $(\mathbf{3}, \mathbf{2})_{\frac{1}{6}}$ representation. Similar to leptons, U_R and D_R are singlets of $SU(2)$ and they are in representations $(\mathbf{3}, \mathbf{1})_{-\frac{1}{3}}$ and $(\mathbf{3}, \mathbf{1})_{\frac{2}{3}}$ respectively. Because of this special chiral structure, quarks and leptons can't get mass without breaking the $SU(2)$ gauge group. The operators $l_L \bar{l}_R$ and $U_L \overline{U}_R$ are simply not gauge invariant under $SU(2)$. The W^+, W^- and Z bosons are also massive. To explain this structure, a spin-zero Higgs field ϕ, in the $(\mathbf{1}, \mathbf{2})_{\frac{1}{2}}$ representation, is introduced to spontaneously break the electro-weak group to electromagnetism.

1.1.2 The Road to Supersymmetry

The Standard Model of particle physics has been tested by countless experiments during the last 50 years. Recently, another important experimental milestone was reached when the Higgs boson was claimed to be discovered at the Large Hadron Collider (LHC) in 2012

[1]Note here the $U(1)$ factor in $SU(2) \times U(1)$ is not the gauge group for electro-magnetism, and a particle's charge under this $U(1)$ is called its hypercharge Y. The electric charge Q is given by $Q = T_3 + Y$ with T_3 the diagonal generator of $SU(2)$ weak isospin group

[4]. Despite its huge success, the Standard Model of particle physics is still some way from becoming a theory of everything in particle physics, since there are still some questions it can not answer. For example, people are still trying to understand the tiny mass of the neutrinos in the framework of the Standard Model and the search for candidates of dark matter, both experimentally and theoretically, are still continuing (for an up to date introduction on this direction, please see [5]). Besides experimentally motivated questions such as those listed above, there are also questions from the theoretical side which motivate us to search for answers and finally these lead us to supersymmetry. In addition, the existence of supersymmetry may also provide more possibilities to understand the questions motivated from experiments. There are lots of good textbooks on supersymmetry, for example [6, 7, 8].

One of the theoretical problems leading us to supersymmetry is the hierarchy problem. The mass of the Higgs boson, as the experiments showed in 2012, is 125 GeV. To get the mass of the Higgs boson in the framework of field theory, we need to calculate the loop corrections to the Higgs mass and this is where the problem comes from. The problem is that, unlike other Standard Model particles where the leading loop correction to the mass is only logarithmically divergent $\delta m^2 \sim \mathrm{Log}\Lambda$, the Higgs boson has a quadratic correction to its mass $\delta m^2 \sim \Lambda^2$. Λ here is the energy scale at which new physics will show up and the effective field theory of the Standard Model breaks down. We know there are very high scales in nature, for example the gravitational scale of approximately 10^{18} GeV. So how can we get a number of magnitude 125 from a number of magnitude 10^{18}? It seems very unnatural and requires fine tuning. Supersymmetry can solve this problem in the following way. Because of supersymmetry, fermions and bosons will exist in pairs. In other words, every particle will have its super-partner. For each correction term to the mass of the Higgs boson, the square divergence is created by a virtual fermion interacting with the Higgs particle. If there is supersymmetry, then the fermion's super-partner will also have an interaction with the

Higgs particle, and its contribution to the mass correction is $\delta m^2 \sim -\Lambda^2$, which will exactly cancel that of the fermion's, at least above the scale at which supersymmetry is broken.

Another motivation for supersymmetry is the unification of gauge couplings. One of the basic goals of the Standard Model, as it was originally proposed, is to unify the strong, weak and electro-magnetic interactions into one framework. But actually, the gauge group at low energy is effectively SU(3) $\times$ SU(2) $\times$ U(1), with SU(3) the gauge group of the strong interaction, SU(2) $\times$ U(1) the gauge group of electro-weak interaction. In addition, the gauge couplings of the 3 interactions are different at low energy. One way in which these different interactions could be unified, at some higher energy scale Λ_{GUT}, would be for the gauge group to become a Grand Unified (or "GUT") group such as SU(5) or SO(10) [9, 10]. Accompanying the unification of gauge group, the gauge couplings of the 3 interactions should also be the same at Λ_{GUT}. The problem for models without supersymmetry is that the 3 gauge couplings will actually miss each other when the energy scale gets higher and higher, and the 3 interactions can not truly be unified. With a minimal supersymmetric extension of the standard model, the gauge couplings actually do meet at one value at a scale around 10^{16} GeV.

Beyond the above two motivations, there are actually other benefits for a model with supersymmetry. For example, more particles will be introduced and some of them could be candidates for dark matter. Beyond those phenomenological reasons, supersymmetric gauge theories in general are very interesting in the sense that in some situations they are exactly solvable and they also have various connections with topics in mathematics.

1.1.3 Gravity and String Theory

In the former subsections, we only talked about 3 fundamental forces, the electro-weak and strong interactions. One of the goals of modern theoretical physics is to find a unified framework to explain all the known interactions in our universe, including gravity in addition to the 3 interactions mentioned above. An early attempt to unify gravity with other interactions was proposed by Kaluza and Klein [11, 12]. They tried to unify gravity and the electro-magnetic force, the only known interactions at their time, by introducing one more space-time dimension. Even though this was not a successful theory, it gave lots of inspiration for many physics ideas that are in use today. The success of using quantum gauge field theory in the Standard Model of particle physics motivates people to have a hope that maybe we can also have a similar framework for gravity and thus all the 4 interactions can be understood in a similar way. However, a fundamental theory of gravity based on quantum field theory seems problematic because the most straightforward attempts at such constructions are non-renormalizable. That is, they always lose predictive power above a given energy scale.

One attempt at a theory unifying gravity with the other forces is string theory. String theory was originally proposed to be a theory of the strong interaction. However people soon found that, after quantization, string theory naturally contains gravity and issues of non-renormalizability do not arise in this context. Thus people started to consider seriously using string theory to unify gravity and the Standard Model interactions. For string theories having supersymmetry, which we call superstring theories, the space-time dimension is constrained to be 10. Local gauge symmetry and general coordinate invariance are the organizing principles for quantum gauge field theories and gravity respectively. One of the huge obstacles in the early development of superstring theory was that it seemingly contains an anomaly that will destroy those symmetries at the quantum level, rendering the theory inconsistent. In 1984, Green and Schwarz found a mechanism that causes string theory to be anomaly free [13]. Shortly after this discovery, Witten et al started the journey to hide (or "compactify") some of the dimensions of the ten dimensional string theory to get four dimensional physics [14]. Today there has been a lot of progress in this direction, the attempt to obtain an extended Standard Model with supersymmetry from string theory.

In ten dimensional space-time, there are actually 5 types of string theories, type I, type IIA/B, and heterotic string theories with $E_8 \times E_8$ or $SO(32)$ gauge group. In the 1990s people found various dualities between them and also that they are related to M-theory in eleven dimensions and F theory in twelve dimensions in certain limits(for a general introduction to string theory, see books [15, 16, 17, 18, 19, 20]). Therefore, we now strongly believe that all the theories are in some sense a limit of a unified theory of which we still don't know the final form. In this dissertation we will consider the heterotic string theory with $E_8 \times E_8$ gauge group. This theory has several phenomenological advantages, including its direct link to GUT gauge groups, since we can break E_8 to $SU(5)$, $SO(10)$ or E_6 easily.

If string theory is really a theory which describes Nature, then certainly it should reproduce all the low energy theory, general relativity and the Standard Model, that we have experimentally observed. The objective of this dissertation is to show some small progresses towards this whole picture. We want to show how some of the mathematics of the six dimensional compact internal space can influence particle physics in four dimensions. We will give an overview in the next section of how the content will be distributed between the chapters.

1.2 Overview of the Dissertation

In this section, we will give a brief overview of the whole dissertation, how it fits into the larger picture of string phenomenology and the logical relation between the chapters.

As we know string theory lives in ten dimensional space-time and the world we see is four dimensional. The extra six dimensions must be small and compact. In Chapter 2 we will go into some detail of how the requirement of four dimensional supersymmetry can constrain the six dimensional manifold to be a Calabi-Yau threefold. Furthermore, we will see that the gauge fields present in the theory are connections on a stable holomorphic vector bundle over that Calabi-Yau manifold. Then, based on this, we will talk about how to relate the Standard Model particle spectrum to certain vector bundle cohomologies and also see how to relate Yukawa couplings between those degrees of freedom to intersection numbers of certain forms on the Calabi-Yau manifold.

Based on the introduction of Chapter 2, in Chapter 3 we will study in detail how the mathematics of Calabi-Yau manifolds can influence physics. In the first part of this chapter we will study how the spectrum of particle physics could be influenced by the choice of the complex structure, that is by the precise shape, of the Calabi-Yau manifold. Based on a computer scan over an entire class of heterotic string theory standard models, we will show that mechanisms which are used to fix the complex structure of the Calabi-Yau threefold influence model building in a non-trivial way. In the second part of this chapter we will show, using a new method for calculating Yukawa couplings in this setting, that a large proportion of the interactions between the standard model fields actually vanish.

In Chapter 4 we will study an important term in the string theory action related to the vacuum stability in heterotic theory: the superpotential contribution from the Chern-Simons term. We will show explicitly that this contribution vanishes in a large class of geometries and we will also construct an explicit example to show that this term is not trivial in more complicated cases. Finally, in the last chapter of this dissertation we will briefly conclude.

Chapter 2

Heterotic String Compactification

To be a successful theory unifying gravity and particle physics, string theory needs to reproduce the Standard Model of particle physics at low energy scales. In this chapter, we will summarize two of the main tools used to try and achieve this in the context of heterotic string theory compactification on Calabi-Yau manifolds. In Section 2.1 we will talk about why we use Calabi-Yau manifolds in string compactification. Based on this, we will provide some of the details of how we can relate quantities in four dimensional physics to the mathematics of the six dimensional internal manifold in Section 2.2. Useful general references for this review chapter are [20, 21, 22].

2.1 Supersymmetry and Calabi-Yau Manifolds

In this section we will be concerned with how four dimensional physics arises from ten dimensional physics in heterotic string theory compactifications. To understand what this means physically, let us consider the energy scales associated with Standard Model physics, GUT physics, the compactification and finally string theory physics. Typically we take the Higgs mass, 125 GeV, as the approximate energy scale of the Standard Model. The running of the gauge couplings in minimal supersymmetric extensions of the standard model indicate that the scale of GUT physics is somewhere around 10^{16} GeV. The string theory scale in the models we will consider is approximately 10^{17} GeV and the compactification scale, set by the size of the extra dimensions, is actually the same as the GUT scale. At the string theory scale, the full ten dimensional space-time can be detected. Then when the energy goes down under the inverse of the Calabi-Yau manifold volume scale, the Calabi-Yau manifold can not be resolved anymore and only four dimensional space-time can be detected. This is where string compactification happens, and from here, the mathematical properties of the Calabi-Yau manifold are reflected in the low energy physics. Thus, by low energy we will frequently just mean energy scales which are lower than the compactification scale, but which could still be almost as large as 10^{16} GeV! Below the compactification scale, since energies are also below the GUT scale, in realistic models the GUT theory is broken down to the gauge group of the standard model of particle physics. Then from the GUT scale to the Standard Model scale, the theory undergoes running of its coupling constants and spontaneous symmetry breaking of its gauge groups. In the rest of this section we will first discuss why supersymmetry in four dimensions leads to a six dimensional internal space which is a Calabi-Yau manifold. Then we will describe how to relate four dimensional physics to the mathematics of that Calabi-Yau manifold.

2.1.1 Why Calabi-Yau Manifolds

From the discussion of the last chapter we know that, for phenomenological and theoretical reasons, it would be desirable to have a model with supersymmetry. At low energy, space-time is four dimensional. Locally, it is Minkowski space. As string theory lives in ten dimensional space-time and at low energy only a four dimensional space-time M_4 presents, it is natural to assume that the ten dimensional space-time is in the form $M_4 \times X$, with X a compact six dimensional space. Therefore, the basic objective here is to see what the manifold X should be under the constraint that there is supersymmetry in the four dimensional theory associated to the locally Minkowski space-time M_4 [14, 20].

To understand what kind of geometry X should be taken to be, we consider the heterotic string action in ten dimensional space-time and then study it's vacuum solutions. We will work at low curvatures and string coupling so that the effective supergravity description of the theory can be used. As we have discussed, we would like to preserve some supersymmetry in the four dimensional theory. The condition for preserving a given supersymmetry is that the supersymmetric variation of the fermionic fields in the heterotic string action should vanish in ten dimensions. As we will see soon, this is actually a strong constraint on both M_4 and the internal manifold X. Before we start the detailed analysis, let us first introduce the notation we will use. We use capital latin letters like I and J to represent the index for ten dimensional space-time. For four dimensional space-time, we use Greek letters like μ and ν. And finally, for the compact six dimensional space, we use lower case latin letters like i and j. In the context of heterotic string theory, the fermionic fields are the gravitino ψ_I, the dilatino λ and the gaugino, χ^α(here α is a gauge index). Their variations under supersymmetric transformation lead to the following equations in ten dimensions [20]:

$$\delta\psi_I = \frac{1}{\kappa}D_I\eta + \frac{\kappa}{32g^2\phi}(\Gamma_I^{JKL} - 9\delta_I^J\Gamma^{KL})\eta H_{JKL} + \cdots = 0; \tag{2.1}$$

$$\delta\chi^\alpha = \frac{1}{4g\sqrt{\phi}}\Gamma^{IJ}F_{IJ}^\alpha\eta + \cdots = 0; \tag{2.2}$$

$$\delta\lambda = -\frac{1}{\sqrt{2}\phi}(\Gamma\cdot\partial\phi)\eta + \frac{\kappa}{8\sqrt{2}g^2\phi}\Gamma^{IJK}\eta H_{IJK} + \cdots = 0. \tag{2.3}$$

Here ϕ is the dilaton field, F is the Yang-Mills gauge field strength, H is the field strength for the anti-symmetric field B, η is the spinor parameter for the supersymmetric transformation. Γ^I's are the ten dimensional gamma matrices which satisfy the Clifford algebra. $\Gamma^{I\ldots J}$ is the antisymmetric product of the above gamma matrices. We will not require these equations to hold for every possible supersymmetry parameter η of the higher dimensional theory, but rather for simply a subset corresponding to the four supercharges of $N = 1$ supersymmetry in four dimensions.

Besides the above constraints, F and H should also obey the Bianchi identity,

$$dH = \alpha'(\mathrm{Tr}R \wedge R - \mathrm{Tr}F \wedge F) \; . \tag{2.4}$$

If there are 5-branes in the theory, then this equation will get modified. Basically, as a source for flux, 5-branes will also contribute on the right hand side of this equation and this contribution is characterized by forms dual to the holomorphic two cycle on which the 5-branes are wrapped [21, 22]. In the summary section at the end of this chapter we will give a topological formula which captures this correction.

For simplicity, as an initial analysis, we will take H to be zero and ϕ a constant [20]. We will also assume that the gauge fields do not break four dimensional Poincare invariance. In this situation, equations (2.1), (2.2), (2.3) can be simplified to:

$$\delta\psi_I = D_I\eta = 0 \; , \tag{2.5}$$

$$\delta\chi^{\alpha} = \Gamma^{ij}F^{\alpha}_{ij}\eta = 0 \; . \tag{2.6}$$

Let us analyze the above equations one by one. From equation (2.5) we can get $[D_I, D_J]\eta = 0$, which implies $R_{IJKL}\Gamma^{KL}\eta = 0$ with R_{IJKL} the Riemann tensor. If M_4 is assumed to be maximally symmetric, this condition, with I and J taken to be on M_4, says that the Ricci scalar $R = 0$ on M_4, which means M_4 is Minkowski space. Since M_4 is Minkowski, $D_I\eta = 0$

automatically implies that $\partial_\mu \eta = 0$ on M_4, which means that the ten dimensional spinor is actually independent of the four dimensional space-time. On the internal compact space X, η is a covariantly constant spinor. Next we will see that the existence of a covariantly constant spinor actually requires that X should have an $SU(3)$ holonomy group.

Let us assume that we have a Riemannian manifold M and there is a physical field $\psi(x)$ on M. If we parallel transport ψ along a contractible curve which starts and ends at the same point p on M, then in general the field ψ we start with will not be the same as the one ψ' we end with, and there is a relation $\psi = U\psi'$. For different contractible curves γ there will be different U, and all the U together will form a group, called the holonomy group of M. For a n dimensional Riemannian manifold without further restrictions, its holonomy group is $SO(n)$. For the manifold we are going to study, the six dimensional compact manifold X, the holonomy group in general is $SO(6)$. But since we require the existence of a covariantly constant spinor η on X, we will see that the holonomy group should actually be a subgroup of $SO(6)$.

Then what should this holonomy group be? Actually this is related to the number of supersymmetries we want to preserve in four dimensions. It can be shown that the requirement of $N = 1$ supersymmetry on M_4 will precisely restrict the holonomy group of X to be $SU(3)$ and the reason is the following [20]. The sixteen dimensional positive chirality spinor η of $SO(1,9)$ will decompose as $\mathbf{16} \to (\mathbf{2} \otimes \mathbf{4}) \oplus \left(\mathbf{2}' \otimes \overline{\mathbf{4}}\right)$ under the sub-group $SO(1,3) \times SO(6)$. An $N = 1$ supersymmetry on M_4 exactly corresponds to a single real Majorana spinor, which is just one $\mathbf{2} \oplus \mathbf{2}'$. Thus we want just a single element of the $\mathbf{4}$ and $\overline{\mathbf{4}}$ six dimensional spinors to be covariantly constant. Recall that the Lie algebras of $SO(6)$ and $SU(4)$ are the same. Without losing any generality we can assume that η_6, a six dimensional spinor, has positive chirality and it belongs to the $\mathbf{4}$ of $SU(4)$. By using a suitable group transformation, η_6 can always be written in the following form:

$$\eta_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \eta_{6_0} \end{pmatrix}. \tag{2.7}$$

From this form we can see easily that an $SU(3)$ holonomy group will leave one covariantly constant spinor invariant and it exactly corresponds to $N = 1$ supersymmetry in four dimensions. To have a covariantly constant spinor is equivalent to have a constant 3-form $\Omega_{ijk} = \eta_{6_0}^T \Gamma_{ijk} \eta_{6_0}$, which we will use a lot later.

From the above analysis stemming from (2.5), we can see that $N = 1$ supersymmetry on M_4, together with some other simplifying choices, constrains M_4 to be Minkowski and the internal compact manifold to be a manifold with $SU(3)$ holonomy. The manifolds satisfying this condition and widely used in string theory literature are Calabi-Yau manifolds, which we will give an introduction to in the next sub-section.

2.1.2 Basic Introduction to Calabi-Yau Manifolds

A Calabi-Yau manifold is a Kahler manifold with $SU(3)$ holonomy [23]. Calabi-Yau manifolds have both a complex structure and a Kahler structure and here in this introduction we will give a very basic introduction to them. A good reference for this section is [20]

For any real $2n$ dimensional manifold, if locally we can define an almost complex structure, a tensor whose square is the minus identity:

$$\mathcal{J}^i_j \mathcal{J}^j_k = -\delta^i{}_k, \tag{2.8}$$

then we will call this manifold an almost complex manifold. If this tensor can be defined globally on the manifold, then this almost complex structure will be called a complex structure and the manifold will be called a complex manifold. The necessary and sufficient condition for a $2n$ dimensional manifold to be complex is that its Nijenhuis tensor vanishes [20]:

$$N^k_{ij} = \mathcal{J}^l_i \partial_l \mathcal{J}^k_j - \mathcal{J}^l_i \partial_j \mathcal{J}^k_l - \mathcal{J}^l_j \partial_l \mathcal{J}^k_i + \mathcal{J}^l_j \partial_i \mathcal{J}^k_l = 0. \tag{2.9}$$

A complex manifold can have many complex structures.

Once a complex structure is specified on a complex manifold, complex coordinates, up to a holomorphic transformation, are defined on the manifold. Complex coordinates can locally be defined by given the coordinates z^a's and their complex conjugates $\bar{z^a}$'s. A basic notion here is that of a tensor of (r,s) type. For a tensor $T^{a_1 \ldots a_r \ldots \bar{b_1} \ldots \bar{b_s}}$ of (r,s) type, its transformation under a holomorphic coordinate transformation $z^{a\prime} = z^{a\prime}(z)$ is:

$$(T^{a_1 \ldots a_r \ldots \bar{b_1} \ldots \bar{b_s}})' = T^{a_1 \ldots a_r \ldots \bar{b_1} \ldots \bar{b_s}} \frac{\partial z^{a_1'}}{\partial z^{a_1}} \cdots \frac{\partial z^{a_r'}}{\partial z^{a_r}} \frac{\partial z^{\bar{b_1}'}}{\partial z^{\bar{b_1}}} \cdots \frac{\partial z^{\bar{b_s}'}}{\partial z^{\bar{b_s}}}. \tag{2.10}$$

On a complex manifold, we can always construct a metric tensor g_{ij} satisfying the condition $g_{ij} = J^k_i J^l_j g_{kl}$, or, in terms of holomorphic and anti-holomorphic indices, $g_{\bar{a}b} = \overline{g_{a\bar{b}}}$ and $g_{ab} = g_{\bar{a}\bar{b}} = 0$. This metric is called an Hermitian metric. There is a $(1,1)$ form $J = g_{a\bar{b}} dz^a \wedge d\bar{z}^b$ associated with a Hermitian metric $g_{a\bar{b}}$. If on a manifold this form is a closed form, $dJ = 0$, then the manifold is a Kahler manifold and J is referred to as the Kahler form.

In general, a Kahler manifold has a $U(n)$ holonomy group. For a $2n$ dimensional manifold, the vector of $SO(2n)$ can be decomposed as $n \oplus \overline{n}$, a holomorphic part and anti-holomorphic part, under $U(n)$. If the manifold has $U(n)$ holonomy, a vector in $n(\overline{n})$ will still fall into $n(\overline{n})$ after parallel transport around a closed path. So the almost complex structure $\mathcal{J}^i_j$ is covariantly constant, thus the Nijenhuis tensor vanishes, so the manifold is naturally complex.

Since the (1,1) form J can be constructed from the covariantly constant metric and $\mathcal{J}^i_j$, this too is covariantly constant and therefore closed. Thus, the $U(n)$ holonomy also implies the manifold is Kahler [20]. In fact a Kahler manifold can be defined as one which admits an metric of $U(n)$ holonomy. Calabi-Yau manifolds have holonomy group $SU(n)$. Notice that, since $U(n) = SU(n) \times U(1)$, the condition for a Kahler manifold to be a Calabi-Yau is that the $U(1)$ part of the holonomy group is trivial. The $U(1)$ component of the spin connection is an abelian gauge field which we can denote by A. Its field strength is $F = dA$, which is just the representative of a topological invariant called the first Chern class $c_1(TX)$ [24]. Requiring the $U(1)$ part of the holonomy group to be trivial is equivalent saying that the first Chern class of the manifold is trivial. So a Calabi-Yau manifold is a Kahler manifold with a trivial first Chern class. As we saw previously, another useful property of a $2n$ dimensional Calabi-Yau manifold is that there is a globally defined nowhere vanishing holomorphic n form Ω on it.

For a given Calabi-Yau manifold, all of the possible complex structures form a space, called its complex structure moduli space. The dimension of this space is given by a topological invariant, the Hodge number $h^{2,1}$ of the Calabi-Yau manifold. Roughly speaking, different complex structures correspond to putting complex coordinates on the Calabi-Yau in different ways. By using an analysis similar to those we will see in the next section, it can be shown that complex structure moduli give rise to massless scalar fields in the four dimensional effective theory [20] . In order to get rid of these unobserved massless scalars, we need to fix the complex structure moduli. In Chapter 3 we will see in detail how we can fix the complex structure moduli and how this process can influence model building.

Just like the possible complex structures, all the Kahler structures, that is choices of Kahler form, also form a space called Kahler moduli space. The dimension of this space is given by the Hodge number $h^{1,1}$. In principle we should also fix the Kahler moduli to get rid of the massless scalars they correspond to in the low energy effective theory.

2.1.3 Constraints on the Vector Bundle

After introducing the properties of Calabi-Yau manifolds, we now have the necessary ingredients to analyze equation (2.6). Since X is a Calabi-Yau manifold which is also a Kahler manifold, there are always local coordinates on open sets of X such that the metric components $g^{\overline{ab}}$ and g^{ab} both vanish. In this situation, the only non-trivial anti-commutation relation between Γ matrices will be $\{\Gamma^a, \Gamma^{\bar{b}}\} = g^{a\bar{b}}$. An interesting fact here is that from this relation we can take $\Gamma^{\bar{b}}$ as creation operators and Γ^a as annihilation operators [20]. So in general spinors on the Calabi-Yau manifold can be spanned by the $\Gamma^{\bar{b}}$ acting on some empty state. An analysis of the consequences of covariant constancy of the six dimensional spinor η_0 and its conjugate reveal that these are the completely empty and full states respectively. Spinors $\psi_6(y^j)$ on the Calabi-Yau manifold can then be associated to $(0, n)$ forms. The reason for this is that, a spinor $\psi_6(y^j)$ can be written as

$$\psi_6(y^j) = \phi(y^j)\eta_0 + \phi_{\bar{b}}(y^j)\Gamma^{\bar{b}}\eta_0 + \phi_{\bar{b}\bar{c}}(y^j)\Gamma^{\bar{b}}\Gamma^{\bar{c}}\eta_0 + \dots \tag{2.11}$$

with the $\phi_{\bar{a}}$ and $\phi_{\bar{a}\bar{b}}$ fields being the relevant $(0, 1)$ and $(0, 2)$ forms respectively (for more details see [20]). Clearly this idea generalizes to the omitted higher forms. This fact will be crucial in the analysis of zero modes of Dirac operators in the next section. Returning to (2.6), we can rewrite that equation using $\Gamma^{\bar{b}}$ and Γ^a in the following way:

$$\Gamma^{ij}F_{ij}\eta = (F_{\overline{ab}}\Gamma^{\overline{ab}} + F_{ab}\Gamma^{ab} + F_{a\bar{b}}\Gamma^{a\bar{b}})\eta = 0. \tag{2.12}$$

If we take Γs as creation and annihilation operators, then all three terms in the above formula should vanish separately since they take different index structure. For the term $F_{\overline{ab}}\Gamma^{\overline{ab}}\eta = 0$, if we take η to be the empty state, then $\Gamma^{\overline{ab}}\eta$ is non-trivial so $F_{\overline{ab}}$ has to vanish. To obtain $F_{ab} = 0$, we can take η to be a fully filled state and use similar reasoning. For the final term, if we take η to be the empty state then $\Gamma^{a\bar{b}}$ will create a fermion and then annihilate it again and introduce an extra factor $g^{a\bar{b}}$. So from it we obtain the condition $g^{a\bar{b}}F_{a\bar{b}} = 0$. The condition $F_{ab} = 0$ together with $F_{\overline{ab}} = 0$ indicates that V should be a holomorphic vector bundle. The condition $g^{a\bar{b}}F_{a\bar{b}} = 0$ further requires that this holomorphic vector bundle should also be stable, by the Donaldson-Uhlenbeck-Yau theorem [25, 26].

Since the tangent bundle of a Calabi-Yau manifold is naturally holomorphic and stable, one natural solution for these equations is to take the vector bundle V to be the tangent bundle of X. The $SU(3)$ structure group of the tangent bundle is embedded into one E_8 factor of the ten dimensional gauge group, referred to as the visible sector, and the other factor is then referred to as the hidden sector. It is easy to see that in this way the low energy visible sector gauge group will be the commutant of $SU(3)$ inside E_8, E_6. The above way to get low energy physics, embedding the tangent bundle of Calabi-Yau manifold directly into the E_8 gauge group, is called the standard embedding.

A natural question to ask is whether we can get other GUT gauge theories, like $SU(5)$ or $SO(10)$, directly from heterotic string compactification? Such solutions are necessarily more complicated than the standard embedding. In the standard embedding case, the Bianchi identity (2.4) is satisfied with $H = 0$. If the gauge fields are a connection on a different vector bundle then it will no longer true that $\mathrm{tr}R \wedge R = \mathrm{tr}F \wedge F$ and the Bianchi identity will force $H \neq 0$. In such a case, the simple analysis of supersymmetry conditions that we have described in this chapter is modified. These issues were explored in [27, 28] where it was shown that the essential conditions that the manifold be Calabi-Yau and the gauge bundle be stable and holomorphic are unchanged in this situation. This follows from an analysis of the killing spinor equations that proceeds order by order in an expansion with respect to α'/r^2, with r the radius of Calabi-Yau manifold.

2.2 Topology of Calabi-Yau Manifolds and Physics of the Standard Model

In the proceeding section we saw that $N = 1$ supersymmetry in four dimensions requires that the internal manifold be a Calabi-Yau manifold and the gauge field a connection on a holomorphic stable vector bundle over the Calabi-Yau manifold. Upon the compactification of ten dimensional space-time to four dimensional space-time, some of the mathematical quantities of the compact internal manifold will correspond to certain physical quantities of the four dimensional theory. In this section, we will explain why this happens in general and then, based on what will be required to understand the content of later chapters, will give a gentle introduction of how we can relate the cohomology of vector bundles on the Calabi-Yau manifolds to the spectrum of particle physics, and also how we can relate Yukawa couplings to intersection numbers of certain differential forms on Calabi-Yau manifolds. General references of this section are [20, 21, 22].

2.2.1 Cohomology of Vector Bundles and Particle Spectrum

Recall that the particle content of the Standard Model has very special properties. There are exactly 3 generations of particles. In addition, the particles exhibit a very special chiral structure [20]. How does this particle content combine into appropriate multiplets at the GUT scale? To answer this, let us first see group theoretically how the Standard Model particles fit into representations of the GUT gauge group. Here we take $SU(5)$ as an example [9, 10]. For $SO(10)$ or E_6 the analysis is similar. The $\overline{\mathbf{5}}$ and $\mathbf{10}$ multiplets in $SU(5)$ decompose under $SU(5) \to SU(3) \times SU(2) \times U(1)$ as:

$$\overline{\mathbf{5}} \to (\overline{\mathbf{3}}, \mathbf{1})_{\frac{1}{3}} \oplus (\mathbf{1}, \mathbf{2})_{-\frac{1}{2}}; \tag{2.13}$$

$$\mathbf{10} \to (\mathbf{3}, \mathbf{2})_{\frac{1}{6}} \oplus (\overline{\mathbf{3}}, \mathbf{1})_{-\frac{2}{3}} \oplus (\mathbf{1}, \mathbf{1})_1. \tag{2.14}$$

As we can see here, all the particle content in a Standard Model family fits very well into one $\overline{\mathbf{5}} \oplus \mathbf{10}$ multiplet (the right handed particles fit here through their charge conjugates, which are also left handed). Since all the particles are left handed in this multiplet, we call this representation V_L. We see, therefore, that the chiral pattern of particles at the Standard Model level is mirrored at the GUT scale. The whole particle content fits into a chiral multiplet V_L [20]. At the GUT scale, all of the particles in the standard model should be taken to be massless given the huge scale difference between the GUT and standard model scales. Massive particles at the GUT scale won't be seen in low energy since they are too heavy to be produced.

So in summary, up to now we see that the matter content of particle physics at the GUT scale can have a chiral asymmetry and all the matter particles should be taken to be massless. The question that we now want to ask is can string theory reproduce this special matter content: 3 generations of left handed massless particles? To answer this, we split the analysis into two steps, we first explain how massless particles in four dimensions emerges from string theory. After this, we will explain how the chiral asymmetry is associated to a topological invariant of the compactification. When we have all the elements ready, we will finally relate the particle spectrum of GUT theories with different gauge group to the cohomology of certain vector bundles.

The following analysis are from [20] and more details can be found there. The massless matter fields which are intrinsically spinors in four dimensional space-time should come from ten dimensional massless spinors. These spinors, which we will denote collectively by ψ_{10}, should satisfy the Dirac equation $i\not{D}_{10}\psi_{10} = 0$, with $\not{D}_{10}$ the ten dimensional Dirac operator $\not{D}_{10} = \sum_{I=1}^{10} \Gamma^I D_I$. $\not{D}_{10}$ here can be decomposed as $\not{D}_{10} = \not{D}_4 + \not{D}_6$, with $\not{D}_4 = \sum_{i=\mu}^{4} \Gamma^\mu D_\mu$ and $\not{D}_6 = \sum_{j=5}^{10} \Gamma^j D_j$. With this decomposition, the Dirac equation for ψ_{10} becomes $i(\not{D}_4 + \not{D}_6)\psi_{10} = 0$. Now the $\not{D}_6\psi_{10}$ part really looks somewhat like a mass term for $\not{D}_4$. But since $\not{D}_4$ and $\not{D}_6$ anti-commute with each other, they do not have common eigenstates. So at this stage, we can't claim that $\not{D}_6\psi_{10}$ is a mass term for $\not{D}_4$. The way out of this is that we multiply both $\not{D}_4$ and $\not{D}_6$ by $\Gamma^{(4)} = i\Gamma_1\Gamma_2\Gamma_3\Gamma_4$ to get $\tilde{\not{D}}_4 = \Gamma^{(4)}\not{D}_4$ and $\tilde{\not{D}}_6 = \Gamma^{(4)}\not{D}_6$. Now $\tilde{\not{D}}_6$ and $\tilde{\not{D}}_4$ commute with each other, and we can assume that ψ_{10} can be expanded in terms of the eigenstates of $\tilde{\not{D}}_6$ and $\tilde{\not{D}}_4$,

$$\psi_{10} = \sum_\alpha \psi_{4\alpha}(x^\mu)\psi_{6\alpha}(y^j). \tag{2.15}$$

Here $\psi_{4\alpha}(x^\mu)$ only depends on four dimensional space time coordinates x^μ and $\psi_{6\alpha}(y^j)$ only depends on six dimensional coordinates y^j. Now the Dirac equation for ψ_{10} becomes

$$\sum_\alpha (\tilde{\not{D}}_4 + \tilde{\not{D}}_6)\psi_{4\alpha}(x^\mu)\psi_{6\alpha}(y^j) = 0. \tag{2.16}$$

If

$$\tilde{\not{D}}_6\psi_{6\alpha}(y^j) = \lambda_i\psi_{6\alpha}(y^j), \tag{2.17}$$

we will have

$$(\tilde{\not{D}}_4 + \lambda_\alpha)\psi_{4\alpha}(x^\mu) = 0 \tag{2.18}$$

and if $\lambda_\alpha = 0$ there will be a massless fermion $\psi_{4\alpha}(x^\mu)$ in four dimensions. So when we are looking for massless fermions in four dimensions, what we are really after are the zero modes of the Dirac operator $\not{D}_6$. For each zero mode of the Dirac operator $\not{D}_6$, there is a corresponding four dimensional massless fermion. Notice here that since we are interested in the zero modes of $\not{D}_6$, for any ψ_6 with the condition that $\not{D}_6\psi_6(y^j) = 0$ it is also true that $\tilde{\not{D}}_6\psi_6(y^j) = 0$. So the multiplication of $\Gamma^{(4)}$ won't really influence the analysis.

What can we say about the chirality asymmetry? Let us first recall the definition of chirality operators. In ten dimensional space time we define the chirality operator as $\Gamma^{(10)} = \Gamma_1\Gamma_2 \ldots \Gamma_{10}$ with Γ_I's ten dimensional gamma matrices. For four dimensional space time and the compact internal space, the chirality operators are defined as $\Gamma^{(4)} = i\Gamma_1\Gamma_2\Gamma_3\Gamma_4$ and $\Gamma^{(6)} = -i\Gamma_5\Gamma_6 \ldots \Gamma_{10}$ respectively. There operators satisfy the relation that

$$(\Gamma^{(10)})^2 = (\Gamma^{(4)})^2 = (\Gamma^{(6)})^2 = 1. \tag{2.19}$$

As we know (for example see [20]) the ten dimensional massless spinors we are considering here satisfy a chirality condition, and we can take our convention to be $\Gamma^{(10)} = 1$ when acting upon them. Since $\Gamma^{(10)} = \Gamma^{(4)}\Gamma^{(6)}$, it is easy to deduce that $\Gamma^{(4)} = \Gamma^{(6)}$ when acting on these states. Recall also that, four dimensional massless spinors correspond to the zero modes of $\not{D}_6$. Combining with this fact with $\Gamma^{(4)} = \Gamma^{(6)}$ we can conclude that the chiral asymmetry in four dimensional space-time is equivalent to the chiral asymmetry of $\not{D}_6$'s zero modes. The chiral asymmetry of $\not{D}_6$'s zero modes is actually a topological invariant called the Dirac index of $\not{D}_6$. To see why it is a topological invariant and how the chiral asymmetry of zero modes can emerge, let us define an operator $H = (i\not{D}_6)^2$. Notice that H and $\Gamma^{(6)}$ commute, so the eigenstates of H could at the same time be the eigenstates of $\Gamma^{(6)}$. For a ψ with $H\psi = E\psi$, we also have $H\not{D}_6\psi = E\not{D}_6\psi$. So ψ and $\not{D}_6\psi$ are degenerate as H's eigenstates: they have the same eigenvalue. But because $\Gamma^{(6)}$ and $\not{D}_6$ anti-commute, ψ and $\not{D}_6\psi$ have opposite chirality unless $\not{D}_6\psi = 0$. So basically, H's eigenstates with non-zero eigenvalues will show up in pairs, one with positive chirality, one with negative chirality. But for the zero eigenstates, which are also the zero modes of $\not{D}_6$, the number of positive chirality modes n_+ and the number of negative modes n_- may not be the same. $n_+ - n_-$ measures the difference between the number of positive modes and number of negative modes and it is called the Dirac index. If there is any change of H's spectrum, some eigenstates with non-zero eigenvalue may become zero modes, or some zero modes may have non-zero eigenvalue. But the key fact here is that they always change in pairs, so $n_+ - n_-$ always stay the same and is an invariant.

Note that in heterotic theory, when we talk about the zero modes of the Dirac operator on the Calabi-Yau manifold, the Dirac operator we use should be schematically of the form $\not{\partial} + \not{A}$, with A a connection on the holomorphic stable vector bundle on the Calabi-Yau manifold. With this correction, the above Dirac index analysis is still true and further more, via the help of the Atiyah-Singer index theorem (for a good reference see [24]), can be computed as follows:

$$n_+ - n_- = \mathrm{Ind}(V) = \sum_{i=1}^{3} (-1)^i h^i(X, V) \ . \tag{2.20}$$

Here $h^i(X, V)$ is the dimension of the cohomology group $H^i(X, V)$(for details of cohomology group see [24])

From the above analysis, we can understand the possible origin of 3 generations of massless particles in the four dimensional Standard Model from the ten dimensional field theory point of view. We are interested in more than just the chiral index, however. We also want to know the exact number of generations and anti-generations in the compactified theory. To study this, let us take the example of an $E_8 \times E_8$ heterotic string theory compactified to an $SU(5)$ GUT theory to see how exactly the above analysis can help us to relate the particle generations to certain vector bundle cohomologies.

First, we need to get the $SU(5)$ gauge group from $E_8 \times E_8$. The way to accomplish this is to take the holomorphic stable vector bundle V on the Calabi-Yau manifold to be an $SU(5)$ bundle with a gauge field taking expectation values in that group. Then the unbroken gauge group is the commutant of $SU(5)$ inside E_8, which is also an $SU(5)$. Since at the string theory level, all the matter we have is in **248** multiplets we are interested in how such matter descends to $SU(5)$ degrees of freedom. Under $E_8 \to SU(5) \times SU(5)_{GUT}$, the **248** will decompose as:

$$\mathbf{248} \to (\mathbf{1}, \mathbf{24}) \oplus (\mathbf{5}, \mathbf{10}) \oplus (\overline{\mathbf{5}}, \overline{\mathbf{10}}) \oplus (\mathbf{10}, \overline{\mathbf{5}}) \oplus (\mathbf{5}, \overline{\mathbf{10}}) \oplus (\mathbf{24}, \mathbf{1}) \ . \tag{2.21}$$

The GUT theory matter content is in the $\overline{\mathbf{5}}$ and **10** multiplets. From the above analysis we can see that they correspond to the Dirac operator zero modes in the **10** and **5** respectively. Recalling that the $\Gamma^{\overline{a}}$ matrices on a Calabi-Yau manifold can be seen as creation operators, the spinor ψ transforming as a **5** multiplet is nothing but a collection of bundle V valued $(0,1)$ differential forms. The zero modes of them, which satisfy the condition $\not{D}_6 \psi_6 = 0$, form the cohomology group $H^1(X,V)$. The dimension of $H^1(X,V)$, which is denoted by $h^1(X,V)$, exactly represents the number of **10** representations we have in four dimensions. Similarly $H^2(X,V)$, which is isomorphic to $H^1(X,V^*)$, whose dimension is $h^1(X,V^*)$, just counts the number of generations of $\overline{\mathbf{10}}$. For the $\overline{\mathbf{5}}$ representations under the GUT group, the massless fermions correspond to zero modes valued in the **10** representation of the $SU(5)$

$G \times H$	Breaking Pattern: $\mathbf{248} \to$	Particle Spectrum
$SU(3) \times E_6$	$(\mathbf{1}, \mathbf{78}) \oplus (\mathbf{3}, \mathbf{27}) \oplus (\overline{\mathbf{3}}, \overline{\mathbf{27}}) \oplus (\mathbf{8}, \mathbf{1})$	$n_{27} = h^1(V)$ $n_{\overline{27}} = h^1(V^*) = h^2(V)$ $n_1 = h^1(V \otimes V^*)$
$SU(4) \times SO(10)$	$(\mathbf{1}, \mathbf{45}) \oplus (\mathbf{4}, \mathbf{16}) \oplus (\overline{\mathbf{4}}, \overline{\mathbf{16}}) \oplus (\mathbf{6}, \mathbf{10}) \oplus (\mathbf{15}, \mathbf{1})$	$n_{16} = h^1(V)$ $n_{\overline{16}} = h^1(V^*) = h^2(V)$ $n_{10} = h^1(\wedge^2 V)$ $n_1 = h^1(V \otimes V^*)$
$SU(5) \times SU(5)$	$(\mathbf{1}, \mathbf{24}) \oplus (\mathbf{5}, \mathbf{10}) \oplus (\overline{\mathbf{5}}, \overline{\mathbf{10}}) \oplus (\mathbf{10}, \overline{\mathbf{5}}) \oplus (\overline{\mathbf{10}}, \mathbf{5}) \oplus (\mathbf{24}, \mathbf{1})$	$n_{10} = h^1(V)$ $n_{\overline{10}} = h^1(V^*) = h^2(V)$ $n_5 = h^1(\wedge^2 V^*)$ $n_{\overline{5}} = h^1(\wedge^2 V)$ $n_1 = h^1(V \otimes V^*)$

Table 2.1: *A vector bundle V with structure group G can break the E_8 gauge group of the heterotic string into a GUT group H. The low-energy representation are found from the branching of the* **248** *adjoint of E_8 under $G \times H$ and the low-energy spectrum is obtained by computing the indicated bundle cohomology groups. This table is taken from [21].*

structure group of V. These correspond to $\wedge^2 V$ valued forms. Finally, the $\mathbf{5}$ GUT multiplets correspond to elements of $H^1(X, \wedge^3 V) = H^2(X, \wedge^2 V^*)$.

It should be noted that, given the above discussion, the chiral asymmetry in the $\mathbf{10}$'s is given by $h^1(X, V) - h^2(X, V)$ and that in the $\overline{\mathbf{5}}$'s is given by $h^1(X, \wedge^2 V) - h^2(X, V)$, in agreement with (2.20) (the zeroth and third cohomologies involved all vanish for a stable bundle V). In fact, in order for the standard model matter to appear in complete families, these two indices should match. This is guaranteed by the properties of third Chern classes under exterior products, and the Hirzebruch Riemann Roch theorem [21, 24]. We can perform a similar analysis for the other GUT groups and the result is summarized in table 2.1.

2.2.2 Intersection Numbers and Yukawa Couplings

Yukawa couplings are important in low energy particle physics, being related to the interactions between particles and at the same time accounting for their mass via the Higgs mechanism. Given this, in this section we want to present a relation between Yukawa couplings at low energy and a mathematical quantity associated to Calabi-Yau manifolds: intersection numbers of differential forms. In doing this we will arrive at a quasi-topological formula for the Yukawa couplings. General references for this section are [20, 21, 22].

At low energy in four dimensional space time, suppose the GUT group we have is G and we have bosons $\phi_{4\,i}^{\alpha_i}(x)$(from here and the following, we use x and y to represent 4 dimensional and 6 dimensional coordinates dependence respectively for convenience and omit the upper index, which won't bring further confusion in this context). Here the α_i are possible gauge indices associated to the representation of the fields with respect to G, the sub-index 4 just means these are four dimensional fields, and the index i runs over the different massless particles in the theory. We could equally well consider some of the particles to be fermions in what follows. However, since the bosons and fermions in the theory are paired by supersymmetry, it suffices to consider the bosonic couplings. Imagine that we have a gauge singlet combination $\lambda^{ijk}_{\alpha_i\alpha_j\alpha_k}\phi_{4\,i}^{\alpha_i}(x)\phi_{4\,j}^{\alpha_j}(x)\phi_{4\,k}^{\alpha_k}(x)$, where the indices α_i, α_j and α_k run over the $SU(5)$ GUT representations $\overline{\mathbf{5}}$, $\overline{\mathbf{5}}$ and $\mathbf{10}$ respectively, and the coefficients λ are chosen to project to the gauge invariant quantity. Then in principle we could have an interaction term in the four dimensional superpotential given by this quantity. This term is the Yukawa couplings between these fields.

We wish to understand the stringy origin of $\lambda^{ijk}_{\alpha_i\alpha_j\alpha_k}$. Recall from the analysis of the last section that every massless matter field $\phi_{4\,i}^{\alpha_i}(x)$ in four dimensions is associated to a six dimensional bundle valued one form $\omega_6^{\alpha_i' i}(y)$ on the Calabi-Yau manifold.In terms of dimensional reduction, this association takes place via the expansion of the higher dimensional gauge field, from which all gauge charged four dimensional scalar matter descends:

$$A_{10}(x^\mu, y) = \phi_{4\,i}^{\alpha_i}(x)\omega_6^{\alpha_i' i}(y)T_{\alpha_i\alpha_i'} + \dots . \tag{2.22}$$

Here the $T_{\alpha_i\alpha'_i}$ are some of the generators of E_8 associated to the branching under the decomposition to the GUT group G and structure group H.

$$E_8 \to G \times H : \mathbf{248} \to (\alpha_1, \alpha'_1) \oplus (\alpha_2, \alpha'_2) \oplus \dots . \tag{2.23}$$

The four dimensional Gukov-Vafa-Witten superpotential of heterotic theories contains the following relevant contribution [29]:

$$W \ni \int_X \mathrm{tr}\left(dA_{10} \wedge A_{10} + \frac{2}{3}A_{10}^3\right) \wedge \Omega. \tag{2.24}$$

Plugging (2.22) into (2.24) and performing the integral over the Calabi-Yau manifold, we obtain the superpotential Yukawa coupling term in four dimensions that we have been discussing. In carrying out this computation we find the following expression for the coupling parameters λ:

$$\lambda^{ijk}_{\alpha_i\alpha_j\alpha_k} = \int_X \left(\omega_6^{\alpha'_i i} \wedge \omega_6^{\alpha'_j j} \wedge \omega_6^{\alpha'_k k} \wedge \Omega\right) \mathrm{tr}\left(T_{\alpha_i\alpha'_i} T_{\alpha_i\alpha'_i} T_{\alpha_i\alpha'_i}\right), \tag{2.25}$$

with Ω the holomorphic $(3,0)$ form of the Calabi-Yau manifold.

Formula (2.25) is actually a quasi-topological invariant. In general it does not rely on the representatives of the six dimensional one forms, but just on their cohomology class. To see this, let us imagine adding an exact piece to one of the one forms: $\omega_6^{\alpha'_i i} \to \omega_6^{\alpha'_i i} + (\overline{D}\epsilon^i)^{\alpha'_i}$. Then the change to λ is:

$$\delta\lambda^{ijk}_{\alpha_i\alpha_j\alpha_k} = \int_X \left((\overline{D}\epsilon^i)^{\alpha'_i} \wedge \omega_6^{\alpha'_j j} \wedge \omega_6^{\alpha'_k k} \wedge \Omega\right) \mathrm{Tr}\left(T_{\alpha_i\alpha'_i} T_{\alpha_i\alpha'_i} T_{\alpha_i\alpha'_i}\right) . \tag{2.26}$$

Because all the forms involved here are closed, we can see $\delta\lambda_{ijk}$ is zero by integrating by parts. For a similar and more complete analysis of Yukawa coupling, please see [20].

From the above calculations we can see how the superpotential Yukawa coupling is related to a quasi-topological quantity defined on the Calabi-Yau manifold. In phenomenological applications, (2.25) is very useful but still limited. The limitation comes from the field normalizations of the kinematic terms in the Lagrangian. The four dimensional kinetic terms that arise in such string compactifications are not canonical. Rather, the kinetic terms are specified by a Kahler potential K [6, 7, 8], via a field space metric of the form $g_{i\bar{j}} = \frac{\partial K}{\partial \phi^i \partial \overline{\phi^j}}$, with ϕ^i the scalar field in a chiral supermultiplet.In order to evaluate the physical Yukawa couplings at a given point in field space, one would have to perform a field redefinition to make this metric the identity matrix at that point. Unfortunately, unlike for the quasi-topological Yukawa coupling, we need an explicit expression for the Calabi-Yau metric to obtain the value of $g_{i\bar{j}}$. Up to now, there has been some progress in calculating the Calabi-Yau metric numerically and using machine learning techniques (see [30, 31, 32, 33, 34] and references therein). Thus, starting from (2.25) we still can not get the physical Yukawa coupling at low energy since we don't know the required field normalizations. Nevertheless, there are some selection rules which can tell us which couplings will vanish based differential geometry. We will show explicitly how this comes about in Chapter 3.

It should be noted that the formula (2.25) is protected from perturbative corrections because of a strong non-renormalization theorem(see [20] and references therein). All the perturbative corrections for W are in powers of α'/r^2, with r the radius of the Calabi-Yau manifold. r, as a modulus of the Calabi-Yau manifold, is actually a massless scalar in four dimensions. If r is included in W, so should be its super-partner p, which is a pseudoscalar zero mode of the ten dimension antisymmetric two form field B_{MN}. The key fact here is that r and p should be included in W in a holomorphic way, as a combination $r + ip$. But p has an important axionic shift symmetry which is exact at all orders, $p \to p + c$, with c a constant. This forbids $r + ip$ from appearing perturbatively in W and hence, there is no α' corrections to W and so the Yukawa coupling. This is the non-renormalization theorem of W and we can see how powerful its application is. Despite the above fact, the Yukawa coupling could indeed receive non-perturbative corrections proportional to factors such as $e^{-\alpha'/r^2}$. However, since r^2 is much greater than α', this contribution is highly suppressed and the Yukawa coupling we get from the field theory approximation should be accurate.

2.3 Wilson Lines and Discrete Quotients of Calabi-Yau manifolds

We are not interested in obtaining GUT physics from our string compactifications, but rather we need to further break the gauge group to that of the Standard Model. In addition, in obtaining the spectrum of particles using the types of compactification we have been discussing, we often get a very high number of families. For example, in the standard embedding on the quintic (a particularly simple Calabi-Yau manifold), the family number we get is $h^{2,1} - h^{1,1} = 100$. This is far more than the three observed in nature. Therefore it would be useful to find a way to reduce the original family numbers we get. If a Calabi-Yau manifold has a free quotient with respect to some discrete symmetry, then there are possible solutions to both of the above two issues [20, 21, 22].

First let us recall some facts and definitions concerning free quotients [20]. Suppose we have a group of actions Γ acting on a manifold X. If for any x on X we have that $\gamma x \neq x$ for any non-trivial element γ in Γ, then Γ is a free action on X. From X and Γ we can get a quotient space $\frac{X}{\Gamma}$. Points in $\frac{X}{\Gamma}$ are equivalent classes of points in X. For example, if $x' = \gamma x$ in X then they will be taken as the same point in $\frac{X}{\Gamma}$. Defined in this way $\frac{X}{\Gamma}$ is naturally not simply connected and this information is carried by the first fundamental group $\pi_1(\frac{X}{\Gamma})$. Imagine that we have a curve C connecting x and γx in X, then this curve will become a closed curve in $\frac{X}{\Gamma}$, since x and γx are taken to be equivalent in $\frac{X}{\Gamma}$. This curve can not be shrinkable otherwise x and γx will literally be the same in X, which is by definition not true for what we are considering now. In general, the topological class of the above closed curves are related to the γ working on the starting point. If X is originally simply connected, then the $\pi_1(\frac{X}{\Gamma})$ is the same as Γ.

How will the above facts help us with the symmetry breaking? The key fact here is that we can define Wilson lines, which are defined as $U_C = \mathrm{Pexp} \oint_C A \cdot dx$, along the non-shrinkable closed curves mentioned above. Because the closed curves are not shrinkable, the Wilson lines are not trivial. Also, because they are pure gauge, Wilson lines don't influence the field strength and as a result they won't influence the field equations or conditions for unbroken supersymmetry. If chosen correctly, these Wilson lines can break the GUT group into the Standard Model group we want. In Chapter 3 we will give more details on carrying this out in worked examples. So in this way, we can embed Wilson lines into the GUT gauge group $SU(5)$, $SO(10)$ or $E6$ to break them into the Standard Model gauge group. To use quotients of Calabi-Yau manifold to get Wilson lines and then break the GUT group is a widely used methodology in heterotic model building.

Quotients of Calabi-Yau manifolds are also helpful to reduce the family numbers. Recall that family numbers are given by $\mathrm{Ind}(V)$, the index of the holomorphic stable vector bundle V. In order to descend properly to $\frac{X}{\Gamma}$, the vector bundle V should be equivariant with respect to the symmetry Γ, that is it must respect the symmetry. If this is the case, the index of the bundle that V descends to on the quotient is given by $\mathrm{Ind}(V)/|\Gamma|$, with $|\Gamma|$ the order of the discrete group Γ. We will see worked examples of this and more details of the concept of equivariant line bundles in Chapter 3. So in string model building the existence of free quotients of Calabi-Yau manifolds really opens more possibilities for us to get the right particle content of the Standard Model at low energy.

2.4 Elements of Heterotic Model Building

In the last two sections we have discussed some theoretical issues as to why Calabi-Yau manifolds arise in string compactifications, how we can relate low energy physics to the topological data of the Calabi-Yau manifold and vector bundles over it and briefly how to get the Standard Model gauge group by using Wilson lines on free quotients of Calabi-Yau manifolds. In this section we will give a short summary of some relevant facts, to act as a convenient reference for later chapters.

We want to get a low energy effective theory with the Standard Model physics from heterotic string theory which has an $E_8 \times E_8$ gauge group. The space-time structure at the string theory level is assumed to be in the form $M_4 \times X$ with M_4 a maximally symmetric manifold and X a six dimensional compact manifold. In our work we will choose X to be a Calabi-Yau mainfold which has $c_1(TX) = 0$.

We also need to specify a vector bundle V on X to break E_8 into GUT gauge group $SU(5)$, $SO(10)$ or E_6. This vector bundle should be holomorphic and stable, which is in formulas:

$$F_{ab} = F_{\bar{a}\bar{b}} = 0, \quad g^{a\bar{b}} F_{a\bar{b}} = 0. \tag{2.27}$$

Given the form the expected GUT gauge groups, the structure group we persue should be $SU(5)$, $SU(4)$ and $SU(3)$, of which the corresponding vector bundle should satisfy the condition that $c_1(V) = 0$.

The vector bundle should also satisfy the anomaly cancellation condition, which is an integrability condition of (2.4) together with possible corrections from 5-branes. With the condition that $c_1(TX) = c_1(V) = 0$, this can be expressed as a topological formula:

$$c_2(TX) - c_2(V) - c_2(\tilde{V}) = [W], \tag{2.28}$$

here $\tilde{V}$ is the hidden sector vector bundle and $[W]$ is the effective class of the holomorphic two cycles where the 5-branes wrap on.

Finally, we need to choose Wilson lines in an appropriate way such that we can break the GUT gauge group to get the Standard Model group. In addition, all of this structure needs to be such as to give exactly 3 generations of particles in the model we build.

Chapter 3

Jumping Spectra and Vanishing Yukawa Couplings In Heterotic Line Bundle Standard Models

In the last chapter we have seen the general picture of how to relate the mathematics of Calabi-Yau manifolds to physics in four dimensional space time. Here in this chapter we are going to apply these ideas to explicit calculations. The content of this chapter is based on our paper [35].

In the last ten to fifteen years a lot of progress has been made in understanding supersymmetric four dimensional effective theories, descending from smooth Calabi-Yau compactifications of heterotic M-theory. In terms of model building, solutions to the theory which give rise to a charged matter spectrum identical to that of the Minimal Supersymmetric Standard Model (MSSM) have been obtained [14, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59]. These were first constructed in small numbers in the context of irreducible higher rank bundles with non-abelian structure groups [43, 44, 46, 47, 48]. Later, the concept of Line Bundle Standard Models was introduced: it was realized that simple sums of line bundles could be phenomenological viable in this context [49, 50]. This work is of course complemented by extensive model building efforts in other heterotic constructions, see for example [60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77].

This lead to very large numbers of heterotic models being produced with exactly the standard models charged matter content. In another advance, that will be directly relevant to this chapter, good progress has been made in understanding Yukawa couplings in this context [36, 39, 41, 42, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89]. Algebraic methods for computing tree-level superpotential trilinear couplings have long been understood [36, 39, 41, 42, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100]. Recently, however, techniques based upon differential geometry have been developed [88, 89] which, perhaps surprisingly, can be more powerful in many situations. In particular, this work provides a very strong vanishing theorem on these tree-level Yukawa couplings and also makes the computation of the moduli dependence of these quantities more tractable in many contexts.

Although heterotic compactifications have traditionally proven to be extremely promising from the point of view of particle physics model building, they have struggled more in the context of moduli stabilization. Nevertheless there have been a number of recent advances in understanding the $\mathcal{N} = 1$ effective theories associated to these compactifications which have lead to new moduli stabilization mechanisms in this context. Of particular note for the current dissertation, it has been realized that the holomorphic poly-stable slope zero vector bundles that appear in this context can stabilize the complex structure moduli of the base Calabi-Yau manifold [101, 102, 103, 104, 105, 106]. It is important to note in this context that concrete examples of this effect have been provided. While it is still difficult to fix one final over-all modulus in a controlled manner in heterotic compactifications (see [105] for example), it is clear that progress is being made. In addition, there is much that is still not understood about the effective theories' potential - particularly at higher order in curvature expansions.

Given this progress in model building and moduli stabilization it is natural to take the analysis of these models to a finer level of detail. In this chapter we wish to achieve this in two particular regards. First, we wish to begin a study of how modern moduli-stabilization mechanisms in Calabi-Yau compactifications of heterotic M-theory interact with model building concerns. More specifically, we will examine the interplay of the moduli stabilization of [103, 104] with Line Bundle Standard Model building [49, 50]. Using hidden sector vector bundles to stabilize complex structure moduli, as was proposed in [103, 104], forces the base Calabi-Yau threefold to a computable sub-locus of its moduli space. Given this concrete knowledge as to where in complex structure moduli space the system is forced, one can investigate how this stabilization mechanism affects model building considerations. In particular, the bundle valued cohomologies that determine particle spectra in heterotic theories are only quasi-topological in nature. They can jump in dimension at higher co-dimensional loci in complex structure moduli space causing the matter spectrum of the associated four dimensional effective theory to jump in an index preserving manner [107, 108]. If the moduli stabilization mechanism of [103, 104] happens to force the system to a locus where the bundle cohomologies associated to standard model degrees of freedom jump, then that mechanism and model building considerations can not be divorced.

This effect can be either good or bad. If the jump causes the addition of an extra standard model family degree of freedom and its partner from a mirror family, then the moduli stabilization mechanism will have forced the addition of standard model exotics - a phenomenologically undesirable result. In contrast to this, one could envisage a situation where a model which had no Higgs, Higgs conjugate pair, was forced to a locus where the cohomologies of such degrees of freedom where forced to jump. This would render previously unviable models phenomenologically interesting.

One might think that such effects would be extremely rare in heterotic models, given the relatively uncoupled nature of the visible and hidden sector vector bundles. Nevertheless, we will show that, in the class of models we study, this interaction of moduli stabilization and model building considerations occurs rather frequently. More precisely, we find that, in cases where the particle spectrum of the standard model bundle is capable of jumping, such phenomena are common in the known examples of Line Bundle Standard Models. This indicates that one should be aware, in pursuing studies that divorce model building from moduli stabilization, that including the latter concern may be relevant to many of the models obtained.

It should be noted that this effect, where the system is driven to a locus in moduli space where extra degrees of freedom occur, might be naively thought to be rather similar in nature to the work presented in [109, 110, 111, 112, 113, 114, 115]. In fact the phenomena being considered here are completely distinct to that work, being rather different in nature and not as ubiquitous in effect.

The second issue we will consider in this chapter concerns vanishing of Yukawa couplings. As was mentioned above, in [88, 89] a vanishing theorem was presented wherein tree-level trilinear couplings that are consistent with all of the obvious gauge symmetries of the four dimensional effective theory are nevertheless zero due to seemingly topological restrictions. We will investigate to what degree this vanishing theorem comes in to effect in the known set of Line Bundle Standard Models [49, 50]. By the simple method of direct computation in every model in this data set, we discern how many of the couplings that are consistent with the symmetries of these theories, as presented in [49, 50], are actually vanishing due to this theorem. In total 17.9% of the potentially allowed couplings are actually zero, with some forms of interaction vanishing at the 35.4% level. This is therefore, once again, a significant effect which should be borne in mind when constructing heterotic standard models with an eye toward phenomenological viability. That this effect is common was anticipated in [88, 89] - here we compute exact numbers in a standard model building context. In addition to this straight forward computation we briefly suggest, based on the work of [86, 87], a gauge-theoretic mechanism which may underly these severe restrictions on the Yukawa-Couplings of these heterotic effective theories. It will be important to understand whether this conjecture is correct going forwards as, if it is indeed responsible for these vanishings, then one could expect many higher order couplings to suffer a similar fate.

The structure of the rest of this chapter is as follows. In Section 3.1 we briefly review Line Bundle Standard Models in Calabi-Yau threefold compactifications of heterotic theories. We then review, in Section 3.2, the mechanism by which hidden sector bundles can stabilize complex structure moduli in this context. In Section 3.3 we present our work combining moduli stabilization and model building considerations in heterotic Line Bundle Standard Models. Section 3.4 of this chapter contains our analysis of topological vanishing of Yukawa couplings in Line Bundle Standard Models. Finally, in Section 4.5 we present our conclusions. Two appendices contain details of the results from our two lines of investigation which complement the summary data given in the main text.

3.1 Heterotic Line Bundle Standard Models

Traditionally, in constructing a heterotic Calabi-Yau compactifications designed to give rise to physics close to the MSSM, one chooses a gauge bundle V_{SM} with a non-abelian structure group, for example $SU(3)$, $SU(4)$ or $SU(5)$. The low energy gauge group in the visible sector is then simply the commutant of this structure group inside E_8, that is E_6, $SO(10)$ or $SU(5)$ respectively for the examples mentioned in the previous sentence. These precursor 'GUT' groups are then broken down to the standard model gauge group by Wilson lines associated with the fundamental group of the Calabi-Yau threefold.

Line Bundle Standard Models are constructed somewhat differently. Instead of focussing on a non-abelian structure group, the gauge bundle $V_{\rm SM}$ is chosen to be a simple sum of line bundles. Taking a sum of five such objects as an example, we have the following

$$V_{\rm SM} = \bigoplus_{i}^{5} \mathcal{L}_i. \tag{3.1}$$

The structure group of such a bundle is $S(U(1)^5) \cong U(1)^4$. The commutant of this group inside E_8 is $SU(5) \times U(1)^4$ which is therefore, naively, the low energy gauge group. However, the four $U(1)$ factors are all typically Green-Schwarz massive, at least in examples with a Kähler moduli space of high enough dimension, and thus at low energies this approach can also give us viable GUT groups that can then be broken to $SU(3) \times SU(2) \times U(1)$ by an appropriate Wilson line.

The advantage of working with Line Bundle Standard Models over more conventional approaches to heterotic model building largely center around proving that the gauge fields in the compactification preserve supersymmetry. Showing that an irreducible, higher rank bundle is slope-stable can be a time consuming and complicated affair, involving the consideration of an infinite number of possible sub-sheafs of $V_{\rm SM}$. In the case of a simple sum of line bundles such as (3.1) proving that supersymmetry is preserved is much simpler. The equivalent condition in this case is slope poly-stability and for such a sum we need only check that the slope of each line bundle is the same (and in fact vanishes in physical examples). This simplification leads to a huge increase in the number of models that can be constructed with thousands of Line Bundle Standard Models being known [49, 50] while only a few irreducible higher rank gauge bundles have ever been constructed which give rise to the exact charged spectrum of the MSSM [38, 43, 47, 48].

The spectrum of a Line Bundle Standard Model is determined in a two step process. Firstly, an exercise in group theory tells us what matter can possibly appear in the four dimensional effective theory. Secondly, what matter actually does appear is computed in terms of bundle valued cohomology groups.

In terms of group theory, the representations of the four dimensional gauge group that can appear are simply determined by branching rules and the fact that all of the charged matter in ten dimensions is valued in the adjoint representation. Thus, in the $SU(5)$ case for example we find the following decomposition of representations under a maximal subgroup:

$$\begin{aligned} E_8 &= SU(5) \times SU(5), \\ \mathbf{248} &= (\mathbf{24}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{24}) \oplus (\mathbf{5}, \mathbf{10}) \oplus (\overline{\mathbf{5}}, \overline{\mathbf{10}}) \oplus (\mathbf{10}, \overline{\mathbf{5}}) \oplus (\overline{\mathbf{10}}, \mathbf{5}). \end{aligned} \tag{3.2}$$

If we take the first $SU(5)$ factor to be the low energy GUT group and the second $SU(5)$ factor to be that in which the structure group of the bundle resides we can then read off what representations we can possibly obtain in four dimensions. Here, for example, we could potentially obtain the $\mathbf{24}, \mathbf{1}, \mathbf{5}, \overline{\mathbf{5}}, \mathbf{10}$ and $\overline{\mathbf{10}}$ representations of $SU(5)$. In the case of Line Bundle Standard Models, we can also, of course associate a series of $U(1)$ charges to the matter multiplets which we have omitted in (3.2) in the interests of keeping the expressions uncluttered. We follow the convention of including all five $U(1)$ charges associated to $S(U(1)^5)$ despite the fact that only four of these gauge factors are independent as this simplifies many of the resulting equations.

In order to see how many copies of each representation we obtain in the low energy spectrum (if any) we must compute the appropriate bundle cohomology groups. In fact, we wish to incorporate a Wilson line and work out the spectrum at the level of the four dimensional theory with standard model gauge group, just like what we introduced in the last chapter. Since most Calabi-Yau that we know how to construct are simply connected, this we typically obtain a compactification manifold with non-trivial fundamental group that can support a

Wilson line by quotienting some 'upstairs' space X by an appropriate freely acting discrete symmetry Γ. The bundle must be chosen to be equivariant with respect to this symmetry in order that it too is compatible with the quotient. Indeed, following [49, 50] we will consider the case where each line bundle $\mathcal{L}_i$ in V_{SM} is equivariant individually. The spectrum on the 'downstairs' quotient manifold $\hat{X} = X/\Gamma$ can then be given in terms of just a few pieces of data.

As described in [49, 50], if the discrete group Γ is a product of abelian factors of the form $\Gamma = \bigotimes_r \mathbb{Z}_{m_r}$ (as will be considered here), then the definition of the Wilson line proceeds via the choice of two sets of integers p_r and $\tilde{p}_r$. These integers must satisfy the conditions

$$3p_r + 2\tilde{p}_r = 0 \text{ mod } m_r \ \forall r \ \text{ such that } \ p_r \neq \tilde{p}_r \ \text{ for at least one } r. \tag{3.3}$$

We can then define some representations $W(g) = \bigotimes_r e^{p_r g 2\pi i/m_r}$ and $\tilde{W}(g) = \bigotimes_r e^{\tilde{p}_r g 2\pi i/m_r}$. These representations encode all of the information we require about the Wilson line in order to complete a spectrum computation. Indeed, if we combine this information with the characters of Γ, χ_i^*, which define the equivariant structure associated to the line bundle $\mathcal{L}_i$, we can write down the spectrum of the Line Bundle Standard Model associated to these choices, as given in Table 3.1. Note that in this table we use the same notation for the (potentially anomalous) $U(1)$ charges as given in [49, 50]. That is, the $\mathbf{e_i}$ are unit vectors such that, for example $10_{\mathbf{e_1}}$ has a single unit of positive charge under the first abelian factor. Note that, because these five $U(1)$ factors are related in $S(U(1)^5)$, a combination of fields that has a single unit of charge under each factor is a gauge invariant. We will frequently specify the spectrum of a Line Bundle Standard Model by giving a set of GUT multiplets with $U(1)$ charges. Such a notation is consistent because, despite the fact that the different standard model degrees of freedom that would form a single irreducible $SU(5)$ multiplet all descend from different ten-dimensional antecedents and thus no such symmetry is present

$SU(5)$ repr.	G_{SM} repr.	name	cohomology
$\mathbf{10}_{\mathbf{e}_i}$	$(\mathbf{3},\mathbf{2})_1$	Q_i	$h^1(X,\mathcal{L}_i,\chi_i\otimes W^*\otimes\tilde{W}^*)$
	$(\bar{\mathbf{3}},\mathbf{1})_{-4}$	u_i	$h^1(X,\mathcal{L}_i,\chi_i\otimes W^*\otimes W^*)$
	$(\mathbf{1},\mathbf{1})_6$	e_i	$h^1(X,\mathcal{L}_i,\chi_i\otimes\tilde{W}^*\otimes\tilde{W}^*)$
$\bar{\mathbf{5}}_{\mathbf{e}_i+\mathbf{e}_j}$	$(\bar{\mathbf{3}},\mathbf{1})_2$	$d_{i,j}$, $T_{i,j}$	$h^1(\mathcal{L}_i\otimes\mathcal{L}_j,\chi_i\otimes\chi_j\otimes W)$
	$(\mathbf{1},\mathbf{2})_{-3}$	$L_{i,j}$, $H_{i,j}$	$h^1(\mathcal{L}_i\otimes\mathcal{L}_j,\chi_i\otimes\chi_j\otimes\tilde{W})$
$\mathbf{5}_{-\mathbf{e}_i-\mathbf{e}_j}$	$(\mathbf{3},\mathbf{1})_{-2}$	$\bar{T}_{i,j}$	$h^2(\mathcal{L}_i\otimes\mathcal{L}_j,\chi_i\otimes\chi_j\otimes W)$
	$(\mathbf{1},\mathbf{2})_3$	$\bar{H}_{i,j}$	$h^2(\mathcal{L}_i\otimes\mathcal{L}_j,\chi_i\otimes\chi_j\otimes\tilde{W})$
$\mathbf{1}_{\mathbf{e}_i-\mathbf{e}_j}$	$(\mathbf{1},\mathbf{1})_0$	$S_{i,j}$	$h^1(\mathcal{L}_i\otimes\mathcal{L}_j^\vee,\chi_i\otimes\chi_j^*)$

Table 3.1: *Cohomologies which determine the downstairs spectrum of Line Bundle Standard Models. The cohomological notation including a representation after a comma simply denotes that only the piece of the cohomology forming that representation under the discrete group Γ should be considered. The representations W, $\tilde{W}$ and χ_i are described in the text. The number of mirror particles is determined by the second cohomology valued in the same bundles and representations.*

in the four-dimensional theory, the standard model multiplets do arise in complete GUT multiplets.

A key point for the latter sections of this chapter is that the cohomologies appearing in Table 3.1 are complex structure dependent. At higher codimension loci in complex structure moduli space, the dimensions of these cohomology groups, and thus the matter spectrum of the resulting four dimensional theory can jump in an index preserving manner.

We will consider a particular existent Line Bundle Standard Model data set [49, 50] built over Calabi-Yau manifolds which can be described as quotients of complete intersections in products of projective spaces (CICYs) [116, 117, 118, 119, 120, 121, 122]. Note that analogous constructions could be pursued over different base spaces, such as quotients of gCICYs [123] or toric hypersurfaces [124, 125, 126, 127, 128]. It would be interesting to see if such constructions mirror the structure that we will describe in this chapter.

A family of CICYs can be represented by a configuration matrix of the following form:

$$X = \left[\begin{array}{c|cccc} \mathbb{P}^{n_1} & q_1^1 & q_2^1 & \dots & q_k^1 \\ \mathbb{P}^{n_2} & q_1^2 & q_2^2 & \dots & q_k^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{P}^{n_m} & q_1^m & q_2^m & \dots & q_k^m \end{array} \right] . \tag{3.4}$$

Here, the first column specifies the ambient space A in which the Calabi-Yau manifold X will be defined, $A = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. The manifold X is defined within this ambient space as the common zero locus of a set of k defining polynomials. The remaining columns each determine the multi-degree of one of these defining polynomials. In a given column each row specifies the degree of that defining relation with respect to the homogeneous coordinates of the corresponding ambient space factor. Throughout this chapter we will denote by $x_{r,a}$ the a^{th} homogeneous coordinate on the r^{th} ambient space projective factor.

The dimension of a complete intersection described by a configuration matrix of the form (3.4) is simply $\sum_r n_r - k$. That is, the dimension is simply given as the dimension of the ambient space minus the number of constraints being imposed. The condition for a vanishing first Chern class for X, meaning that the manifold is indeed Calabi-Yau, can be achieved if the following condition is met:

$$n_r + 1 = \Sigma_{l=1}^{k} q_l^r \ \ \forall \, r. \tag{3.5}$$

The CICYs are all simply connected and therefore, in order to accommodate Wilson line breaking, quotients of these manifolds by appropriate freely acting discrete symmetries are considered. Braun has classified all such actions, allowing for a set of defining relations which respect the symmetry while remaining transverse, that descend from a linear action on the ambient spaces A that appear in the original classifying list of such constructions [129].

Having specified the Calabi-Yau manifolds to be utilized $\hat{X}$ in the above way, in [49, 50] the authors then produce Heterotic Line Bundle Standard Models by specifying appropriate sums of line bundles on X. These are chosen to be equivariant under the symmetries by which the manifolds are quotiented and to give rise to spectra on $\hat{X}$ which precisely match that of the standard model in the sector charged under $SU(3) \times SU(2) \times U(1)$. If one works with favorable CICYs, where all of $H^{1,1}(X)$ descends from forms dual to divisors on the ambient space, general line bundles on X can be specified by the following notation

$$\mathcal{L} = \mathcal{O}_{\mathbf{X}}(p_1, p_2, \ldots, p_k) \Leftrightarrow c_1(\mathcal{L}) = \sum_r p_r J_r. \tag{3.6}$$

Here J_r is the Kähler forms of the r^{th} ambient space factor, restricted to the Calabi-Yau threefold. These are the restriction of the analogous line bundles on the ambient space $\mathcal{O}_{\mathbb{A}}(p_1, p_2, \ldots, p_k))$ to X. The bundle V_{SM} is then taken to be an equivariant sum of five such objects. In fact, in the data set of [49, 50], each line bundle in V_{SM} is taken to be equivariant individually. As we will see in concrete examples in later sections, the cohomology of various products of these line bundles can be computed using a combination of a theorem due to Bott, Borel and Weil and the Koszul sequence [21, 130]. For a discussion of equivariance in this setting, and induced symmetry actions on cohomology see for example the appendices of [50]. Once the cohomology, and its representation content, of the line bundles is known, the spectrum of the associated heterotic theory can be read off from Table 3.1.

Using such a construction, in [49, 50], a data set of 2012 Line Bundle Standard Models was produced. It is properties of this data set that will be examined in the rest of this chapter. It would certainly be interesting to apply a similar analysis to larger data sets of this type which could be obtained by extending the work of [55], for example.

3.2 Moduli Stabilization, Potentials and Couplings

The moduli stabilization mechanism we will consider in this chapter concerns the complex structure degrees of freedom and was presented in [103, 104] (a similar description of moduli stabilization in Type II was considered in [131]). We will be particularly interested in the mechanism for fixing these particular moduli in the current work as the cohomology groups determining the spectrum of a model, as presented in the previous section, depend upon these degrees of freedom. The basic mechanism is as follows.

As we know from Chapter 2, an $\mathcal{N} = 1$ compactification of heterotic string theory on a Calabi-Yau threefold X includes a gauge connection on a gauge bundle V which satisfies the Hermitian Yang-Mills equations:

$$g^{a\bar{b}} F_{a\bar{b}} = 0 \ , \ F_{ab} = F_{\bar{a}\bar{b}} = 0. \tag{3.7}$$

Starting with a good solution to these equations, one can consider a perturbation of all of the degrees of freedom of the problem about that vacuum. In particular, focusing on the holomorphy condition $F_{\bar{a}\bar{b}} = 0$, we can perturb the complex structure and gauge connection and ask what constraints maintaining supersymmetry places on those variations. The following condition is obtained [103, 104]:

$$\delta \mathcal{J}_{[\bar{a}}{}^{d} F^{(0)}_{\bar{b}]d} + 2i D^{(0)}_{[\bar{a}} \delta A_{\bar{b}]} = 0. \tag{3.8}$$

Here $\delta\mathcal{J} \in H^1(TX)$ is a variation of the complex structure tensor, δA is the perturbation in the gauge connection and objects with a superscript (0) are constructed from unperturbed quantities. What (3.8) states is that a complex structure fluctuation is a true low energy degree of freedom only if there exists a gauge field fluctuation which solves this constraint.

Otherwise, such a variation of complex structure will necessarily cause the bundle to become non-holomorphic, breaking supersymmetry.

Equation (3.8) can be interpreted cohomologically as saying that the complex structure moduli of the base Calabi-Yau threefold that are true massless degrees of freedom of the four dimensional effective theory are given as the following kernel:

$$\ker\left(H^1(TX) \xrightarrow{F^{(0)}} H^2(\mathrm{End}_0(V))\right). \tag{3.9}$$

The allowed deformations of the connection are much easier to understand. A gauge field fluctuation living in the usual cohomology describing the bundle moduli, $\delta A \in H^1(\mathrm{End}_0(V))$, satisfies (3.8) for a vanishing $\delta\mathcal{J}$ and is therefore always consistent with holomorphy as one would expect.

The permitted combined deformations of the base complex structure and bundle moduli of holomorphic vector bundles is in fact very well studied in the mathematics literature. Indeed, the above discussion is simply a field theory manifestation of Atiyah's discussion of the tangent to the moduli space of holomorphic bundles [132]. Atiyah states that the allowed deformations are given by $H^1(\mathcal{Q})$ where the bundle $\mathcal{Q}$ is defined by the following short exact extension sequence

$$0 \to \mathrm{End}_0(V) \to \mathcal{Q} \to TX \to 0. \tag{3.10}$$

Analyzing the long exact sequence in cohomology associated to (3.10) one then finds the following,

$$H^1(\mathcal{Q}) = H^1(\mathrm{End}_0(V)) \oplus \ker\left(H^1(TX) \longrightarrow H^2(\mathrm{End}_0(V))\right)\ , \tag{3.11}$$

which agrees with the field theoretic analysis given above.

The above discussion shows in general terms a choice of bundle can restrict complex structure moduli via the requirement of holomorphy of that object. However, it will be crucial for the purposes of this chapter to construct explicit examples of such bundles and compute to exactly which locus in complex structure moduli space the system is constrained.

Fortunately such examples have indeed been provided in the literature [103, 104, 106]. Perhaps the simplest such examples take the form of bundles of $SU(2)$ structure group which are constructed as extensions of a line bundle and its dual. To see how this works it is simplest to look at an explicit case. The example that follows was first presented in [106].

As a base manifold, let us consider a freely acting quotient $\hat{X}$ of the the following CICY,

$$X = \left[\begin{array}{c|c} \mathbb{P}^1 & 2 \\ \mathbb{P}^1 & 2 \\ \mathbb{P}^1 & 2 \\ \mathbb{P}^1 & 2 \end{array}\right] , \tag{3.12}$$

by the following $\mathbb{Z}_2 \times \mathbb{Z}_4$ symmetry action:

$$\begin{aligned} \gamma_1 \quad &: \quad x_{r,a} \to (-1)^{a+r+1} x_{r,a}, \\ \gamma_2 \quad &: \quad x_{r,a} \to x_{\sigma(r),a+r+1} \text{ where } \sigma = (12)(34). \end{aligned} \tag{3.13}$$

It will useful going forward to know the most general form of the polynomial defining relation for X that is consistent with the symmetry (3.13). This is explicitly given by the following expression:

$$
\begin{aligned}
p \;=\;& \mathrm{c}_1 x_{1,0}x_{1,1}x_{2,0}x_{2,1}x_{3,0}x_{3,1}x_{4,0}x_{4,1} + \mathrm{c}_9 \left(x_{1,0}^2 x_{3,0}x_{3,1}x_{4,0}x_{4,1}x_{2,0}^2 + x_{1,1}^2 x_{3,0}x_{3,1}x_{4,0}x_{4,1}x_{2,0}^2\right.\\
&\left.+x_{1,0}^2 x_{2,1}^2 x_{3,0}x_{3,1}x_{4,0}x_{4,1} + x_{1,1}^2 x_{2,1}^2 x_{3,0}x_{3,1}x_{4,0}x_{4,1}\right) + \mathrm{c}_3 \left(x_{1,1}^2 x_{2,0}x_{2,1}x_{4,0}x_{4,1}x_{3,0}^2\right.\\
&\left.+x_{1,0}x_{1,1}x_{2,1}^2 x_{3,1}x_{4,0}^2 x_{3,0} + x_{1,0}x_{1,1}x_{2,0}^2 x_{3,1}x_{4,1}^2 x_{3,0} + x_{1,0}^2 x_{2,0}x_{2,1}x_{3,1}^2 x_{4,0}x_{4,1}\right) +\\
&\mathrm{c}_4 \left(x_{1,0}x_{1,1}x_{2,0}^2 x_{4,0}x_{4,1}x_{3,0}^2 + x_{1,1}^2 x_{2,0}x_{2,1}x_{3,1}x_{4,0}^2 x_{3,0} + x_{1,0}^2 x_{2,0}x_{2,1}x_{3,1}x_{4,1}^2 x_{3,0}\right.\\
&\left.+x_{1,0}x_{1,1}x_{2,1}^2 x_{3,1}^2 x_{4,0}x_{4,1}\right) + \mathrm{c}_5 \left(x_{1,0}x_{1,1}x_{2,1}^2 x_{4,0}x_{4,1}x_{3,0}^2 + x_{1,0}^2 x_{2,0}x_{2,1}x_{3,1}x_{4,0}^2 x_{3,0}\right.\\
&\left.+x_{1,1}^2 x_{2,0}x_{2,1}x_{3,1}x_{4,1}^2 x_{3,0} + x_{1,0}x_{1,1}x_{2,0}^2 x_{3,1}^2 x_{4,0}x_{4,1}\right) + \mathrm{c}_6 \left(x_{1,0}^2 x_{2,0}x_{2,1}x_{4,0}x_{4,1}x_{3,0}^2\right.\\
&\left.+x_{1,0}x_{1,1}x_{2,0}^2 x_{3,1}x_{4,0}^2 x_{3,0} + x_{1,0}x_{1,1}x_{2,1}^2 x_{3,1}x_{4,1}^2 x_{3,0} + x_{1,1}^2 x_{2,0}x_{2,1}x_{3,1}^2 x_{4,0}x_{4,1}\right) +\\
&\mathrm{c}_7 \left(x_{1,1}^2 x_{2,1}^2 x_{3,0}^2 x_{4,0}^2 + x_{1,0}^2 x_{2,1}^2 x_{3,1}^2 x_{4,0}^2 + x_{1,1}^2 x_{2,0}^2 x_{3,0}^2 x_{4,1}^2 + x_{1,0}^2 x_{2,0}^2 x_{3,1}^2 x_{4,1}^2\right) +\\
&\mathrm{c}_8 \left(x_{1,0}^2 x_{2,1}^2 x_{3,0}^2 x_{4,0}^2 + x_{1,0}^2 x_{2,0}^2 x_{3,1}^2 x_{4,0}^2 + x_{1,1}^2 x_{2,1}^2 x_{3,0}^2 x_{4,1}^2 + x_{1,1}^2 x_{2,0}^2 x_{3,1}^2 x_{4,1}^2\right) +\\
&\mathrm{c}_2 \left(x_{1,0}x_{1,1}x_{2,0}x_{2,1}x_{3,0}^2 x_{4,0}^2 + x_{1,0}x_{1,1}x_{2,0}x_{2,1}x_{3,1}^2 x_{4,0}^2 + x_{1,0}x_{1,1}x_{2,0}x_{2,1}x_{3,0}^2 x_{4,1}^2\right.\\
&\left.+x_{1,0}x_{1,1}x_{2,0}x_{2,1}x_{3,1}^2 x_{4,1}^2\right) + \mathrm{c}_{10} \left(x_{1,1}^2 x_{2,0}^2 x_{3,0}^2 x_{4,0}^2 + x_{1,1}^2 x_{2,1}^2 x_{3,1}^2 x_{4,0}^2 + x_{1,0}^2 x_{2,0}^2 x_{3,0}^2 x_{4,1}^2\right.\\
&\left.+x_{1,0}^2 x_{2,1}^2 x_{3,1}^2 x_{4,1}^2\right) + \mathrm{c}_{11} \left(x_{1,0}^2 x_{2,0}^2 x_{3,0}^2 x_{4,0}^2 + x_{1,1}^2 x_{2,0}^2 x_{3,1}^2 x_{4,0}^2 + x_{1,0}^2 x_{2,1}^2 x_{3,0}^2 x_{4,1}^2\right.\\
&\left.+x_{1,1}^2 x_{2,1}^2 x_{3,1}^2 x_{4,1}^2\right) .
\end{aligned}
\tag{3.14}
$$

Here the coefficients c are general constants which form a redundant description of the complex structure moduli of the manifold.

Over this base we construct the extension,

$$0 \to \mathcal{L} \to V \to \mathcal{L}^\vee \to 0 \,, \tag{3.15}$$

where $\mathcal{L}$ is the line bundle that descends from the object $\mathcal{O}_X(-2,-2,1,1)$ on the covering space in the language outlined in the previous subsection. This line bundle is equivariant with respect to the $\mathbb{Z}_2 \times \mathbb{Z}_4$ symmetry and thus the construction does indeed respect the symmetry being quotiented by. The non-trivial nature of the bundle (3.15) is controlled by extension group $\text{Ext}^1(\mathcal{L}^\vee, \mathcal{L}) = H^1(X, \mathcal{L}^2)$, or rather by the appropriately transforming piece of this that survives in the downstairs theory. For the line bundle specified here, this cohomology vanishes for a generic enough choice of complex structure of X. As such, generically, the only extension of the form (3.15) is the split bundle which has structure group $S(U(1) \times U(1))$. However, at higher codimension loci in complex structure moduli space the cohomology $H^1(X, \mathcal{L}^2)$ jumps in dimension to non-zero values. At such loci, one can define a non-split $SU(2)$ bundle of the form (3.15).

The essential idea, then is to start with a background wherein the complex structure is fixed to a jumping locus of $H^1(X, \mathcal{L}^2)$ and the vector bundle V is taken to be an irreducible rank 2 object of the form (3.15). One would expect that complex structure fluctuations that took the system off of this loci would not lie in the kernel (3.9) as there would then be no appropriate $SU(2)$ bundle to perturb to and going to the split bundle would be more than an infinitesimal perturbation of the gauge connection. It was shown in [104] that this is indeed the case. In such a situation, the requirement of bundle holomorphy stabilizes the system to the jumping locus of the extension group.

In fact, the computations that one performs to explicitly find the stabilization locus associated to such a bundle reveal an extremely rich structure. To perform such calculations one examines the Koszul sequence which, in the current example, takes the following form:

$$0 \to \mathcal{N}^\vee \otimes \mathcal{L}_A^2 \to \mathcal{L}_A^2 \to \mathcal{L}_X^2 \to 0. \tag{3.16}$$

Performing sequence chasing on the long exact sequence in cohomology associated to (3.16) and using some facts associated to the specific example we have described above one can find that the cohomology group describing the extension classes of (3.15) is given by the following expression:

$$H^1(X, \mathcal{L}^2) = \ker\left(H^2(A, \mathcal{N}^\vee \otimes \mathcal{L}_A^2) \to H^2(A, \mathcal{L}_A^2)\right) . \tag{3.17}$$

In the case at hand, we can denote a general element of the cohomology $H^2(A, \mathcal{N}^\vee \otimes \mathcal{L}_A^2) = H^2(A, \mathcal{O}(-6,-6,0,0))$, in polynomial language via the Bott-Borel-Weil theorem, as follows,

$$\begin{aligned} & s_1\left(\frac{1}{x_{1,0}^2 x_{1,1}^2 x_{2,0}^2 x_{2,1}^2}\right) + s_3\left(\frac{1}{x_{1,0}^4 x_{2,0}^2 x_{2,1}^2} + \frac{1}{x_{1,1}^4 x_{2,0}^2 x_{2,1}^2} + \frac{1}{x_{1,0}^2 x_{1,1}^2 x_{2,0}^4} + \frac{1}{x_{1,0}^2 x_{1,1}^2 x_{2,1}^4}\right) \\ & + s_2\left(\frac{1}{x_{1,0}^3 x_{1,1} x_{2,0}^3 x_{2,1}} + \frac{1}{x_{1,0} x_{1,1}^3 x_{2,0}^3 x_{2,1}} + \frac{1}{x_{1,0}^3 x_{1,1} x_{2,0} x_{2,1}^3} + \frac{1}{x_{1,0} x_{1,1}^3 x_{2,0} x_{2,1}^3}\right) \\ & + s_4\left(\frac{1}{x_{1,0}^4 x_{2,0}^4} + \frac{1}{x_{1,1}^4 x_{2,0}^4} + \frac{1}{x_{1,0}^4 x_{2,1}^4} + \frac{1}{x_{1,1}^4 x_{2,1}^4}\right) . \end{aligned} \tag{3.18}$$

Here the s_k are arbitrary coefficients. Given this, via (3.17) any potential extension class can be represented by an object of the form (3.18).

By performing an explicit computation, examples of which can be found in [103, 104, 106] or later sections of this chapter, one can obtain a set of loci, that is a reducible algebraic variety, in a combined space of complex structure moduli and potential extension classes of

(3.15). In the case at hand, the generators of this reducible variety are as follows:

$$c_7s_1 + c_2s_2 + c_8s_3 + c_{10}s_3 + c_{11}s_4, c_8s_1 + c_2s_2 + c_7s_3 + c_{11}s_3 + c_{10}s_4, c_9s_1 + c_1s_2 + 2c_9s_3$$
$$+c_9s_4, c_{10}s_1 + c_2s_2 + c_7s_3 + c_{11}s_3 + c_8s_4, c_{11}s_1 + c_2s_2 + c_8s_3 + c_{10}s_3 + c_7s_4, c_3s_1 + c_4s_2 + c_5s_2$$
$$+c_6s_3, c_4s_1 + c_3s_2 + c_6s_2 + c_5s_3, c_5s_1 + c_3s_2 + c_6s_2 + c_4s_3, c_6s_1 + c_4s_2 + c_5s_2 + c_3s_3,$$
$$c_{10}s_1 + c_2s_2 + c_7s_3 + c_{11}s_3 + c_8s_4, c_7s_1 + c_2s_2 + c_8s_3 + c_{10}s_3 + c_{11}s_4, c_9s_1 + c_1s_2 + 2c_9s_3$$
$$+c_9s_4, c_{11}s_1 + c_2s_2 + c_8s_3 + c_{10}s_3 + c_7s_4, c_8s_1 + c_2s_2 + c_7s_3 + c_{11}s_3 + c_{10}s_4, c_5s_1 + c_3s_2 + c_6s_2$$
$$+c_4s_3, c_3s_1 + c_4s_2 + c_5s_2 + c_6s_3, c_6s_1 + c_4s_2 + c_5s_2 + c_3s_3, c_4s_1 + c_3s_2 + c_6s_2 + c_5s_3, c_2s_1$$
$$+c_7s_2 + c_8s_2 + c_{10}s_2 + c_{11}s_2, c_2s_1 + c_7s_2 + c_8s_2 + c_{10}s_2 + c_{11}s_2, c_1s_1 + 4c_9s_2, c_2s_1 + c_7s_2$$
$$+c_8s_2 + c_{10}s_2 + c_{11}s_2, c_2s_1 + c_7s_2 + c_8s_2 + c_{10}s_2 + c_{11}s_2, c_4s_1 + c_3s_2 + c_6s_2 + c_5s_3, c_6s_1$$
$$+c_4s_2 + c_5s_2 + c_3s_3, c_3s_1 + c_4s_2 + c_5s_2 + c_6s_3, c_5s_1 + c_3s_2 + c_6s_2 + c_4s_3, c_8s_1 + c_2s_2 + c_7s_3$$
$$+c_{11}s_3 + c_{10}s_4, c_{11}s_1 + c_2s_2 + c_8s_3 + c_{10}s_3 + c_7s_4, c_9s_1 + c_1s_2 + 2c_9s_3 + c_9s_4, c_7s_1$$
$$+c_2s_2 + c_8s_3 + c_{10}s_3 + c_{11}s_4, c_{10}s_1 + c_2s_2 + c_7s_3 + c_{11}s_3 + c_8s_4, c_6s_1 + c_4s_2 + c_5s_2 + c_3s_3,$$
$$c_5s_1 + c_3s_2 + c_6s_2 + c_4s_3, c_4s_1 + c_3s_2 + c_6s_2 + c_5s_3, c_3s_1 + c_4s_2 + c_5s_2 + c_6s_3, c_{11}s_1 + c_2s_2$$
$$+c_8s_3 + c_{10}s_3 + c_7s_4, c_{10}s_1 + c_2s_2 + c_7s_3 + c_{11}s_3 + c_8s_4, c_9s_1 + c_1s_2 + 2c_9s_3 + c_9s_4,$$
$$c_8s_1 + c_2s_2 + c_7s_3 + c_{11}s_3 + c_{10}s_4, c_7s_1 + c_2s_2 + c_8s_3 + c_{10}s_3 + c_{11}s_4.$$

Essentially, if one substitutes in a specific complex structure into these equations then the possible solutions for the s_k specify all of the possible extensions classes at that point in moduli space in terms of the description given in (3.18).

Next, this reducible algebraic variety can be primary decomposed to find its irreducible pieces. A discussion of the methods that we use for computations such as this can be found in, for example, [133, 134]. We utilized the specific implementations found in [135, 136] in this work. Each of these pieces can then be processed further by an elimination of the variables

s_k describing the possible extension classes. This provides a set of irreducible varieties in complex structure moduli space which are the loci to which the associated choices of extension classes stabilize the system. There can be a great many such loci. For example, in [106], it was shown for the example described above that there are 25 such loci in complex structure moduli space to which one could be stabilized, varying from points to 7 dimensional surfaces. The specific loci that were found in that work are reproduced in Table 3.2.

Equations	**Dimension**	**Singular**
$c_3 - c_4 - c_5 + c_6 = c_2 - c_7 - c_8 - c_{10} - c_{11} = c_1 - 4c_9 = 0$	7	singular
$c_3 + c_4 + c_5 + c_6 = c_2 + c_7 + c_8 + c_{10} + c_{11} = c_1 + 4c_9 = 0$	7	singular
$c_9 = c_2 = c_1 = c_7 + c_8 + c_{10} + c_{11} = c_4 + c_5 = c_3 + c_6 = 0$	4	singular
$c_7 - c_8 - c_{10} + c_{11} = c_4 - c_5 = c_3 - c_6 = c_2 = c_1 = 0$	5	singular
$c_7 - c_8 - c_{10} + c_{11} = c_6 = c_5 = c_4 = c_3 = c_1c_8 - 2c_2c_9 + c_1c_{10} = 0$	4	singular
$c_{11} = c_{10} = c_9 = c_8 = c_7 = 0$	5	singular
$c_9 = c_6 = c_5 = c_4 = c_3 = c_2 = c_1 = c_8 + c_{10} = c_7 + c_{11} = 0$	1	singular
$c_9 = c_2 = c_1 = c_8 + c_{10} = c_7 + c_{11} = c_5 + c_6 = c_4 + c_6 = c_3 - c_6 = 0$	2	singular
$c_9 = c_2 = c_1 = c_8 + c_{10} = c_7 + c_{11} = c_5 - c_6 = c_4 - c_6 = c_3 - c_6 = 0$	2	singular
$c_{11} = c_{10} = c_9 = c_8 = c_7 = c_2 = c_1 = c_3 - c_4 - c_5 + c_6 = 0$	2	singular
$c_{11} = c_{10} = c_9 = c_8 = c_7 = c_2 = c_1 = c_3 + c_4 + c_5 + c_6 = 0$	2	singular
$c_{11} = c_{10} = c_9 = c_8 = c_7 = c_2 = c_1 = c_4 + c_5 = c_3 + c_6 = 0$	1	singular
$c_{11} = c_{10} = c_9 = c_8 = c_7 = c_2 = c_1 = c_4 - c_5 = c_3 - c_6 = 0$	1	singular
$c_{11} = c_{10} = c_9 = c_8 = c_7 = c_2 = c_1 = c_5 + c_6 = c_4 + c_6 = c_3 - c_6 = 0$	0	singular
$c_{11} = c_{10} = c_9 = c_8 = c_7 = c_2 = c_1 = c_5 - c_6 = c_4 - c_6 = c_3 - c_6 = 0$	0	singular
$c_{10} - c_{11} = c_8 - c_{11} = c_7 - c_{11} = c_6 = c_5 = c_4 = c_3 = 0$	3	singular
$c_{10} - c_{11} = c_8 - c_{11} = c_7 - c_{11} = c_6 = c_5 = c_4 = c_3 = c_2c_9 - c_1c_{11} = 0$	2	singular
$c_{10} - c_{11} = c_8 - c_{11} = c_7 - c_{11} = c_4 + c_5 = c_3 + c_6 = c_2c_9 - c_1c_{11} = 0$	4	smooth
$c_{10} - c_{11} = c_8 - c_{11} = c_7 - c_{11} = c_5 + c_6 = c_4 + c_6 = c_3 - c_6 = c_2c_9 - c_1c_{11} = 0$	3	singular
$c_{10} - c_{11} = c_8 - c_{11} = c_7 - c_{11} = c_5 - c_6 = c_4 - c_6 = c_3 - c_6 = c_2c_9 - c_1c_{11} = 0$	3	singular
$c_8 - c_{10} = c_7 - c_{11} = c_6 = c_5 = c_4 = c_3 = c_2c_9 + 50c_1c_{10} + 50c_1c_{11} = 0$	3	singular
$c_{10} + c_{11} = c_9 = c_6 = c_5 = c_4 = c_3 = c_2 = c_1 = c_8 + c_{11} = c_7 - c_{11} = 0$	0	singular
$c_{10} + c_{11} = c_9 = c_2 = c_1 = c_8 + c_{11} = c_7 - c_{11} = c_4 - c_5 = c_3 - c_6 = 0$	2	singular
$c_{10} + c_{11} = c_9 = c_2 = c_1 = c_8 + c_{11} = c_7 - c_{11} = c_5 + c_6 = c_4 + c_6 = c_3 - c_6 = 0$	1	singular
$c_{10} + c_{11} = c_9 = c_2 = c_1 = c_8 + c_{11} = c_7 - c_{11} = c_5 - c_6 = c_4 - c_6 = c_3 - c_6 = 0$	1	singular

Table 3.2: *A table of results taken from [106] showing the loci in complex structure moduli space to which the Calabi-Yau three-fold $\tilde{X}/(\mathbb{Z}_2 \times \mathbb{Z}_4)$ can be stabilized by the bundle V in equation (3.15). The column "Dimension" denotes the complex dimension of the given locus. The column "Singular" specifies whether the Calabi-Yau three-fold associated with a generic complex structure in each locus is singular or smooth.*

Finally one should check, for each locus that the system could be stabilized to, that for a generic enough choice of complex structure moduli in that set the Calabi-Yau manifold is smooth. This can be quite constraining, especially for quotients of CICYs, and in fact only one of the loci for the example being discussed here, of dimension 4, turns out to correspond to a smooth threefold (see Table 3.2).

3.3 Particle Spectrum Jumping due to Moduli Stabilization

In this section we will consider the interplay between bundles constructed in the visible sector in order to engineer a standard model like spectrum in the low energy theory, and bundles inserted into the hidden E_8 in order to stabilize complex structure moduli. In particular, we will be investigating to what degree hidden sector bundles can force the complex structure of the Calabi-Yau threefold to a locus in moduli space where the visible sector is forced to jump. Such an effect could be either beneficial (in introducing a Higgs doublet pair into a model which previously had none for example) or undesired (for example in causing additional generations and anti-generations to appear).

There are several possibilities for intersection of the jumping locus of the standard model sector and hidden sector bundles in complex structure moduli space. These are depicted in Figure 3.1. In the first situation depicted in the figure, the locus of jumping of the standard model bundle lies inside the locus of jumping of the hidden sector one. In the second situation the two loci intersect at a higher codimension locus in complex structure moduli space. In both of these cases, the moduli stabilization mechanism does not force the standard model matter sector to change from that found at an arbitrary point in complex structure moduli space, where most such models are analyzed during their construction. In both situations,

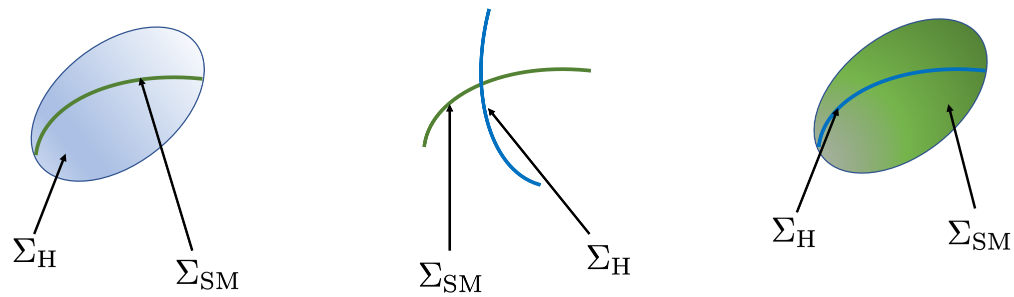

Figure 3.1: *Three possible situations involving intersection between the locus in moduli space Σ_H where the hidden sector bundle stabilizes the complex structure and the locus Σ_{SM} where the spectrum of the standard model bundle jumps. The only case in which the moduli stabilization mechanism forces the standard model spectrum to change is the last, as discussed further in the text.*

if the stabilization mechanism forces the complex structure to a generic enough point in Σ_{H}, then Σ_{SM} will miss this point.

On the other hand, the third situation in Figure 3.1, or its extreme limit where $\Sigma_{\mathrm{H}} = \Sigma_{\mathrm{SM}}$ is of interest to us here. In this case, wherever we are on the hidden sector locus the standard model spectrum jumps from that which is observed at a generic point in complex structure moduli space. As such, if model building was carried out without thinking about the moduli stabilization mechanism, then incorrect conclusions would be reached about the particle content of the four dimensional effective theory.

Naively, one might think that such a phenomenon would be extremely rare. After all, the visible and hidden E_8's of heterotic string or M-theory are rather separate in nature and are only coupled to each other gravitationally. Given this, why should the locus of jumping of a bundle in one sector lie exactly inside that of another ($\Sigma_{\mathrm{H}} \subset \Sigma_{\mathrm{SM}}$)? There are some conditions linking the two bundles, however, and we will find that these are strong enough to make the phenomenon we are talking about surprisingly common.

The first condition we will consider is the standard one following from requiring integrability of the Bianchi Identity:

$$\mathrm{ch}_2(V_{\mathrm{H}})_a + \mathrm{ch}_2(V_{\mathrm{SM}})_a - \mathrm{ch}_2(TX)_a + [W]_a = 0 \ \ \forall a. \tag{3.19}$$

This formula is equivalent to formula (2.28) we gave at the end of Chapter 2 by using the fact that the first Chern classs of all the vector bundles here vanish. Here the indices $a = 1, \ldots, h^{(1,1)}(X)$ label the harmonic $(2,2)$ forms on the Calabi-Yau threefold, V_{SM} and V_{H} are the visible and hidden sector bundles respectively, X is the Calabi-Yau manifold and $[W]$ is a form proportional to the dual of the class of the (in general reducible) curve wrapped by NS 5-branes/M5 branes in the vacuum configuration being considered (in the heterotic string/heterotic M-theory respectively). Allowing for M5 branes that preserve supersymmetry, and thus lead to a class $[W]$ that is effective, (3.19) leads us to the following inequality:

$$\mathrm{ch}_2(V_{\mathrm{H}})_a + \mathrm{ch}_2(V_{\mathrm{SM}})_a \leq \mathrm{ch}_2(TX)_a \ \ \forall \ \ a. \tag{3.20}$$

In addition to the second Chern character constraint (3.20), there is the constraint that both the visible and hidden sector bundles must be slope poly-stable and slope zero for the same choice of four dimensional Kähler moduli. Due to the warping of heterotic M-theory, there is a slight difference between the polarizations experienced between the two bundles, but nevertheless this is easy to account for. As with (3.20), providing that both bundles are indeed stable in reasonably large chambers of the Kähler cone, this constraint is not seemingly too difficult to satisfy.

Although the inequality (3.20) and the requirement for simultaneous stability of the hidden and visible sector gauge bundles may not seem like a very strong set of constraints, in some cases it can become so once one considers quotienting the Calabi-Yau manifold in order to introduce Wilson lines[1]. The issue is that equivariance constraints, ensuring that the gauge bundles are consistent with the symmetry by which the Calabi-Yau threefold is being quotiented, can mean that quite a few bundles are not available in building models and hidden sectors. The resulting combination of equivariance, stability and second Chern class constraints can be quite restrictive.

3.3.1 A systematic investigation of a class of bundle constructions

To illustrate the issues discussed above, and to obtain an idea of how commonly moduli stabilization affects the visible sector spectrum, at least in a class of examples, we will look at specific types of construction of visible and hidden sector bundles. The visible sector will be taken to be a sum of line bundles (more specifically a line bundle standard model). The hidden sector will be taken to be a simple extension of two line bundles of the following form

$$0 \to \mathcal{L} \to V_{\mathrm{H}} \to \mathcal{L}^{\vee} \to 0. \tag{3.21}$$

Constructions of the type (3.21) are perhaps the simplest types of bundles that can lead to complex structure stabilization of the type described in Section 3.2. They have structure group $SU(2)$ and the only simpler possibility, that of an abelian structure group, is ruled out by the fact that sums of line bundles do not exhibit this phenomenon.

In considering examples of such hidden and visible sector bundles, one immediately sees one compatibility constraint that arises. Consider a line bundle sum V_{SM} in the visible sector containing a line bundle $\mathcal{L}_1$. That same line bundle can not be used in creating an extension of the form (3.21) for the hidden sector bundle V_{H}. The issue is simply one of stability. As

[1] Which has been shown to be essentially the only way to break the GUT group in such compactifications [137].

can be seen from the defining sequence (3.21), if $\mathcal{L} = \mathcal{L}_1$ then that line bundle injects into $V_{\rm H}$. It must therefore be of negative slope if $V_{\rm H}$ is to be stable. However, on the locus in Kähler moduli space where $V_{\rm SM}$ is poly-stable $\mu(\mathcal{L}_1) = 0$ and thus the visible and hidden sectors can't simultaneously preserve supersymmetry in such a situation. Similarly one can not set $\mathcal{L} = \mathcal{L}_1^\vee$. In such a situation we find an exactly analogous situation when we consider the stability of $V_{\rm H}^\vee$. Since $V_{\rm H}$ is stable iff $V_{\rm H}^\vee$ is, this leads us to the same conclusion.

Overall, the observation of the previous paragraph can be quite a big constraint on the possible hidden sectors that can be included to complete a line bundle standard model and stabilize complex structure moduli. As stated earlier, there are frequently not many choices of equivariant line bundles that can be included in an extension such as (3.21) in the hidden sector without violating the bound on $\text{ch}_2(V_{\rm H})$ imposed by integrability of the Bianchi Identity and supersymmetry. Given that all of the line bundles that appear in the line bundle standard model (and their duals) are ruled out on grounds of stability, in some cases one can be left with very few, or even no, possibilities.

Assuming simultaneously stable hidden and visible sector bundles can be found we must then study the relevant jumping loci in complex structure moduli space and compare them. Ideally the procedure would be as follows, using the discussion of Section 3.2.

1. Find the locus in complex structure/potential extension space to which the hidden sector bundle forces the system.

2. Primary decompose that locus to find its irreducible components. For each individual locus, eliminate the degrees of freedom corresponding to potential extensions to obtain a variety living purely in complex structure moduli space Σ_I^H. We will denote the reducible variety composed of all of these irreducible components $\Sigma_H = \bigcup_I \Sigma_I^H$.

3. In a similar manner, find the locus in complex structure moduli space, $\Sigma_{\rm SM}$ on which

the visible sector matter spectrum jumps.

4. For each irreducible piece of $\Sigma_{\rm H}$ ask if that locus is contained in $\Sigma_{\rm SM}$. I.e. check whether there exists an Σ_I^H such that $\Sigma_I^H \subseteq \Sigma_{\rm SM}$.

Unfortunately, in practice the above procedure is often prohibitively computationally intensive. The problem is that the jumping cohomology of relevance for $\Sigma_{\rm H}$ is $H^1(\mathcal{L}^2)$ where $\mathcal{L}$ is an equivariant line bundle. This cohomology very often involves large numbers in the first Chern class of $\mathcal{L}^2$ and this leads to a primary decomposition which is extremely costly in the second step in the list just given.

If at all computationally feasible, we use the above procedure when analyzing examples. However, if this computation can not be completed in practice, then we carry out the following analysis instead.

1. Find a set of example points in complex structure moduli space lying on $\Sigma_{\rm SM}$.

2. Determine if these points also lie on $\Sigma_{\rm H}$.

3. For those that do, if any, perform a linear perturbation analysis around that point in complex structure moduli space to determine if the hidden sector locus Σ_I^H on which it lies is localized within $\Sigma_{\rm SM}$.

We will give more details as to how this is achieved in the examples we will present going forward. In this manner, we can check whether any of the random points in $\Sigma_{\rm SM}$ that were picked lie on a component of the hidden sector locus such that $\Sigma_I^H \subset \Sigma_{\rm SM}$. When we do find such points it is most likely that we have found a case where the equidimensional hulls of the two reducible varieties in complex structure moduli space coincide. We should mention in addition that, throughout the work presented in this section, we check the smoothness of the Calabi-Yau threefolds involved at both the specific points we chose and the loci we consider in complex structure moduli space.

Applying the procedure described above provides us with a, presumably rather weak, lower bound on the frequency with which the hidden sector bundle can cause the standard model matter content to jump. We will see later that this is already good enough to illustrate one must be cautious in combining moduli stabilization and model building. We note that we start by finding points on $\Sigma_{\rm SM}$ rather than $\Sigma_{\rm H}$ here because the line bundles involved then tend to have smaller entries in their first Chern class. This leads to a more tractable computation.

3.3.2 An example

Let us illustrate the above general discussion with a concrete example. We will work on a freely acting $\mathbb{Z}_2$ quotient of CICY number 6777 which is described by the following configuration matrix,

$$X = \left[\begin{array}{c|cccc} \mathbb{P}^1 & 1 & 1 & 0 & 0 \\ \mathbb{P}^1 & 0 & 0 & 0 & 2 \\ \mathbb{P}^1 & 0 & 0 & 2 & 0 \\ \mathbb{P}^1 & 2 & 0 & 0 & 0 \\ \mathbb{P}^3 & 1 & 1 & 1 & 1 \end{array}\right] . \tag{3.22}$$

We label the homogeneous coordinates of the four ambient space $\mathbb{P}^1$ factors as $x_{r,a}$ where $r = 1, \ldots, 4$ runs over the projective spaces and $a = 0, 1$ labels the homogeneous coordinates on each factor. The homogeneous coordinates of the $\mathbb{P}^3$ factor are labeled as $x_{5,\alpha}$ where $\alpha = 0, \ldots, 3$. Given this notation, we can write the ambient space coordinate action of the

$\mathbb{Z}_2$ symmetry by which we will quotient as follows:

$$\Gamma^x_{\mathbb{Z}_2} : (x_{r,a} : x_{5,\alpha}) \to ((-1)^{a+1} x_{r,a} : (-1)^{\max(2\alpha,3)} x_{5,\alpha}). \tag{3.23}$$

In addition, the symmetry has a non-trivial normal bundle action, or equivalently action on the defining polynomials. Labeling the four defining relations corresponding to the columns of (3.22) as p_A where $A = 1, \ldots, 4$, we have the following

$$\Gamma^p_{\mathbb{Z}_2} : (p_1, p_2, p_3, p_4) \to (-p_1, p_2, p_3, -p_4). \tag{3.24}$$

Given the action (3.23) and (4.62), the most general defining relations for the configuration matrix (3.22) that are compatible with the symmetry are as follows:

$$\begin{aligned}
p_1 &= c_{1,7}x_{1,1}x_{5,0}x_{4,0}^2 + c_{1,8}x_{1,1}x_{5,1}x_{4,0}^2 + c_{1,1}x_{1,0}x_{5,2}x_{4,0}^2 + c_{1,2}x_{1,0}x_{5,3}x_{4,0}^2 \\
&\quad + c_{1,3}x_{1,0}x_{4,1}x_{5,0}x_{4,0} + c_{1,4}x_{1,0}x_{4,1}x_{5,1}x_{4,0} + c_{1,9}x_{1,1}x_{4,1}x_{5,2}x_{4,0} + c_{1,10}x_{1,1}x_{4,1}x_{5,3}x_{4,0} \\
&\quad + c_{1,11}x_{1,1}x_{4,1}^2x_{5,0} + c_{1,12}x_{1,1}x_{4,1}^2x_{5,1} + c_{1,5}x_{1,0}x_{4,1}^2x_{5,2} + c_{1,6}x_{1,0}x_{4,1}^2x_{5,3}, \\
p_2 &= c_{2,1}x_{1,0}x_{5,0} + c_{2,2}x_{1,0}x_{5,1} + c_{2,3}x_{1,1}x_{5,2} + c_{2,4}x_{1,1}x_{5,3}, \\
p_3 &= c_{3,1}x_{5,2}x_{3,0}^2 + c_{3,2}x_{5,3}x_{3,0}^2 + c_{3,3}x_{3,1}x_{5,0}x_{3,0} + c_{3,4}x_{3,1}x_{5,1}x_{3,0} + c_{3,5}x_{3,1}^2x_{5,2} + c_{3,6}x_{3,1}^2x_{5,3}, \\
p_4 &= c_{4,1}x_{5,0}x_{2,0}^2 + c_{4,2}x_{5,1}x_{2,0}^2 + c_{4,3}x_{2,1}x_{5,2}x_{2,0} + c_{4,4}x_{2,1}x_{5,3}x_{2,0} + c_{4,5}x_{2,1}^2x_{5,0} + c_{4,6}x_{2,1}^2x_{5,1}.
\end{aligned} \tag{3.25}$$

In these expressions, the c's are arbitrary coefficients associated to the complex structure moduli space. We call the manifold obtained by quotienting X by the symmetry action (3.23) and (4.62) $\hat{X}$.

On the quotient manifold described above we now define the visible sector bundle (first constructed in [49, 50]). On X we define,

$$V_{\rm SM} = \bigoplus_{i=1}^{5} \mathcal{L}_i \ , \tag{3.26}$$

where

$$\mathcal{L}_1 = \mathcal{O}(1,-1,1,-1,0) \ , \ \mathcal{L}_2 = \mathcal{O}(0,1,1,1,-1) \ , \ \mathcal{L}_3 = \mathcal{O}(0,1,-2,1,0) \ , \\ \mathcal{L}_4 = \mathcal{O}(0,-1,0,-2,1) \ , \ \mathcal{L}_5 = \mathcal{O}(-1,0,0,1,0) \ . \tag{3.27}$$

Each line bundle in $V_{\rm SM}$ is individually equivariant, and thus this does indeed define a line bundle standard model, with bundle $\hat{V}_{\rm SM}$, on the quotient $\hat{X}$. We take the parameters defining the Wilson line and equivariant structure on the sum of line bundles, as described in Section 3.1 to be $W = 1$, $\tilde{W} = -1$, $\chi_i = 1$ for $i = 1, 3 \ldots, 5$ and $\chi_2 = -1$. With these choices, the downstairs standard model charged matter spectrum, expressed concisely in terms of GUT multiplets as described earlier, is as follows at a general point in complex structure moduli space,

$$\left\{2\,\mathbf{10}_{\mathbf{e}_3}, \mathbf{10}_{\mathbf{e}_4}, 2\,\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_4}, \overline{\mathbf{5}}_{\mathbf{e}_2,\mathbf{e}_3}, \mathbf{5}_{-\mathbf{e}_1,-\mathbf{e}_2}, \overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2}\right\} . \tag{3.28}$$

The number of $\overline{\mathbf{5}}_{\mathbf{e}_2,\mathbf{e}_3}$ representations in this example has the potential to jump (along with the number of $\mathbf{5}_{-\mathbf{e}_2,-\mathbf{e}_3}$ multiplets in an index preserving manner) at higher dimensional loci in complex structure moduli space. To see this, we must consider the cohomology $H^1(\hat{X}, \hat{\mathcal{L}}_2 \otimes \hat{\mathcal{L}}_3) = H^1(\hat{X}, \hat{\mathcal{O}}(0,2,-1,2,-1))$, the dimension of which counts the multiplicity of these degrees of freedom (here hatted bundles correspond to projections of the associated upstairs objects). To compute this jumping, we work on the covering space X with the cohomology of the associated equivariant bundles and then pick out the relevant subspace (which descends to the cohomology on $\hat{X}$) by comparing representation content of that space with the Wilson line and equivariant structure.

To calculate the cohomology $H^1(X, \mathcal{L}_2 \otimes \mathcal{L}_3) = H^1(X, \mathcal{O}(0, 2, -1, 2, -1))$ we make use of the Koszul sequence

$$0 \to \wedge^4 \mathcal{N}^\vee \otimes \mathcal{L}_2 \otimes \mathcal{L}_3 \to \wedge^3 \mathcal{N}^\vee \otimes \mathcal{L}_2 \otimes \mathcal{L}_3 \to \cdots \to \mathcal{L}_2 \otimes \mathcal{L}_3|_A \to \mathcal{L}_2 \otimes \mathcal{L}_3|_X \to 0. \quad (3.29)$$

This long exact sequence can be broken into several short exact sequences as follows:

$$\begin{aligned} 0 \to \wedge^4 \mathcal{N}^\vee \otimes \mathcal{L}_2 \otimes \mathcal{L}_3 \to \wedge^3 \mathcal{N}^\vee \otimes \mathcal{L}_2 \otimes \mathcal{L}_3 \to \mathcal{K}_1 \to 0, \\ 0 \to \mathcal{K}_1 \to \wedge^2 \mathcal{N}^\vee \otimes \mathcal{L}_2 \otimes \mathcal{L}_3 \to \mathcal{K}_2 \to 0, \\ \vdots \\ 0 \to \mathcal{K}_3 \to \mathcal{L}_2 \otimes \mathcal{L}_3 \to \mathcal{L}_2 \otimes \mathcal{L}_3|_X \to 0. \end{aligned} \quad (3.30)$$

Here the $\mathcal{K}_i$ where $i = 1, \dots 3$ are kernels and cokernels of the relevant maps. The ambient space cohomologies of all of the line bundles appearing in the sequences (3.30) (excluding those of the $\mathcal{K}$'s) are vanishing with two exceptions: $h^5(A, \wedge^4 \mathcal{N}^\vee \otimes \mathcal{L}_2 \otimes \mathcal{L}_3) = 8$ and $h^5(A, \wedge^3 \mathcal{N}^\vee \otimes \mathcal{L}_2 \otimes \mathcal{L}_3) = 6$. Chasing the associated long exact sequences in cohomology we find the following:

$$H^1(X, \mathcal{L}_2 \otimes \mathcal{L}_3) \cong \ker \left(H^5(A, \wedge^4 \mathcal{N}^\vee \otimes \mathcal{L}_2 \otimes \mathcal{L}_3) \to H^5(A, \wedge^3 \mathcal{N}^\vee \otimes \mathcal{L}_2 \otimes \mathcal{L}_3) \right). \quad (3.31)$$

Thus, at a generic enough point in complex structure moduli space, we find that $h^1(X, \mathcal{L}_2 \otimes \mathcal{L}_3) = 2$, leading to the single $\overline{\mathbf{5}}_{\mathbf{e}_2, \mathbf{e}_3}$ representation in (3.28), after quotienting by the $\mathbb{Z}_2$ symmetry, by applying the correspondence of Table 3.1.

In order to present concrete formula which are concise, we will focus on calculating the locus in complex structure moduli space where the subspace $H^1(X, \hat{\mathcal{L}}_2 \otimes \hat{\mathcal{L}}_3, \chi_2 \otimes \chi_3 \otimes \tilde{W}) \in H^1(X, \hat{\mathcal{L}}_2 \otimes \hat{\mathcal{L}}_3)$ jumps in dimension (corresponding to a jump in the number of left handed $SU(2)$ doublets in the four dimensional effective theory). To do this, we need to study the map in (3.31) in more detail. We now form an explicit description of the cohomologies in (3.31) as polynomials in ambient space coordinates, take the relevant subset of such objects that are picked out in the downstairs cohomology of interest by the choice of equivariant structure and Wilson line, and study the map in more detail.

A general element of the relevant subspace of the source cohomology group in (3.31) can be written as follows:

$$S = \frac{s_1}{x_{3,0}x_{5,0}} + \frac{s_2}{x_{3,0}x_{5,1}} + \frac{s_3}{x_{3,1}x_{5,2}} + \frac{s_4}{x_{3,1}x_{5,3}}. \tag{3.32}$$

Here the s_k are arbitrary coefficients. Note that $h^6(A, \wedge^4\mathcal{N}^\vee \otimes \mathcal{L}_2 \otimes \mathcal{L}_3) = 8$ and we are dividing by a $\mathbb{Z}_2$ symmetry, so the four dimensional space obtained in (3.32) is as expected.

Next we consider the target space in (3.31). We have that

$$\begin{aligned} \wedge^3\mathcal{N}^\vee \otimes \mathcal{L}_2 \otimes \mathcal{L}_3 \quad = \quad & \mathcal{O}(-2,0,-1,0,-4) \oplus \mathcal{O}(-2,2,-3,0,-4) \\ & \oplus \mathcal{O}(-1,0,-3,2,-4) \oplus \mathcal{O}(-1,0,-3,0,-4)\ . \end{aligned} \tag{3.33}$$

Given this, the only contribution to $h^6(A, \wedge^3\mathcal{N}^\vee \otimes \mathcal{L}_2 \otimes \mathcal{L}_3)$ comes from $h^6(A, \mathcal{O}(-2,2,-3,0,-4))$, with the other three cohomologies vanishing. The map from (3.32) to this cohomology is given by multiplication by the fourth defining relation, followed by deleting terms in the resulting polynomial that are not of the correct degree to appear in $h^6(A, \mathcal{O}(-2,2,-3,0,-4))$. Performing this computation we obtain the following image of the general source element

(3.32):

$$\text{Im}(S) = \frac{s_1 c_{4,1} x_{2,0}^2}{x_{3,0}} + \frac{s_2 c_{4,2} x_{2,0}^2}{x_{3,0}} + \frac{s_3 c_{4,3} x_{2,1} x_{2,0}}{x_{3,1}} + \frac{s_4 c_{4,4} x_{2,1} x_{2,0}}{x_{3,1}} + \frac{s_1 c_{4,5} x_{2,1}^2}{x_{3,0}} + \frac{s_2 c_{4,6} x_{2,1}^2}{x_{3,0}}. \tag{3.34}$$

In order to find the kernel of the map, we then simply require that the coefficients of each of the rationomes in (3.34) vanishes. Doing so we obtain the following constraints on the s_k, in terms of the complex structure choice $c_{A,\gamma}$ in (3.25), in order for an element of the source of the form in (3.32) to be in the kernel

$$s_1 c_{4,5} + s_2 c_{4,6} = 0 \ , \ s_3 c_{4,3} + s_4 c_{4,4} = 0 \ , \ s_1 c_{4,1} + s_2 c_{4,2} = 0. \tag{3.35}$$

Writing these conditions in matrix form we obtain the following:

$$\begin{pmatrix} c_{4,5} & c_{4,6} & 0 & 0 \\ 0 & 0 & c_{4,3} & c_{4,4} \\ c_{4,1} & c_{4,2} & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = 0. \tag{3.36}$$

Given (3.36), it is easy to see that for a generic choice of complex structure the kernel will be one dimensional as stated earlier. However, on the locus ,

$$c_{4,2} c_{4,5} - c_{4,1} c_{4,6} = 0 \ , \tag{3.37}$$

the rank of the matrix in (3.36) changes from 3 to 2, and thus the dimension of the kernel will change from one to two. Therefore, on this special locus the number of $SU(2)$ doublets descending from the $\overline{\mathbf{5}}_{\mathbf{e_2},\mathbf{e_3}}$ representation increases. Naturally, in order for the index to be preserved, there is also an increase in the number of associated anti-doublets on the same locus in complex structure moduli space.

Next we turn our attention to the hidden sector and the bundle which is added to constraint the complex structure of the compactification. In searching for bundles of the form (3.21), we find that the following two possibilities

$$\mathcal{L} = \mathcal{O}(-2,-1,1,1,0) \text{ and } \mathcal{L} = \mathcal{O}(1,-1,1,-2,0) \tag{3.38}$$

are equivariant and satisfy all of the constraints given earlier in this Section.

To examine this in more detail we first note that the second Chern class of X can be presented as a two index quantity, where we expand the $(2,2)$ form in a redundant basis given by products of $(1,1)$ forms spanning $H^{1,1}(X)$. We can then contract this description of the Chern class with the intersection form to get a description of $c_2(X)$ as a vector of length $h^{1,1}(X) = h^{2,2}(X)$. When we do this we obtain the following:

$$c_2(X) = (24, 24, 24, 24, 56). \tag{3.39}$$

The second Chern class of the standard model we are examining here, expressed in the same manner is:

$$c_2(V_{\rm SM}) = (12, 12, 12, 12, 32)\ . \tag{3.40}$$

Finally, the second Chern classes of the extensions (3.21) of the $\mathcal{L}$'s given in (3.38) are respectively the following:

$$c_2(V_{\rm H}) = (4, 12, 4, 4, 20) \;\; \text{or} \;\; c_2(V_{\rm H}) = (4, 12, 4, 4, 20). \tag{3.41}$$

It is easy to see that the $SU(2)$ bundles we are choosing satisfy the second Chern character condition (3.20).

Since all line bundles are equivariant with respect to the $\mathbb{Z}_2$ symmetry we are considering, the only constraint that we have left to consider is that of stability. It is straight forward to show [106] that an extension of the form (3.21) is stable iff $\mu(\mathcal{L}) < 0$, that is if the slope of $\mathcal{L}$ is strictly negative.

We recall the expression for the slope of a line bundle,

$$\mu(\mathcal{L}_i) = \sum_{r,s,t=1}^{h^{1,1}(X)} d_{rst} c_1^r(\mathcal{L}_i) t^s t^t = 0 \; , \tag{3.42}$$

and give a definition of a set of variables σ_r:

$$\sigma_r = \sum_{s,t=1}^{h^{1,1}(X)} d_{rst} t^s t^t \; . \tag{3.43}$$

Then, examining the standard model bundle given in (3.26) and (3.27), we obtain the following conditions for the slopes of the line bundles involved to vanish (a necessary and sufficient condition for its poly-stability):

$$\sigma_1 - \sigma_2 + \sigma_3 - \sigma_4 = 0 \; , \; \sigma_2 + \sigma_3 + \sigma_4 - \sigma_5 = 0 \; , \; \sigma_2 - 2\sigma_3 + \sigma_4 = 0 \; , \tag{3.44}$$
$$-\sigma_2 - 2\sigma_4 + \sigma_5 = 0 \; , \; -\sigma_1 + \sigma_4 = 0.$$

The general solution to these equations is given by the following:

$$\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 \ , \ \sigma_5 = 3\sigma_1 . \tag{3.45}$$

We can now ask about the slope of the possible $\mathcal{L}$'s given in (3.38) on this locus, and thus about the stability of the hidden sector bundles. We find that,

$$\mu(\mathcal{O}(-2,-1,1,1,0)) = -\sigma_1 < 0 \ \ , \ \ \mu(\mathcal{O}(1,-1,1,-2,0)) = -\sigma_1 < 0 \ . \tag{3.46}$$

Thus, the proposed hidden sector extensions are indeed stable on the same locus in Kähler moduli space as the $V_{\rm SM}$ and our last constraint is satisfied.

To proceed further we will focus on $\mathcal{L} = \mathcal{O}(1,-1,1,-2,0)$, although a similar analysis can be followed for the other possibility in (3.38). The next step is to study the jumping locus of the extension group defining (3.21) and compare this jumping locus to that of $V_{\rm SM}$. The extension class of (3.21) lies in $H^1(X,\mathcal{L}^2)$. Performing an analogous chasing of Koszul sequence to the one we performed for the visible sector bundle, we arrive at the following description of this cohomology:

$$H^1(X,\mathcal{L}^2) \cong \ker(H^5(A,\wedge^4\mathcal{N}^\vee\otimes\mathcal{L}^2) \to H^2(A,\mathcal{L}^2)). \tag{3.47}$$

In fact, we will be interested in the associated cohomology on the quotiented manifold $\hat{X}$. For simplicity in this example we choose the trivial equivariant structure on $\mathcal{L}^2$, and thus this will correspond to simply considering the invariant elements in the cohomology groups concerned under the naive transformation induced from the coordinate action of the symmetry. This choice is consistent with non-trivial equivariant structures on the normal bundle such as (4.62) in this example. More generally in this work we consider all possible choices of equivariant structure.

A general element of the down-stairs cohomology describing the source space of the map corresponding to (3.47) is found to be the following:

$$S = \frac{s_1}{x_{2,0}^2 x_{4,0}^4} + \frac{s_4}{x_{2,0}x_{2,1}x_{4,0}^3 x_{4,1}} + \frac{s_2}{x_{2,0}^2 x_{4,0}^2 x_{4,1}^2} + \frac{s_7}{x_{2,1}^2 x_{4,0}^2 x_{4,1}^2} + \frac{s_5}{x_{2,0}x_{2,1}x_{4,0}x_{4,1}^3} + \frac{s_6}{x_{2,1}^2 x_{4,0}^4} + \frac{s_3}{x_{2,0}^2 x_{4,1}^4} + \frac{s_8}{x_{2,1}^2 x_{4,1}^4}, \quad (3.48)$$

where, as in previous examples, the s_k are a set of arbitrary constants. The map itself, from an examination of (3.47), should be built out of a combination of four defining relations. This map is in fact constructed in a somewhat non-trivial fashion as follows:

$$f = \epsilon^{\alpha\beta\gamma\delta} p_{1\alpha} p_{2\beta} p_{3\gamma} p_{4\delta}, \quad (3.49)$$

here $\epsilon^{\alpha\beta\gamma\delta}$ is the totally antisymmetric tensor and $p_{A\alpha}$ denotes the partial differentiation of p_A with respect to the variable $x_{5,\alpha}$ where α runs from 0 to 3. It is easy to see that f then has multi-degree $(2,2,2,2,0)$ which is precisely what is needed to match the source and target degrees in (3.49).

As in the computation of the kernel in (3.31), we can now multiply the general source polynomial (3.48) by the map polynomial (3.49) and demand that all of the coefficients of terms appearing in the target space vanish. When we do so we obtain a very long expression depending upon the s_k in (3.48) and the coefficients in the defining relations $c_{A,\gamma}$. While there are only 14 constraints obtained in this manner, which we will denote by $\mathcal{I}_\alpha$ where $\alpha = 1, \ldots 14$, they are over two pages in length and so we do not reproduce them here.

For general defining relations, the kernel of this map is found to be trivial. For special loci in complex structure moduli space, however, a non-trivial kernel is obtained, and thus the question arises how best to find this locus. As discussed earlier in this subsection, ideally we would like to primary decompose the ideal associated to these constraints and analyze each irreducible component of the associated variety separately. However, in the case at hand, this method is too computationally intensive, especially as part of a large scan over cases.

Given this situation, this is an example where we follow the methodology outlined earlier for cases where primary decomposition is too slow. We begin by finding a set of points in complex structure moduli space lying on the jumping locus Σ_{SM} of the Standard Model sector bundle $V_{\rm SM}$. In other words, denoting the generators of the ideal that define the locus in complex structure moduli space where the cohomology of $V_{\rm SM}$ jumps as $S_\kappa(c_{A,\gamma})$, we find sets of $c_{A,\gamma} = c^0_{A,\gamma}$ such that $S_\kappa(c^0_{A,\gamma}) = 0 \ \ \forall\ \kappa$. This is achievable in almost all cases we encounter, as the standard model bundle ideal is somewhat less complicated than its hidden sector cousin. This is simply due to the fact that the extension classes of (3.21) is the first cohomology of $\mathcal{L}^2$, and the square that appears tends to make the associated ideals larger.

Next we ask whether any of the solutions $c^0_{A,\gamma}$ also lie on the variety describing the kernel of the map (3.49) for some non-vanishing value of the s_k. That is we plug each solution $c_{A,\gamma} = c^0_{A,\gamma}$ into $\mathcal{I}(c_{A,\gamma}, s_k)$ and get a new ideal $\mathcal{I}'(s_k)$:

$$\mathcal{I}(c_{A,\gamma}, s_k) \xrightarrow{c_{A,\gamma}=c^0_{A,\gamma}} \mathcal{I}(c^0_{A,\gamma}, s_k) \equiv \mathcal{I}'(s_k)\ . \tag{3.50}$$

We then find sets of points $s_k = s^0_k$ which lie on the locus described by $\mathcal{I}'(s_k)$, that is, we find a series of associated possible kernel elements of (3.49), if any non-trivial solutions exist. Assuming all of this can be achieved, which it can in the example at hand, we end up with a set of solutions, each comprised of a set of values $c_{A,\gamma} = c^0_{A,\gamma}$, which lie on the jumping locus of both $V_{\rm SM}$ and $V_{\rm H}$, along with some associated non-trivial examples of kernel elements for (3.49) given by the $s_k = s^0_k$.

Given these sets of points common to Σ_{SM} and Σ_H, we must now decide which of the cases given in Figure 3.1 these points lie on. We are most interested in the third possibility depicted in that figure where the component of the hidden sector jumping locus that the starting point we have isolated lies on is a subset of the standard model jumping locus: $\Sigma_I^H \subset \Sigma_{\text{SM}}$. It is in this case that the moduli stabilization mechanism will cause the standard model spectrum to jump.

To ascertain if the situation described in the last paragraph is indeed the one we have, we perform a linearized perturbation analysis of the equations given by setting the generators of the relevant ideal to zero. To do this, we substitute $c_{A,\gamma} = c^0_{A,\gamma} + \delta c_{A,\gamma}$ and $s_k = s^0_k + \delta s_k$ into $\mathcal{I}(c_{A,\gamma}, s_k)$ and keep only the linear terms in $\delta c_{A,\gamma}$ and δs_k to obtain a new set of generators for an ideal $\mathcal{I}''(\delta c_{A,\gamma}, \delta s_k)$. For this ideal, the generators are nothing but a set of linears in the variables $\delta c_{A,\gamma}$ and δs_k, and thus it is very easy to perform an elimination on the variables δs_k and obtain a set of constraints, $\mathcal{S}'(\delta c_{A,\gamma})$, purely in terms of the $\delta c_{A,\gamma}$. Now our task is to compare the two ideals $\mathcal{S}(c_{A,\gamma})$ and $\mathcal{S}'(\delta c_{A,\gamma})$, If all the solutions of $\mathcal{S}'(\delta c_{A,\gamma}) = 0$ solve $\mathcal{S}(c^0_{A,\gamma} + \delta c_{A,\gamma}) = 0$ up to linear terms in $\delta c_{A,\gamma}$, then we can conclude that, at least under infinitesimal perturbation, some irreducible component Σ_I^H of Σ_H lies on Σ_{SM}.

In the case at hand, the locus on the standard model side is as follows:

$$\mathcal{S} = c_{4,2}c_{4,5} - c_{4,1}c_{4,6} = 0. \tag{3.51}$$

Assuming that $c^0_{4,1} \neq 0$ we can then use the following solution for $\mathcal{S}$:

$$c^0_{4,1} = a, c^0_{4,2} = b, c^0_{4,5} = c, c^0_{4,6} = \frac{bc}{a}. \tag{3.52}$$

The other complex structure coefficients $c_{A,\gamma}$ in the problem can be taken to be any number since no constraint arises on them.

Substituting these solutions into the equations $\mathcal{I}$ on the extension side, we obtain an $\mathcal{I}'$ which can easily be seen to have the following solutions for s_k^0:

$$s_4^0 = 0, s_5^0 = 0, s_6^0 = -\frac{a}{c}s_1, s_7^0 = -\frac{a}{c}s_2, s_8^0 = -\frac{a}{c}s_3. \tag{3.53}$$

Here we can take s_1,s_2 and s_3 to be any value.

Now that we have some points in moduli space common to both jumping loci, the next step is the linear perturbation analysis. Substituting $c_{A,\gamma} \to c^0_{A,\gamma} + \delta c_{A,\gamma}$ and $s_k \to s_k^0 + \delta s_k$ into $\mathcal{I}$, keeping up to linear terms in the perturbations and eliminating the δs_k we arrive at the following single constraint on the $\delta c_{A,\gamma}$,

$$-a^2\delta c_{4,6} + ab\delta c_{4,5} + ac\delta c_{4,2} - bc\delta c_{4,1} = 0. \tag{3.54}$$

In principle we now should solve this constraint for, for example $\delta c_{4,6}$ and substitute the result into $\mathcal{S}$ to see if that set of equations is also solved by these fluctuations to linear order. In fact this is not necessary in this case, as it can easily be observed that (3.54) is precisely the linearization of (3.51) around the starting points we have chosen. In this case this hidden sector locus does not merely lie inside Σ_{SM}, it is identical to it.

Thus, even though we don't know the full information about the primary decomposition and elimination of $\mathcal{I}$, it is still possible to show that some of its components lie on the standard model jump locus. In fact, in this example, we find a locus on the extension side which is precisely Σ_{SM}. As a final check, one can verify that for a generic enough choice of complex structure of the form given in (3.52) the cohomology on the extension side does indeed jump, from $h^*(X, \mathcal{O}(2,-2,2,-4,0)) = (0,0,12,0)$ to $h^*(X, \mathcal{O}(2,-2,2,-4,0)) = (0,5,17,0)$. An examination of the representation content of the larger first cohomology group which is obtained shows that three of these five elements survive to the quotient. This is in agreement with the freedom found in (3.53) above.

3.3.3 Results of a systematic scan over a class of Heterotic Line Bundle Standard Models

As we have seen in the last subsection, moduli stabilization can indeed influence the standard model physics we observe in heterotic compactifications. The question we wish to answer is how common are such phenomena in known examples of heterotic standard model compactifications. That is, how often is it the case that the hidden sector bundle can effect the visible sector spectrum in this manner. To investigate this we have run a scan over the known data set of Heterotic Line Bundle Standard Models [49, 50]. To summarize, for each Line Bundle Standard Model in the data set, we do the following.

- First, we scan over all of the standard model multiplets to find those which have the potential to jump by using an analagous calculation to that found in Section 3.3.2.
- Second, for all the standard models which are found in the first step to have spectra which can jump, we find all the extension bundles of the form (3.21) which satisfy the relevant consistency conditions, such as equivariance under the symmetry being considered and (3.20).
- Third, we calculate the jumping locus for the cohomology on the standard model side, and if it is possible, find the jumping locus on the extension side by using primary decomposition and elimination. If this is not practical in a given case, then we employ the linear perturbation analysis described in Section 3.3.2.

By using this process, we scanned over all of the 2012 cases in the data set of [49, 50].

The resulting data detailing which standard model constructions have spectra which can be influenced by moduli stabilization are given in Table 3.3.

Of the 2012 models in the data set, only 100 of them, approximately 5%, can be influenced by the moduli stabilization mechanism. This percentage is not very high but this figure is somewhat misleading. The issue is that in most cases in this list the standard model spectra is based upon line bundle cohomologies that do not jump on any locus in complex structure moduli space. If we focus on the 182 standard models which do have a spectrum that can jump at sub-loci of moduli space (which are listed in Appendix A), 100 is suddenly a large fraction. Perhaps a more useful figure then is that, within this data set, if the standard model spectrum can jump, then there is a 55% chance that it will be forced to by the moduli stabilization mechanism. Clearly, in such a situation one should not consider moduli stabilization and model building separately. One should check if the cohomologies involved in model building can jump, and if they can it is important to check the effect of the hidden sector bundle on the spectrum.

We would like to emphasize that the above figure of 55% can in some respects be regarded as a lower bound on the frequency at which this effect occurs in the line bundle standard model data set. As detailed above, we have not been able to perform a complete primary decomposition analysis of the jumping loci in all examples, and have had to restrict our attention to more crude analyses in many cases. These computations can easily miss loci associated with the hidden sector bundle that force the standard model spectrum jump. As such, interplay between moduli stabilization and model building structures could be even more pronounced than indicated here.

CICY. Num.	Mod. Num	Sym. Num.	Multiplet	Extension Line
6784	6	3,4,5,6	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_5},\ \mathcal{O}(-1,2,2,-1)$	$\mathcal{O}(-1,1,1,-1)$
6784	51	3,4,5,6	$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-3,2,2,-1)$	$\mathcal{O}(-1,1,1,-1)$
6784	54	3,4,5,6	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_5},\ \overline{\mathbf{5}}_{\mathbf{e}_2,\mathbf{e}_5},\ \mathcal{O}(-1,2,2,-1)$	$\mathcal{O}(-1,1,1,-1)$
6828	1	2	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,-3,2,-1)$	$\mathcal{O}(1,-1,1,-1)$
6828	2-5	2	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,-1,2,-1)$	$\mathcal{O}(1,-1,1,-1)$
6828	7	2	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,2,-1,-1)$	$\mathcal{O}(1,1,-1,-1)$
7435	1	2	$\mathbf{10}_{\mathbf{e}_5},\ \mathcal{O}(-2,-2,0,1)$	$\mathcal{O}(0,1,1,-1)$
			$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(4,2,-2,-1)$	$\mathcal{O}(0,1,1,-1)$
			$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_5},\ \mathcal{O}(-3,-2,1,1)$	$\mathcal{O}(1,1,0,-1)$
			$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-3,-2,1,1)$	$\mathcal{O}(1,1,0,-1)$
7435	2	2	$\mathbf{10}_{\mathbf{e}_5},\ \mathcal{O}(-2,0,-2,1)$	$\mathcal{O}(0,1,1,-1)$
			$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(4,-2,2,-1)$	$\mathcal{O}(0,1,1,-1)$
			$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_5},\ \mathcal{O}(-3,1,-2,1)$	$\mathcal{O}(1,0,1,-1)$
			$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-3,1,-2,1)$	$\mathcal{O}(1,0,1,-1)$
7435	3	2	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_5},\ \mathcal{O}(-2,4,2,-1)$	$\mathcal{O}(1,0,1,-1)$
			$\overline{\mathbf{5}}_{\mathbf{e}_2,\mathbf{e}_4},\ \mathcal{O}(1,-3,-2,1)$	$\mathcal{O}(0,1,1,-1)$
			$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_4},\ \mathcal{O}(1,-3,-2,1)$	$\mathcal{O}(0,1,1,-1)$
7435	4	2	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_5},\ \mathcal{O}(-2,2,4,-1)$	$\mathcal{O}(1,1,0,-1)$
			$\overline{\mathbf{5}}_{\mathbf{e}_2,\mathbf{e}_4},\ \mathcal{O}(1,-2,-3,1)$	$\mathcal{O}(0,1,1,-1)$
			$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_4},\ \mathcal{O}(1,-2,-3,1)$	$\mathcal{O}(0,1,1,-1)$
7435	5	2	$\mathbf{10}_{\mathbf{e}_5},\ \mathcal{O}(-2,-2,0,1)$	$\mathcal{O}(0,1,1,-1)$
			$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,4,-2,-1)$	$\mathcal{O}(1,0,1,-1)$
			$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_5},\ \mathcal{O}(-2,-3,1,1)$	$\mathcal{O}(1,1,0,-1)$
			$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-2,-3,1,1)$	$\mathcal{O}(1,1,0,-1)$
7435	6	2	$\mathbf{10}_{\mathbf{e}_5},\ \mathcal{O}(-2,0,-2,1)$	$\mathcal{O}(0,1,1,-1)$
			$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,-2,4,-1)$	$\mathcal{O}(1,1,0,-1)$
			$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_5},\ \mathcal{O}(-2,1,-3,1)$	$\mathcal{O}(1,0,1,-1)$
			$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-2,1,-3,1)$	$\mathcal{O}(1,0,1,-1)$
6732	1-2,34-35	1,2	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_4},\ \mathcal{O}(0,2,2,-1,-1)$	$\mathcal{O}(0,1,1,0,-1)$
6732	3-4	1,2	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,0,2,-1,-1)$	$\mathcal{O}(1,0,1,0,-1)$
6732	15-17	1,2	$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_5},\ \mathcal{O}(2,0,2,-1,-1)$	$\mathcal{O}(1,0,1,0,-1)$
6732	19	1,2	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_5},\ \mathcal{O}(-1,2,0,2,-1)$	$\mathcal{O}(0,1,0,1,-1)$
			$\overline{\mathbf{5}}_{\mathbf{e}_2,\mathbf{e}_4},\ \mathcal{O}(1,-2,1,-3,1)$	$\mathcal{O}(0,1,0,1,-1)$
6732	26-28	1,2	$\overline{\mathbf{5}}_{\mathbf{e}_2,\mathbf{e}_5},\ \mathcal{O}(0,-2,-2,1,1)$	$\mathcal{O}(0,1,1,0,-1)$
6732	30-31	1,2	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,0,2,-1,-1)$	$\mathcal{O}(1,0,1,0,-1)$
6732	32	1,2	$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-2,0,-2,2,1)$	$\mathcal{O}(1,0,1,0,-1)$
6732	33	1,2	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,-1,0,2,-1)$	$\mathcal{O}(1,0,0,1,-1)$
			$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-2,1,1,-3,1)$	$\mathcal{O}(1,0,0,1,-1)$
6732	36	1,2	$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_4},\ \mathcal{O}(0,-2,-2,2,1)$	$\mathcal{O}(0,1,1,0,-1)$
6770	13	1,2	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(1,1,-2,-2,0)$	$\mathcal{O}(-1,-1,0,1,1)$
6770	14	1,2	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(1,1,-2,1,-2)$	$\mathcal{O}(-1,-1,0,1,1)$
6777	17	1,2,3,4	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_3},\ \mathcal{O}(1,1,-2,-3,1)$	$\mathcal{O}(0,0,1,1,-1)$
6777	20	1,2,3,4	$\overline{\mathbf{5}}_{\mathbf{e}_2,\mathbf{e}_3},\ \mathcal{O}(0,2,-1,2,-1)$	$\mathcal{O}(1,-1,1,-2,0)$
6890	1-2 16-17	1,2	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_4},\ \mathcal{O}(0,2,2,-1,-1)$	$\mathcal{O}(0,0,1,1,-1)$
6890	5	1,2	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_3},\ \mathcal{O}(1,1,-2,-3,1)$	$\mathcal{O}(0,0,1,1,-1)$
6890	18-19,22	1,2	$\overline{\mathbf{5}}_{\mathbf{e}_2,\mathbf{e}_5},\ \mathcal{O}(0,-2,-2,1,1)$	$\mathcal{O}(0,0,1,1,-1)$
6890	20,21	1,2	$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-2,1,1,-3,1)$	$\mathcal{O}(1,0,0,1,-1)$
6890	24-27	1,2	$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_4},\ \mathcal{O}(0,-2,-2,2,1)$	$\mathcal{O}(0,1,1,0,-1)$

Table 3.3: *Heterotic line bundle standard models whose spectrum can be forced to jump by a hidden sector bundle of the form (3.21). The first column specifies the CICY number of the manifold involved (according to the standard list [118, 138]). The second and third columns specify the standard model numbers involved and the symmetries that are used in their construction according to the data sets provided in [49, 50, 139] and [129, 138] respectively. The fourth column gives an example of the component of the spectrum which can be forced to jump and the line bundle to which it is associated. Finally, the fifth column gives an example of an $\mathcal{L}$ which, when utilized in (3.21) would result in the change of spectrum being discussed.*

3.4 Topological Vanishing of Yukawa Couplings in Heterotic Line Bundle Standard Models

The tree level superpotential Yukawa couplings of Heterotic compactifications on Calabi-Yau threefolds are given by the following formula:

$$\lambda_{IJK} \propto \int_X \omega_I \wedge \omega_J \wedge \omega_K \wedge \Omega. \tag{3.55}$$

Here I, J, K label the matter fields whose coupling is being computed and the ω's are the bundle valued one forms to which those matter fields are associated. The gauge structure of (3.55) has been suppressed here: it is a gauge invariant combination of the three ω's that appears in the expression. We have left a 'proportional to' sign explicitly in (3.55) to emphasize that the absolute value of such a superpotential coupling is not physically meaningful in absence of knowledge of the Kähler potential. However, this formula can give us some information about the physical Yukawas, particularly concerning vanishings of such couplings.

There are several methods for computing quantities such as (3.55) in the literature. These fall into two main approaches, using algebraic geometry [36, 39, 41, 42, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87] and differential geometry [88, 89] respectively. Here we will focus exclusively on the latter approach, which seems to be more powerful in the case of Line Bundle Standard Models. In particular, the approach of [88, 89] makes it computationally easier to obtain moduli dependence of such couplings and leads to a powerful vanishing theorem. It is this latter result that we will make use of in what follows. We now discuss the statement of this theorem, leaving the details of its proof to the associated literature [88, 89].

Each cohomology group of which the ω_I are elements can be spanned by a basis, each element of which has a well defined "type." Fortunately, in the Line Bundle Standard Model cases we will be interested in, this basis is compatible with the basis corresponding to standard model degrees of freedom. The type of a one form corresponding to a matter field is determined by how it descends from ambient space cohomologies in the Koszul sequence. In particular, if the form descends from a cohomology of the form $H^{\tau}(A, \wedge^{\tau-1}\mathcal{N}^{\vee} \otimes \mathcal{L})$, then it is said to be of type τ.

The vanishing theorem proven in [88, 89] then simply states that if the following condition is satisfied,

$$\tau_I + \tau_J + \tau_K < \dim(A) \;, \tag{3.56}$$

where τ_I is the type of differential form and $\dim(A)$ is the dimension of the ambient space, then the Yukawa coupling will vanish.

Using this result, it is possible to detect vanishings of Yukawa couplings in Heterotic Line Bundle Standard Models without heavy calculations. Such vanishings are, naively, topological in nature and need not be tied to any obvious symmetry property of the low energy effective theory (we will return to this issue at the end of this section). This is clearly of potential phenomenological interest as a mechanism of generating Yukawa textures of various types in such models. Much like the 'forced jumping' phenomena discussed in the previous section, such textures could be good or bad for the phenomenological viability of a given string theory standard model, depending upon their structure. For example, if all Yukawa couplings were found to vanish it might be difficult to achieve a sufficiently massive top quark in such a model. However, if the Yukawa matrix were forced to be rank one then this mechanism might provide a nice explanation as to why we observe one very heavy family in Nature. An example of another effect constraining couplings in such models are discussed in [140].

In what follows we will investigate how common the vanishings of couplings we have discussed here are in the data set of Line Bundle Standard Models provided in [49, 50, 139]. Specifically we will examine those couplings which are consistent with all obvious symmetries of the models and compute which vanish due to (3.56). We will begin with an example in the next sub-section and proceed to a general analysis in the following one.

3.4.1 An example of topologically vanishing Yukawa couplings

Let us illustrate the simple process of applying the vanishing theorem described above in an example. We will work on the manifold with CICY number 5256 according to the standard list [118, 138], which is defined by the following configuration matrix:

$$X = \left[\begin{array}{c|cccc} \mathbb{P}^1 & 1 & 1 & 0 & 0 \\ \mathbb{P}^1 & 2 & 0 & 0 & 0 \\ \mathbb{P}^1 & 0 & 0 & 1 & 1 \\ \mathbb{P}^1 & 0 & 0 & 1 & 1 \\ \mathbb{P}^3 & 1 & 1 & 1 & 1 \end{array}\right] . \tag{3.57}$$

We will quotient X by the fourth discrete symmetry in the canonical list [129, 138], which acts on the homogeneous coordinates in the following manner:

$$\mathbb{Z}_2^{(1)} : \begin{cases} x_{r,a} \rightarrow (-1)^a x_{r,a}, \\ x_{5,\alpha} \rightarrow (-1)^\alpha x_{5,\alpha}, \end{cases} \tag{3.58}$$

$$\mathbb{Z}_2^{(2)} : \begin{cases} x_{r,a} \rightarrow x_{r,a+1}, \\ x_{5,\alpha} \rightarrow x_{5,\alpha+(-1)^\alpha}. \end{cases}$$

Here we make the identifications $x_{r,2} = x_{r,0}$, $\forall i$. In addition to this coordinate action there is a normal bundle action which descends to the following transformations on the defining polynomials:

$$\mathbb{Z}_2^{(1)} : (p_1, p_2, p_3, p_4) \to (p_1, -p_2, p_3, -p_4), \tag{3.59}$$
$$\mathbb{Z}_2^{(2)} : (p_1, p_2, p_3, p_4) \to (p_1, -p_2, p_4, p_3).$$

On $X/\mathbb{Z}_2 \times \mathbb{Z}_2$ a Line Bundle Standard Model can be built of the form $V_{\text{SM}} = \bigoplus_{i=1}^{5} \mathcal{L}_i$ with the following line bundle content [49, 50]:

$$\mathcal{L}_1 = \mathcal{O}_X(1,0,-2,1,0)\ ,\ \mathcal{L}_2 = \mathcal{O}_X(1,-2,1,0,0)\ ,\ \mathcal{L}_3 = \mathcal{O}_X(0,1,1,-2,0)\ , \tag{3.60}$$
$$\mathcal{L}_4 = \mathcal{O}_X(-1,1,0,0,0)\ ,\ \mathcal{L}_5 = \mathcal{O}_X(-1,0,0,1,0).$$

The non-trivial cohomology content of combinations of the line bundles $\mathcal{L}_i$ which are relevant for the standard model spectrum of this Line Bundle Standard Model are as follows:

$$h^*(X, \mathcal{L}_1) = (0,4,0,0); \quad h^*(X, \mathcal{L}_2) = (0,4,0,0); \quad h^*(X, \mathcal{L}_3) = (0,4,0,0); \tag{3.61}$$
$$h^*(X, \mathcal{L}_1 \otimes \mathcal{L}_4) = (0,4,0,0); \quad h^*(X, \mathcal{L}_1 \otimes \mathcal{L}_5) = (0,3,3,0); \quad h^*(X, \mathcal{L}_2 \otimes \mathcal{L}_5) = (0,4,0,0);$$
$$h^*(X, \mathcal{L}_4 \otimes \mathcal{L}_5) = (0,4,0,0); \quad h^*(X, \mathcal{L}_1^\vee \otimes \mathcal{L}_5^\vee) = (0,3,3,0); \quad h^*(X, \mathcal{L}_1 \otimes \mathcal{L}_2^\vee) = (0,12,0,0);$$
$$h^*(X, \mathcal{L}_1 \otimes \mathcal{L}_5^\vee) = (0,3,3,0); \quad h^*(X, \mathcal{L}_2 \otimes \mathcal{L}_3^\vee) = (0,12,0,0); \quad h^*(X, \mathcal{L}_2 \otimes \mathcal{L}_4^\vee) = (0,12,0,0);$$
$$h^*(X, \mathcal{L}_3 \otimes \mathcal{L}_4^\vee) = (0,4,0,0); \quad h^*(X, \mathcal{L}_3 \otimes \mathcal{L}_5^\vee) = (0,16,0,0).$$

These cohomologies correspond respectively to the following multiplets on X (before the quotient):

$$4\,\mathbf{10}_{\mathbf{e}_1};\ 4\,\mathbf{10}_{\mathbf{e}_2};\ 4\,\mathbf{10}_{\mathbf{e}_3};\ 4\,\bar{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_4};\ 3\,\bar{\mathbf{5}}_{\mathbf{e_1},\mathbf{e_5}};\ 4\,\bar{\mathbf{5}}_{\mathbf{e_2},\mathbf{e_5}};\ 4\,\bar{\mathbf{5}}_{\mathbf{e_4},\mathbf{e_5}};\ 3\,\mathbf{5}_{-\mathbf{e_1},-\mathbf{e_5}}; \tag{3.62}$$
$$12\,\mathbf{1}_{\mathbf{e_1},-\mathbf{e_2}};\ 3\,\mathbf{1}_{\mathbf{e_1},-\mathbf{e_5}};\ 3\,\mathbf{1}_{\mathbf{e_5},-\mathbf{e_1}};\ 12\,\mathbf{1}_{\mathbf{e_2},-\mathbf{e_3}};\ 12\,\mathbf{1}_{\mathbf{e_2},-\mathbf{e_4}};\ 4\,\mathbf{1}_{\mathbf{e_3},-\mathbf{e_4}};\ 16\,\mathbf{1}_{\mathbf{e_3},-\mathbf{e_5}}.$$

As in earlier sections, we give spectra in this section in terms of GUT multiplets for conciseness, despite the fact that we use a Wilson line (whose exact form will not be needed here) to break the gauge group to that of the standard model.

Given the spectrum of standard model representations and $U(1)$ charges given in (3.62), one would naively expect the following Yukawa couplings to be present:

$$\bar{\mathbf{5}}_{\mathbf{e_2},\mathbf{e_5}}\mathbf{5}_{-\mathbf{e_1},-\mathbf{e_5}}\mathbf{1}_{\mathbf{e_1},-\mathbf{e_2}}\ ,\ \mathbf{10}_{\mathbf{e_3}}\bar{\mathbf{5}}_{\mathbf{e_1},\mathbf{e_4}}\bar{\mathbf{5}}_{\mathbf{e_2},\mathbf{e_5}}. \tag{3.63}$$

Let us look at these two Yukawa couplings in more detail in the context of the vanishing theorem (3.56). In order to use the theorem, we must first work out which ambient space cohomologies the relevant matter fields descend from in the Koszul sequence. Beginning with the Yukawa coupling $\bar{\mathbf{5}}_{\mathbf{e_2},\mathbf{e_5}}\mathbf{5}_{-\mathbf{e_1},-\mathbf{e_5}}\mathbf{1}_{\mathbf{e_1},-\mathbf{e_2}}$, the line bundles associated to the multiplets which appear are as follows:

$$\bar{\mathbf{5}}_{\mathbf{e_2},\mathbf{e_5}} : \mathcal{O}_X(0,-2,1,1,0)\ ,\ \mathbf{5}_{-\mathbf{e_1},-\mathbf{e_5}} : \mathcal{O}_X(0,0,2,-2,0)\ ,\ \mathbf{1}_{\mathbf{e_1},-\mathbf{e_2}} : \mathcal{O}_X(0,2,-3,1,0). \tag{3.64}$$

A short computation shows that $H^1(X,\mathcal{L})$ for all of these line bundles descends from the associated first cohomology on A, that is $H^1(A,\mathcal{L}_2\otimes\mathcal{L}_5)$, which is four dimensional, $H^1(A,\mathcal{L}_1^\vee\otimes\mathcal{L}_5^\vee)$ which is three dimensional and $H^1(A,\mathcal{L}_1\otimes\mathcal{L}_2^\vee)$, which is twelve dimensional, respectively.

From this analysis we can see that all three of the involved matter fields are of type one, and thus we have,

$$\tau_{\bar{\mathbf{5}}_{\mathbf{e_2},\mathbf{e_5}}} + \tau_{\mathbf{5}_{-\mathbf{e_1},-\mathbf{e_5}}} + \tau_{\mathbf{1}_{\mathbf{e_1},-\mathbf{e_2}}} = 3 < \dim(A) = 7 \; . \tag{3.65}$$

Given this, the vanishing theorem tells us that this Yukawa coupling (or more precisely this set of 144 couplings) vanishes, despite the fact there is no obvious gauge theoretic restriction that would cause it to do so. Given that these upstairs couplings vanish, so do all of the associated downstairs couplings associated to the Line Bundle Standard Model itself.

For the second Yukawa coupling, $\mathbf{10}_{\mathbf{e_3}}\bar{\mathbf{5}}_{\mathbf{e_1},\mathbf{e_4}}\bar{\mathbf{5}}_{\mathbf{e_2},\mathbf{e_5}}$ a similar procedure can be followed. We find that once again all three matter fields are of type 1, and thus the Yukawa coupling vanishes, naively due to topological restrictions with, once again, no obvious gauge theoretic restriction presenting itself.

3.4.2 Scanning over the Line Bundle Standard Models

We now proceed to apply an analysis of the form presented in the previous subsection to every Line Bundle Standard Model in the data set of [49, 50]. The procedure we apply is as follows. For each model, we first look at the multiplets which arise. That is, we examine the cohomology groups,

$$H^1(X, \mathcal{L}_i) \; , \; H^1(X, \mathcal{L}_i \otimes \mathcal{L}_j) \; , \; H^1(X, \mathcal{L}_i^\vee \otimes \mathcal{L}_j^\vee) \; , \; H^1(X, \mathcal{L}_i \otimes \mathcal{L}_j^\vee) \; , \tag{3.66}$$

which, as was detailed in Table 3.1, are the upstairs cohomologies that correspond to the following matter representations:

$$\mathbf{10}_{\mathbf{e_i}} \; , \; \bar{\mathbf{5}}_{\mathbf{e_i},\mathbf{e_j}} \; , \; \mathbf{5}_{-\mathbf{e_i},-\mathbf{e_j}} \; , \; \mathbf{1}_{\mathbf{e_i},-\mathbf{e_j}} . \tag{3.67}$$

Once we have extracted this list of multiplets from the Line Bundle Standard Model data set, we then extract all of the Yukawa couplings that are consistent with the constraints imposed by gauge symmetry. These are all of one of the following three forms:

$$\mathbf{5}_{-\mathbf{e_i},-\mathbf{e_j}}\mathbf{10}_{\mathbf{e_i}}\mathbf{10}_{\mathbf{e_j}}\,, \quad \bar{\mathbf{5}}_{\mathbf{e}_i,\mathbf{e}_j}\mathbf{5}_{\mathbf{e_k},\mathbf{e_l}}\mathbf{10}_{\mathbf{e_m}}\,, \quad \mathbf{1}_{\mathbf{e_i},-\mathbf{e_j}}\mathbf{5}_{-\mathbf{e_i},-\mathbf{e_k}}\bar{\mathbf{5}}_{\mathbf{e_j},\mathbf{e_k}}\,. \tag{3.68}$$

Finally, for each Yukawa coupling that does not vanish due to gauge theoretic considerations, we examine the Koszul sequence associated to each of the line bundles giving rise to the matter multiplets involved and determine the types of the associated forms. We can then use the vanishing theorem (3.56) to determine whether or not these couplings are actually present. We present the full results of this analysis in Appendix B and will content ourselves here with some brief statistics on the results.

The number of Yukawa couplings which are non-zero after gauge theoretic considerations are taken into account is given in Table 3.4. Models are only listed in this table if they have at least one non-vanishing coupling at this level.

Given the data in Table 3.4, the question is now how many of these Yukawa couplings vanish due to the topological vanishing theorem (3.56). The answer to this question is given in Table 3.5. Compiling this data into even more coarse overall figures, we obtain the percentage of the different types of coupling given in (3.67) which would be allowed by gauge invariance but which vanish due to these topological considerations. These figures are presented in Table 3.6.

In the final analysis there is a total of 1481 Standard Models in the data set which have at least one Yukawa coupling that would be expected to be non-zero based upon consideration of the obvious symmetries in the construction. Of these, 442 have have at least one such coupling which turns out to be zero due to the vanishing theorem (3.56). This means that

CICY No.	No. Sym.	No. Models	$\mathbf{5}_{-\mathbf{e_i},-\mathbf{e_j}}\mathbf{10}_{\mathbf{e_i}}\mathbf{10}_{\mathbf{e_j}}$	$\bar{\mathbf{5}}_{\mathbf{e}_i,\mathbf{e}_j}\mathbf{5}_{\mathbf{e_k},\mathbf{e_l}}\mathbf{10}_{\mathbf{e_m}}$	$\mathbf{1}_{\mathbf{e_i},-\mathbf{e_j}}\mathbf{5}_{-\mathbf{e_i},-\mathbf{e_k}}\bar{\mathbf{5}}_{\mathbf{e_j},\mathbf{e_k}}$
6784	4	188	0	120	144
6828	1	2	0	5	0
7862	1	14	18	53	19
5256	6	84	0	126	32
5452	20	800	0	1376	208
6947	1	24	0	24	12
6732	2	28	24	68	12
6770	2	16	16	32	0
6777	4	24	48	64	80
6890	2	22	24	50	12
7447	1	3	0	5	4
7487	4	276	164	580	444

Table 3.4: *The number of Yukawa couplings of various types that are permitted by the gauge symmetries of the set of Line Bundle Standard Models being considered [49, 50, 139]. The first column gives the CICY identification number of the manifold on which the models are based, according to the standard list [118, 138]. The second column details how many symmetries are being considered, and thus the number of downstairs manifolds that each row corresponds to. 'No. Models' gives the number of models with at least one Yukawa coupling that would be consistent with gauge invariance in the data set. The remaining three columns give the number of each type of such couplings that appear in this set of models.*

topological vanishing of Yukawa couplings plays a role in 29.8% of these models.

A few comments are order about these results. Firstly, it is clear that this is not a rare phenomenon. A lot of couplings that would naively be allowed by gauge invariance in the theory actually vanish due to topological considerations. That this effect would be common was anticipated in [88, 89]. It should also be mentioned that these results are reminiscent, for example, of long understood selection rules in orbifold compactifications [141, 142, 143, 144, 145].

Obvious questions include whether or not such vanishings are also so ubiquitous in higher order couplings and whether there is some hidden reason, beyond quasi-topological restrictions, for the phenomenon. For the latter question, a potential hint is given by the results of [86, 87]. There it was shown that stability walls elsewhere in extended bundle and Kähler

CICY No.	No. Sym.	No. Models	$\mathbf{5}_{-\mathbf{e_i},-\mathbf{e_j}}\mathbf{10}_{\mathbf{e_i}}\mathbf{10}_{\mathbf{e_j}}$	$\bar{\mathbf{5}}_{\mathbf{e}_i,\mathbf{e}_j}\mathbf{5}_{\mathbf{e_k},\mathbf{e_l}}\mathbf{10}_{\mathbf{e_m}}$	$\mathbf{1}_{\mathbf{e_i},-\mathbf{e_j}}\mathbf{5}_{-\mathbf{e_i},-\mathbf{e_k}}\bar{\mathbf{5}}_{\mathbf{e_j},\mathbf{e_k}}$
6784	4	0	0	0	0
6828	1	0	0	0	0
7862	1	9	8	5	3
5256	6	32	0	32	24
5452	20	256	0	240	192
6947	1	24	0	24	12
6732	2	0	0	0	0
6770	2	8	16	0	0
6777	4	0	0	0	0
6890	2	0	0	0	0
7447	1	1	0	1	3
7487	4	112	80	32	0

Table 3.5: *The number of Yukawa couplings of various types that are permitted by the gauge symmetries but vanish due to the topological restriction (3.56) for the set of Line Bundle Standard Models being considered [49, 50, 139]. The first column gives the CICY identification number of the manifold on which the models are based, according to the standard list [118, 138]. The second column details how many symmetries are being considered, and thus the number of downstairs manifolds that each row corresponds to. 'No. Models' gives the number of models with at least one Yukawa coupling that vanishes due to this topological consideration. The remaining three columns give the number of each type of such couplings that vanish due to (3.56) in this set of models.*

Yukawa Type	Total Num.	Top. Van. Num.	Percentage
$\mathbf{5}_{-\mathbf{e_i},-\mathbf{e_j}}\mathbf{10}_{\mathbf{e_i}}\mathbf{10}_{\mathbf{e_j}}$	294	104	35.4%
$\bar{\mathbf{5}}_{\mathbf{e}_i,\mathbf{e}_j}\mathbf{5}_{\mathbf{e_k},\mathbf{e_l}}\mathbf{10}_{\mathbf{e_m}}$	2503	334	13.3%
$\mathbf{1}_{\mathbf{e_i},-\mathbf{e_j}}\mathbf{5}_{-\mathbf{e_i},-\mathbf{e_k}}\bar{\mathbf{5}}_{\mathbf{e_j},\mathbf{e_k}}$	967	234	24.2%
In total	3764	672	17.9%

Table 3.6: *The total number of Yukawa couplings of each type in the Line Bundle Standard Model data set studied [49, 50, 139]. The column 'Total Number' details the number of each type of coupling which are consistent with gauge invariance. The column 'Top. Van. Num.' details the number of these couplings that are actually zero due to the vanishing theorem (3.56).*

moduli space could have $U(1)$ symmetries that, while broken for the split bundle being studied, still restrict its couplings hugely. This effect can be very strong, essentially due to the holomorphic nature of the superpotential of the four dimensional theory. The bundles being studied in the Line Bundle Standard Model data set considered here are in larger Kähler cones than the simple examples considered in [86]. In such cases, it is expected that the constraints on couplings will be even more restrictive (due to a larger number of stability walls being present). This could potentially explain the high percentages of topological vanishings found in Table 3.6.

3.5 Conclusions

In this chapter we have studied two effects which can arise in Line Bundle Standard Models [49, 50]. The first of these concerns the interaction of line bundle model building and the moduli stabilization mechanism of [103, 104]. In that work, the hidden sector gauge bundle is used to stabilize the complex structure to some higher co-dimensional sub-locus of moduli space. Here, we have investigated how often the system being forced to this special locus in complex structure moduli space causes the massless charged spectrum of the standard model to jump. The second effect we considered was concerned with the structure of Yukawa couplings. Couplings which are consistent with all obvious symmetries of the four dimensional effective theory can be zero due to seemingly topological restrictions. We have considered the form of topological vanishing presented in [88, 89] and have determined how common such effects are in the known data set of Line Bundle Standard Models.

In our work on the first of these directions we have seen that, in the data set studied, if the standard model field content is capable of jumping, the hidden sector stabilization mechanism has a good chance of forcing it to do so. In particular, there is at least a 55% chance that one (of the usually small number) of $SU(2)$ structure extension hidden sector bundles that can be consistently included in such a compactification will force the standard model bundle to a jumping locus. Such a strong interaction between the visible sector and hidden sector bundles may seem surprising at first. However, for the threefolds that are considered with non-vanishing first fundamental group, the second Chern characters are not that large. Given this, there are then not many choices of equivariant line bundles that can be used in the construction of a hidden sector bundle given any particular Line Bundle Standard Model. The restricted nature of the choices seems to lead to a relatively ubiquitous correlation between jumping loci of the cohomologies governing the hidden sector extension and the standard model spectrum. The basic message of Section specjump of this chapter is thus that such effects are something that should be considered in model building work, if the standard model bundles being considered have cohomologies which are capable of jumping. As we have emphasized in the main text, this effect could be either good or bad. It could force unwanted family/anti-family pairs to appear in the spectrum, but it could equally well force the generation of a Higgs-Higgs bar pair in a model that previously lacked such degrees of freedom.

In our investigation of topological vanishings of Yukawa couplings we have seen a similarly strong effect. We have seen that the vanishing theorem presented in [88, 89] leads to an otherwise permitted Yukawa coupling being zero in 30% of the models presented in [49, 50]. Indeed, almost 18% of the couplings that are allowed by all of the obvious symmetries in these models actually vanish. As with the previous result, this effect can be either good or bad for the phenomenological viability of a model depending on the particular case at hand. It is clear, however, given the ubiquity of the effect, that such vanishings should be taken into account in phenomenological explorations of these constructions. Several natural questions follow from these results. For example, do similar, seemingly topological, vanishings of couplings happen for higher order interactions? We conjecture one possible explanation for the large number of vanishings that would answer this question in the affirmative. As was discussed in [86], stability walls elsewhere in combined bundle and Kähler moduli space, can have strong effects on superpotential couplings in backgrounds where those splittings are not manifest. In particular they can force such vanishings of couplings. Whether this really is the effect that is behind many of the vanishings that we have seen is a study that we leave for future work.

Chapter 4

Chern-Simons Invariants and Heterotic Superpotentials

In this chapter we are going to investigate how the Chern-Simons term in heterotic theory can contribute to the superpotential. The content of this chapter is based on our paper [146].

The literature on model building in smooth Calabi-Yau compactifications of heterotic string theory stretches back nearly 40 years. In the early seminal work on the subject, efforts were focussed on the case of the "standard embedding" where the gauge bundle was taken to be the holomorphic tangent bundle of the Calabi-Yau manifold [14, 40, 41, 42, 43]. In more recent years, advances in the technology used to describe these compactifications has lead to the construction of heterotic standard models with the exact charged spectrum of the MSSM [43, 44, 45, 46, 47, 48, 49, 50]. Most of this model building progress has been achieved by branching out to more general situations where the gauge fields and the spin connection are connections on different holomorphic vector bundles. Despite the sophisticated constructions that lead to carefully chosen charged particle spectra, it has generally been the case that these compactifications give rise to only marginally stable vacua. Nevertheless, this work has been motivated by the hope that at least some of the general lessons learned in this decade-spanning and extensive effort will carry over to more realistic situations where all of the moduli are stabilized.

Compared to string model building focused on particle physics properties, the subject of moduli stabilization is at a much less advanced stage of development in heterotic theories. Although a large literature on this topic does exist, several pieces of the low energy effective theory that would be required for a full analysis of the vacuum space of the theory are still unknown. These include, for example, the Kähler potential, in the case of non-standard embeddings, for fields such as matter and bundle moduli as an explicit function of the $\mathcal{N}=1$ degrees of freedom [147, 148, 149, 150, 151]. In this work *our goal is to compute another such missing component of the theory – namely the vacuum contribution to the superpotential that appears due to the presence of the gauge bundle in heterotic compactifications.* If this quantity is non-vanishing it can potentially destabilize the model, in the absence of other effects. This superpotential due to the heterotic gauge bundle is also a crucial ingredient in moduli stabilization scenarios and so its computation is of great importance.

In heterotic compactifications on smooth Calabi-Yau three-folds, we typically consider a gauge bundle V over some Calabi-Yau manifold X with tangent bundle TX. The Bianchi identity of the ten-dimensional theory, which we gave in Chapter se:HSC as formula dhh, in the absence of 5-branes is[1],

$$dH = \alpha' \left(\mathrm{tr}(R \wedge R) - \mathrm{tr}(F \wedge F) \right) \ . \tag{4.1}$$

This implies that the Neveu-Schwarz three-form field strength can be written, at least locally, as

$$H = H_0 + \alpha' \left(\omega_3(\omega) - \omega_3(A) \right) \ , \quad \omega_3(A) = \mathrm{tr}(dA \wedge A + \frac{2}{3} A^3) \ , \quad \omega_3(\omega) = \mathrm{tr}(d\omega \wedge \omega + \frac{2}{3} \omega^3) \ . \tag{4.2}$$

Here, H_0 is a closed contribution to the field strength that obeys an integer flux quantization condition and it can be locally written as $H_0 = dB$, with the two-form field B. Further, we

[1] Throughout this chapter we will define 'tr' to include a factor of $\frac{1}{8\pi^2}$ to avoid unnecessary cluttering of the formulae with numerical factors.

have introduced the gauge connection A on V and the spin connection ω on TX, along with their respective Chern-Simons forms $\omega_3(A)$ and $\omega_3(\omega)$.

Given such a form for the field strength H, the Gukov-Vafa-Witten superpotential [29] of the four dimensional effective theory can be written as

$$W = \int_X (H + i\, dJ) \wedge \Omega = \int_X H_0 \wedge \Omega + \alpha' \, \mathrm{CS}_{\mathrm{phys}}(A, \omega) + i \int_X dJ \wedge \Omega \, , \tag{4.3}$$

where Ω is the holomorphic $(3,0)$ form on the Calabi-Yau three-fold. The physics literature usually defines the Chern-Simons contribution to this superpotential as

$$\mathrm{CS}_{\mathrm{phys}}(A, \omega) = \int_X \mathrm{tr}\left(\omega_3(A) - \omega_3(\omega)\right) \wedge \Omega \, . \tag{4.4}$$

It is often tacitly assumed that this contribution vanishes in vacuum. In general, however, there is no reason for this to be the case and it must be computed explicitly, even to verify the existence of simple forms of marginally stable Minkowski vacua[2]. It is possible that many of the heterotic standard models in the literature are not, in fact, associated to Minkowski vacua unless other contributions to (4.3) are included. To date, only one of the models with an exact MSSM spectrum mentioned above is known to have a vanishing Chern-Simons contribution to the superpotential [43]. The gauge bundle, in this model, is a holomorphic deformation of the tangent bundle and in such instances, as we will discuss, the vanishing of (4.4) is guaranteed.

Unfortunately, very few techniques exist in either the physics or mathematics literature for explicitly computing the value of (4.4) and only a handful of simple examples have been studied. This is frustrating given its importance for heterotic model building and moduli

[2] We thank E. Witten for pointing this out to us and suggesting we consider this issue in the context of the work [49, 50].

stabilization. The goal of the present work is to improve the tools required for calculating Chern-Simons contributions to the superpotential in relevant heterotic models. In particular, our primary results are the following.

- We develop new computational tools to efficiently calculate the vacuum value of (4.4) in explicit heterotic string compactifications.
- We construct new and non-trivial examples of consistent heterotic compactifications in which the Chern-Simons contribution to the vacuum superpotential can be exactly determined. These include cases with vanishing as well as non-vanishing and fractional Chern-Simons contributions.

It should be noted that Wilson line contributions to the superpotential (4.4) have been frequently considered in the literature [152, 153, 154, 155]. We emphasize that this is *not* what we are doing here. We are interested in all contributions to (4.4), including those from the non-flat bundles. It is this quantity which is of relevance for concrete models, since heterotic compactifications on Calabi-Yau three-folds necessarily require a gauge bundle with non-vanishing curvature. Other recent papers considering this contribution to the superpotential even for non-flat bundles include [156] which utilizes mirror symmetry and [157] which explores deformations of the Hull-Strominger system [158, 159].

To understand the physical consequences for non-vanishing Chern-Simons contributions to the vacuum potential, it is important to keep in mind that the effect being described here could, of course, be cancelled by the other contributions to the superpotential (4.3). However, the Chern-Simons term is somewhat different in nature to the effects from H_0 and dJ. The contributions to (4.3) from H_0 are associated with quantized quantities, which means that the corresponding terms in the superpotential are determined by a set of integers. The same is not necessarily true for the Chern-Simons contribution (4.4). The Chern-

Simons term is determined, as we will discuss, by a set of $2(h^{2,1}(X)+1)$ numbers which may be fractional. The contribution from dJ, by contrast, vanishes for any torsion-free background and thus represents a highly non-trivial modification of the background geometry if present. Such a modification would have to be taken into account in other aspects of the dimensional reduction, for example in model building work. If the Chern-Simons contribution (4.4) is non-zero it will typically be large so that, in the absence of other effects, it will destabilize the theory. On the other hand, any credible scenario for moduli stabilization which may, for example, include additional non-perturbative effects, must include the Chern-Simons contribution. Fractional Chern-Simons contributions obtained from Wilson lines have already been used in some moduli stabilization scenarios [153, 155].

In any eventuality, it is important to understand what values the Chern-Simons term (4.4) takes in compactifications of heterotic string theory, and it is this question that we will try to address in the rest of this chapter.

In the next section we review the proper formulation of holomorphic Chern-Simons terms in heterotic superpotentials, and we explain how the various physical and mathematical notions relate. In Section 4.2 we describe how to use real bundle isomorphisms between the tangent and gauge bundles of heterotic compactifications to compute the holomorphic Chern-Simons invariant (4.4). We also provide a concrete example of such a computation. Section 4.3 reviews an important theorem that explains why the Chern-Simons contribution vanishes in many cases. In Section 4.4 we discuss issues that arise when considering holomorphic Chern-Simons contributions to the superpotential in compactifications on quotient manifolds. In that section we also construct an explicit example of a heterotic compactification with a non-flat gauge bundle that gives rise to a fractional holomorphic Chern-Simons invariant. Finally, in Section 4.5 we briefly conclude and discuss possible future directions of research. The appendices contain several technical results that are necessary for our discussion.

4.1 Basics of Chern-Simons terms

The reader may well be used to defining Chern-Simons invariants in the form discussed in the introduction. In many physical applications such a definition suffices. However, in the case of heterotic compactifications, the non-trivial topological structure of the compactification means that more care is required. In what follows we will compare the definitions of such invariants as they appear in the physics and mathematics literature, and we will describe why caution is required.

4.1.1 Heterotic Chern-Simons terms

The Chern-Simons term $\mathrm{CS}_{\mathrm{phys}}(A, \omega)$ which appears in heterotic theories and forms part of the heterotic superpotential has already been defined in (4.4). How does this Chern-Simons term behave under gauge transformations

$$A \mapsto hAh^{-1} + hdh^{-1} , \qquad \omega \mapsto g\omega g + gdg^{-1} , \tag{4.5}$$

of the gauge connection A and the spin connection ω? A short calculation reveals the Chern-Simons forms change as

$$\omega_3(A) \mapsto \omega_3(A) + \mathrm{tr}\left[d(Adh^{-1}h) - \frac{1}{3}(hdh^{-1})^3\right] , \tag{4.6}$$

and similarly for $\omega_3(\omega)$, but with h replaced by g. It is easy to see that the integrand of the Chern-Simons term $\mathrm{CS}_{\mathrm{phys}}(A, \omega)$ in (4.4) is not invariant under these transformations. This is problematic since contributions to the integral (4.4) from two patches can differ on their overlap. In other words, the integral is not well-defined globally as it depends on the choice of partition of unity that is used in its definition.

In the context of supergravity this problem is addressed by assigning a gauge transformation to the two-form field B which cancels the variation (4.6) so that the field strength H is gauge invariant. This means that the sum of the first and second integral on the right-hand side of (4.3) is gauge invariant, so the superpotential is well-defined as it should be. However, for the purpose of investigating the effect of the Chern-Simons contribution this state of affairs is quite inconvenient. It is desirable to have a well-defined version of the Chern-Simons term and a way to express the superpotential (4.3) in terms of this object. We will define this mathematical version of the Chern-Simons term - the holomorphic Chern-Simons invariant - in the next sub-section and subsequently describe its relationship to the physics Chern-Simons term.

Before we do so, it is important to note that compactifications of heterotic string theory also contains another type of Chern-Simons integral defined over real three-manifolds. Heterotic flux quantization [160, 161] can be stated as the condition

$$\frac{1}{\alpha'}\int_{\mathcal{C}} H - \int_{\mathcal{C}} (\omega_3(\omega) - \omega_3(A)) \in \mathbb{Z} \tag{4.7}$$

for any integral three-cycle $\mathcal{C} \subset X$. Note that this condition involves an integral over a three-cycle in the Calabi-Yau space, in contrast to the Chern-Simons term (4.4) which requires integration over the entire manifold. The integrand in (4.7) is not gauge variant under the transformation (4.6), and is, hence, ill-defined, much as its six-dimensional counterpart (4.4). To formulate flux quantization properly, we will introduce the ordinary Chern Simons invariant and subsequently explain how it enters the physical condition.

4.1.2 Chern-Simons invariants

We begin by formulating the holomorphic Chern-Simons invariant, the object which will provide us with a well-defined version of the Chern-Simons term (4.4) which appears in the heterotic superpotential. Useful discussions of this and related topics, intended for an audience of physicists, can be found here [162, 163, 164]. The set-up requires two connections[3], A and A_0, on the *same* vector bundle V over a base Calabi-Yau manifold X. The connection A will be seen as the argument of the Chern-Simons invariant and A_0 is called a "reference connection". Then, with the adjoint valued one form $a = A - A_0$, the definition of the holomorphic Chern-Simons invariant is as follows [165]:

$$\mathrm{CS}_{A_0}(A) = \int_X \mathrm{tr}\left((\overline{\partial}_{A_0} a \wedge a) + \frac{2}{3} a \wedge a \wedge a + 2a \wedge F_0 \right) \wedge \Omega. \tag{4.8}$$

Here, F_0 is the field strength associated to the connection A_0 and we define the covariant derivative $d_{A_0} a = da + A_0 \wedge a + a \wedge A_0$. Note that, naively, this is quite different from its supposed counterpart (4.4) in the heterotic theory which is defined in terms of two connections *seemingly on different bundles*, the gauge bundle and the tangent bundle. We will review the relationship between the mathematical and physics picture in the next subsection. For now, we note that the holomorphic Chern-Simons invariant can be written in terms of the Chern-Simons forms $\omega_3(A)$ and $\omega_3(A_0)$, defined as in (4.2), as

$$\mathrm{CS}_{A_0}(A) = \int_X \mathrm{tr}\left(\omega_3(A) - \omega_3(A_0) - d(A \wedge A_0)\right) \wedge \Omega \, . \tag{4.9}$$

What happens to the holomorphic Chern-Simons invariant under simultaneous gauge trans-

[3]For brevity of exposition, we will sometimes conflate connections and the local gauge fields they give rise to in discussions where this should not cause confusion.

formations of A and A_0,

$$A \mapsto hAh^{-1} + hdh^{-1} \quad \text{and} \quad A_0 \mapsto hA_0h^{-1} + hdh^{-1} \, , \tag{4.10}$$

with the *same* gauge parameter h? Evidently, under such a gauge transformation all of the quantities appearing in (4.8) transform in a covariant manner and, as a result, the holomorphic Chern-Simons invariant (4.8) is thus manifestly invariant. Note, the transformation (4.10) is different to just performing a gauge transformation on A while keeping A_0 fixed, a perhaps more familiar case which we will discuss shortly. The fact that the integrand in (4.8) is invariant under (4.10) means the integral is well-defined. More specifically, since the values of A and A_0 on overlaps are related by gauge transformations and diffeomorphisms, the value of the integrand is well-defined everywhere. In practice, the integral can then be evaluated by combining the contributions from different patches with any suitable partition of unity.

The holomorphic Chern-Simons invariant satisfies a number of properties which can be directly derived from its definition (4.8) or the equivalent expression (4.9) and which will be useful for our subsequent discussion. First, for three connections A, B and C on the same bundle we have

$$\mathrm{CS}_B(A) = -\mathrm{CS}_A(B) \, , \qquad \mathrm{CS}_C(A) = \mathrm{CS}_B(A) - \mathrm{CS}_B(C) \, . \tag{4.11}$$

Further, the holomorphic Chern-Simons invariant is unchanged under holomorphic deformations of the gauge connection. Consider an infinitesimal deformation, $\delta a = A - A_0$ of the connection A_0 to a connection A, so that

$$\mathrm{CS}_{A_0}(A) = 2 \int_X \mathrm{tr} \left(\delta a \wedge F_0 \right) \wedge \Omega \, . \tag{4.12}$$

Clearly, this expression vanishes if the connection A_0 is holomorphic. Thus holomorphic connections are extrema of the Chern-Simons functional. Any, even finite, deformation A of A_0 which preserves the condition $F_{(0,2)} = 0$ everywhere along a path in connection space from A_0 to A will therefore lead to a vanishing Chern-Simons invariant, $\mathrm{CS}_{A_0}(A) = 0$.

Computing the Chern-Simons invariant in heterotic models will often involve breaking up the deformation from the reference connection A_0 to a connection A into several parts. Specifically, consider the sequence of deformations

$$A_0 \to A_1 \to \cdots \to A_n = A \ . \tag{4.13}$$

Then, (4.11) implies that the holomorphic Chern-Simons invariant is additive, in the sense

$$\mathrm{CS}_{A_0}(A) = \sum_{k=1}^{n} \mathrm{CS}_{A_{k-1}}(A_k) \ . \tag{4.14}$$

Note that any of the partial deformations $A_{k-1} \to A_k$ which is holomorphic satisfies $\mathrm{CS}_{A_{k-1}}(A_k) = 0$ and, hence, does not contribute the above sum.

The other type of Chern-Simons invariant we need to introduce is the ordinary Chern-Simons invariant, defined by

$$\mathrm{OCS}_{A_0}(A, \mathcal{C}) = \int_{\mathcal{C}} \mathrm{tr}\left((d_{A_0} a \wedge a) + \frac{2}{3} a \wedge a \wedge a + 2a \wedge F_0 \right) \ . \tag{4.15}$$

Here $\mathcal{C}$ is some three manifold, which will be a three-cycle within the Calabi-Yau three-fold X in our application, and A and A_0 are connections on a bundle V over $\mathcal{C}$. Following exactly the same logic as for the holomorphic Chern-Simons invariant, this object is invariant under the transformations (4.10) acting on both A and A_0 simultaneously and with the same gauge parameter. This means the integral in (4.15) is well defined.

It will be crucial in Section 4.4 to understand how the two Chern-Simons invariants defined above behave under a different type of gauge transformation, namely one where A changes by a large gauge transformation, while the reference connection A_0 is kept fixed. To this end we reproduce here the standard construction from the literature addressing this issue [166].

Consider constructing a bundle $\mathbb{V}$ on $\mathcal{C} \times S^1$, where the circle is described by the interval $[0, 1]$ with ends identified, by using a large gauge transformation g to glue $\mathcal{V}|_{\mathcal{C}\times\{0\}}$ to $\mathcal{V}|_{\mathcal{C}\times\{1\}}$. We also construct another bundle $\mathbb{V}_0$ on the same four manifold by using the identity group element, rather than g, in the identification. We take $\mathbb{A}$ to be any connection on $\mathbb{V}$ that restricts to be A on $\mathcal{C} \times \{0\}$ and therefore $g(A)$ on $\mathcal{C} \times \{1\}$. We take $\mathbb{A}_0$ to be any connection on $\mathbb{V}_0$ that restricts to A_0 on both $\mathcal{C} \times \{0\}$ and $\mathcal{C} \times \{1\}$. Then, by Stokes' theorem, we have the following:

$$\mathrm{OCS}_{A_0}(g(A), \mathcal{C}) - \mathrm{OCS}_{A_0}(A, \mathcal{C}) = \int_{\mathcal{C}\times S^1} \mathrm{tr}\,(\mathbb{F} \wedge \mathbb{F}) - \mathrm{tr}\,(\mathbb{F}_0 \wedge \mathbb{F}_0)\,. \tag{4.16}$$

This is the usual statement that Chern-Simons invariants change by an integer under large gauge transformations. Note that we are viewing the situation in two different manners here. To use Stokes' theorem we are dropping the gluing with g to simply have a line interval with boundary in order to obtain (4.16). On this space, A and A_0 are connections on the same bundle so that the Chern-Simons invariant is well defined and we can use Stokes' theorem. Then, to claim that the right hand side of (4.16) is an integer we are viewing the situation as the glued geometry described above. We can then properly define the two topological invariants that appear independently as being associated to two different bundles and, since they are integrated over a closed manifold, we can see they are proportional to integers.

Thus, the ordinary Chern-Simons invariant (4.15) changes by integers under large gauge transformations of their argument, with the reference connection fixed. It should also be pointed out that the fact that the reference connection is not transformed to obtain this behaviour is implicit in common applications of Chern-Simons invariants where A_0 is taken to vanish. If this were not the case one would obtain a non-vanishing reference connection after the gauge transformation.

A very similar construction to the one given above can be used to determine how the holomorphic Chern-Simons invariant behaves under a large gauge transformations of its argument with the reference connection fixed [167]. The result is that $CS_{A_0}(A)$ changes by a period of the holomorphic three-form Ω.

The final question we would like to review in this sub-section is how the two types of Chern-Simons invariants, (4.8) and (4.15), are linked. This is a little more difficult to see than in the case of flat bundles where the Chern-Simons invariants are closed [152, 153, 154], but is still straightforward.

We begin with a remark about the structure of the holomorphic Chern-Simons invariant. From (4.8), this invariant contains the form $\mathrm{tr}(\overline{\partial}_{A_0} a \wedge a + \frac{2}{3} a \wedge a \wedge a + 2a \wedge F_0)$ which is not closed. However, by the Hodge decomposition any form can be written as a sum of a closed and a co-exact form. Luckily, in the expression (4.8) for the holomorphic Chern-Simons invariant the co-exact piece does not contribute to the integral. To see this, write this co-exact piece as $d^\dagger \beta_4$ and work out its contribution to the Chern-Simons invariant which is given by

$$\int d^\dagger \beta_4 \wedge \Omega = -i \int d^\dagger \beta_4 \wedge *\Omega = -i \left\langle d^\dagger \beta_4, \overline{\Omega} \right\rangle = -i \left\langle \beta_4, d\overline{\Omega} \right\rangle = 0 \ . \qquad (4.17)$$

Given this, we can treat the holomorphic Chern-Simons invariant as an integral over a wedge product of two closed forms. In fact this is, in part, why the Chern-Simons term is so hard to compute. The information we have easy access to - the relation of $\omega_3(A)$ to $\mathrm{tr}(F \wedge F)$ - drops out of the integral defining the invariant.

This being the case, let us take the usual symplectic basis of the third cohomology of X, (α_i, β^i), and the associated dual basis of three-cycles $(\mathcal{A}^i, \mathcal{B}_i)$. These quantities obey the following standard special geometry relations

$$\begin{aligned} \int_X \alpha_i \wedge \beta^j &= \int_{\mathcal{A}^j} \alpha_i = \delta_i^j, & \int_X \alpha_i \wedge \alpha_j &= 0, \\ \int_X \beta^j \wedge \alpha_i &= \int_{\mathcal{B}_i} \beta^j = -\delta_i^j, & \int_X \beta^i \wedge \beta^j &= 0. \end{aligned} \tag{4.18}$$

In terms of the cohomology basis, we can expand the holomorphic three-form Ω as

$$\Omega = \mathcal{Z}^i \alpha_i - \mathcal{G}_i \beta^i \;, \tag{4.19}$$

where $\mathcal{Z}^i$ are the usual coordinates on the complex structure moduli space and the $\mathcal{G}_j$ are the derivatives of the pre-potential with respect to these variables. Given this set-up, the holomorphic Chern-Simons invariant can now be expressed in terms of ordinary Chern-Simons invariants associated to the basis three-cycles $(\mathcal{A}^i, \mathcal{B}_i)$. A short calculation shows that

$$\mathrm{CS}_{A_0}(A) = b_i \mathcal{Z}^i - a^i \mathcal{G}_i \;, \qquad a^i = \mathrm{OCS}_{A_0}(A, \mathcal{A}^i) \;, \qquad b_i = \mathrm{OCS}_{A_0}(A, \mathcal{B}_i) \;. \tag{4.20}$$

Hence, the ordinary Chern-Simons invariants, carried out over a basis of three-cycles, determine the holomorphic Chern-Simons invariant. Note that this result is consistent with the above discussion of how these objects behave under a large gauge transformation. Under such large gauge transformations, the ordinary Chern-Simons invariants and, hence, the numbers a_i and b^i, change by integers. Equation (4.20) then implies that the holomorphic Chern-Simons invariant changes by a period, in agreement with the earlier discussion.

We can now be more precise about what we mean by a "fractional holomorphic Chern-Simons invariant". This terminology indicates a holomorphic Chern-Simons invariant for which at least one of the numbers (a^i, b_i) in (4.20) is not an integer. An analogous definition has been used in related work studying flat bundles [152, 153, 154].

4.1.3 Chern-Simons invariants in heterotic theories

We now have the tools to examine how the Chern-Simons invariants introduced in the previous sub-section relate to the Chern-Simons terms which appear in heterotic theories and how we can use the former to calculate the latter. This correspondence will form the basis of our subsequent calculations.

The first difference to resolve is the apparent discrepancy in the set-up of vector bundles. While the physical Chern-Simons term (4.4) depends on connections with independent gauge transformations on apparently different bundles, the tangent bundle TX and the gauge bundle V on X, the holomorphic Chern-Simons term (4.8) depends on a connection and a reference connection, both defined on the *same* bundle. What comes to the rescue is the fact that two E_8 bundles are the same *as real bundles*[4] if and only if their second Chern characters, as elements of $H^4(X, \mathbb{Z})$, match [168, 169]. In the case where no 5-branes are present and we only have a bundle in one E_8 factor of the heterotic gauge group the second Chern characters for the tangent bundle and the gauge bundle must be equal [5]. This follows from the integrability condition on the heterotic Bianchi identity (4.1) for the Chern-Characters with real coefficients, and from global worldsheet anomaly considerations for the extension to include torsion [170, 171]. Therefore the tangent bundle and gauge bundle are the same

[4]Note that we use the term "real bundle" here to refer to the underlying smooth structure (i.e. real bundle as opposed to complex bundle), not to refer to a real structure on a holomorphic bundle (e.g. special orthogonal or symplectic structures).

[5]Note that, in order for heterotic theory to be well defined more generally, there must be a generalization of the consistent mathematical definition of the Chern-Simons invariant to include more complicated cases, such as those involving contributions to the Bianchi Identity from M5 branes. Such generalizations would certainly be interesting to pursue, both from the perspective of physics and mathematics, but are beyond the scope of the current work.

as real bundles in such a situation. Note this does not mean that they are the same as holomorphic objects. Indeed this could not be possible in the case, for example, where the third Chern class of the gauge bundle differs from that of the tangent bundle.

Given this discussion, the physical and mathematical versions of the Chern-Simons invariant start to look somewhat similar. Only two differences remain. The first is that in the holomorphic Chern-Simons invariant (4.8), both A and A_0 are defined relative to the same trivialization of the bundle. This is in distinction to (4.4) where they would be written with respect to two different trivializations adapted to the holomorphic structure on the gauge and tangent bundles respectively. Second, (4.8) contains one additional term relative to (4.4) which, despite initial appearances, cannot be integrated by parts to obtain zero.

To address these differences, we recall that the H-part of the superpotential

$$W_H = \int_X H \wedge \Omega = \int_X H_0 \wedge \Omega + \alpha' \mathrm{CS}_{\mathrm{phys}}(A, \omega) \tag{4.21}$$

is actually gauge invariant due to the gauge invariance of H. The invariance is achieved by cancelling the non-vanishing variation of the Chern-Simons term against the variation of H_0 which can be written as dB locally. Hence, we can choose a gauge where A and ω, which we can now think of as connections on the same bundle, are described relative to the same trivialization. Further, in order to remove the additional term which appears in (4.8) relative to (4.4) we can use a gauge where at least one of the connections A and A_0 has a vanishing $(0,1)$ component. This is always possible because the connections of physical interest can be written as Chern-connections in appropriate trivializations. Having fixed a gauge in this manner, we can express W_H in terms of the holomorphic Chern-Simons invariant as

$$W_H = \int_X H_0 \wedge \Omega + \alpha' \mathrm{CS}_{\omega}(A) \; . \tag{4.22}$$

We already know from (4.20) that this superpotential can be expressed in term of ordinary Chern-Simons invariants, associated to the symplectic basis $(\mathcal{A}^i, \mathcal{B}_i)$ of three-cycles. More precisely, the two terms in (4.22) can be written as

$$\frac{1}{\alpha'}\int_X H_0 \wedge \Omega = m_i \mathcal{Z}^i - n^i \mathcal{G}_i \ , \qquad \mathrm{CS}_\omega(A) = b_i \mathcal{Z}^i - a^i \mathcal{G}_i \ , \tag{4.23}$$

where

$$\begin{aligned} a^i &= \mathrm{OCS}_\omega(A, \mathcal{A}^i), & b_i &= \mathrm{OCS}_\omega(A, \mathcal{B}_i), \\ n^i &= \tfrac{1}{\alpha'}\textstyle\int_{\mathcal{A}^i} H - \mathrm{OCS}_\omega(A, \mathcal{A}^i), & m_i &= \tfrac{1}{\alpha'}\textstyle\int_{\mathcal{B}_i} H - \mathrm{OCS}_\omega(A, \mathcal{B}_i). \end{aligned} \tag{4.24}$$

Consequently, the full superpotential W_H is given by

$$\frac{1}{\alpha'} W_H = (m_i + b_i)\mathcal{Z}^i - (n^i + a^i)\mathcal{G}_i \ . \tag{4.25}$$

The flux quantization condition (4.7), properly expressed in terms of ordinary Chern-Simons invariants, takes the form

$$\frac{1}{\alpha'}\int_{\mathcal{C}} H - \mathrm{OCS}_\omega(A, \mathcal{C}) \in \mathbb{Z} \tag{4.26}$$

and it shows that the quantities n^i and m_i in (4.25) are, in fact, integers. In other words, the integers n^i and m_i describe the harmonic flux in H_0 while the potentially fractional quantities a^i and b_i describe the holomorphic Chern-Simons invariant. It is the latter, which are the main subject of this chapter.

As is clear from the above discussion, a key ingredient in computing these quantities in actual heterotic models is knowing the isomorphism between the tangent bundle and the vector bundle explicitly. This is needed to write the connections on the tangent and gauge bundles relative to the same trivialization. Given that this isomorphism is typically not holomorphic it is not easy to find and we will describe in Section 4.2 and Appendix C how this can be done in certain cases.

4.2 Calculating Chern-Simons invariants

4.2.1 General approach

After the general discussion of the last section we return to the central goal of this chapter. We want to compute the holomorphic Chern-Simons invariant (4.8) for specific connections over Calabi-Yau three-folds X that appear in heterotic compactifications. In particular, we are interested in the case where $A_0 = \omega$ is the spin connection on TX and A the gauge connection on a bundle $V \to X$ solving the Hermitian Yang-Mills equations, which we introduced in Chapter 2,

$$F_{\bar{a}\bar{b}} = 0 \, , \qquad g^{a\bar{b}} F_{a\bar{b}} = 0 \, . \tag{4.27}$$

We will choose to write these connections in "math gauge" as Chern connections. Note that different gauge transformations would be needed on A and A_0 in order to write them in the gauge, more prevalent in the physics literature, where these fields are real. Given the transformation properties discussed in Section 4.1, this means that the result we will obtain will generically change by an integer if we chose to do this. Obviously, such an integer shift cannot change whether or not a Chern-Simons invariant is fractional, which is a main point of interest here.

Typically, the holomorphic structure of the gauge and tangent bundles in a heterotic compactification are different. The Chern connection solving (4.27) and the spin connection would be written in terms of different local trivializations respecting these structures. Nevertheless, we could compute the holomorphics Chern-Simons invariant (4.8) if we had the requisite real bundle isomorphisms between the two bundles. Let us discuss how such a computation would proceed.

Let us phrase this discussion more generally, in terms of two bundles $V \to X$ and $V' \to X$ over a Calabi-Yau three-fold X and a (possibly non-holomorphic) bundle isomorphism $f : V' \to V$. One might imagine taking $V' = TX$, for example, given the above discussion. However, we wish to keep our notation more general because, as we will see, in practice this might be required for the computation. We assume that we have connections ∇_0 and ∇' on V and V', respectively, as well as local frames s_a and s'_a associated to some given open set in the base. Then, relative to these local frames, the gauge fields A_0 and A' associated to ∇_0 and ∇' are obtained from

$$\nabla_0 s_a = A^b{}_{0a} s_b \ , \qquad \nabla' s'_a = A'^b{}_a s'_b \ . \tag{4.28}$$

We can use the bundle morphism f to "transport" the connection ∇' on V' to a connection on V, which we will denote by ∇. This connection is defined by

$$\nabla(s) := f \circ \nabla'(f^{-1} \circ s) \ , \tag{4.29}$$

where s is a section of V. The bundle morphism f can also be used to map the frame s'_a of V' to a frame $\tilde{s}_a := f \circ s'_a$ of V. Now we have two frames, s_a and $\tilde{s}_a$ on V and thus there is a gauge transformation

$$s_a = P^b{}_a \tilde{s}_b \tag{4.30}$$

relating them. We can work out the gauge field which corresponds to ∇ relative to the frame $\tilde{s}_a$ and the frame s_a. The result is

$$\nabla(\tilde{s}_a) = A'^b{}_a \tilde{s}_b \ , \qquad \nabla(s_a) = A^b{}_a s_b \quad \text{where} \quad A = P^{-1} A' P + P^{-1} dP \ . \tag{4.31}$$

In other words, relative to the frame $\tilde{s}_a$, obtained by transporting the frame on V' to V, the gauge field remains unchanged, that is, it is given by A' for both the frames s'_a on V' and $\tilde{s}_a$ on V. For the frame s_a on V, on the other hand, the gauge field is obtained from A' by the above gauge transformation.

Now that we have phrased matters in terms of two connections on the same bundle, we can work out the holomorphic Chern-Simons term more explicitly. Suppose that both initial connections A' and A_0 are Chern connections. (But note that A, being obtained from A' by a potentially non-holomorphic bundle morphism, does not need to be a Chern connection.) Then, the term $d(A \wedge A_0) \wedge \Omega$ in (4.9) vanishes simply by index structure arguments and a quick calculation shows the remaining terms satisfy

$$\omega(A) - \omega(A_0) = \omega(A') - \omega(A_0) + \mathrm{tr}\left(\theta\, dA' - A'\theta^2 - \frac{1}{3}\theta^3\right) , \qquad \theta := dP\, P^{-1} . \tag{4.32}$$

In particular, in our case where A' and A_0 happen to be $(1,0)$ gauge fields, we have

$$[\omega(A) - \omega(A_0)]_{(0,3)} = -\frac{1}{3}\mathrm{tr}(\theta^3)_{(0,3)} \qquad \text{so that} \qquad \mathrm{CS}_{A_0}(A) = -\frac{1}{3}\int_X \mathrm{tr}(\theta^3) \wedge \Omega . \tag{4.33}$$

Thus we see from (4.33) that if the real isomorphism f is known, so that P can be obtained from its action on frames via (4.30), then we can compute the Chern-Simons invariant associated to Chern connections A and A_0, even if we do not know the explicit form of these connections themselves. Clearly this his helpful given the non-constructive nature of the Yau [172] and Donaldson-Uhlenbeck-Yau theorems [173, 174].

It should be noted that, given the form of the result (4.33), one might expect this Chern-Simons invariant to not be fractional since the integrand looks like the wedge product with Ω of the standard integrand giving a winding number. In fact, the holomorphic Chern-Simons invariant can be fractional as we will demonstrate in Section 4.4.3. Nevertheless, even if this were to be the case an integral result here would still be important. A non-zero Chern-Simons invariant $\mathrm{CS}_{A_0}(A)$ of this type would destabilize the usual meta-stable vacuum in the absence of other effects. In addition, non-zero integer results can lead to fractional Chern-Simons invariants in quotients, as we will discuss in Section 4.4.

4.2.2 Finding the isomorphism

It is clear from the proceeding discussion that the key quantity we need to compute is the real bundle isomorphism f between V' and V and the associated gauge transformations P. How do we describe such an isomorphism practically? Since this is somewhat technical, a full description of this topic is relegated to Appendix C. Here we will content ourselves with a summary of the essential ideas, along with a simple illustrative example.

By definition, vector bundles locally look like a direct product of an open set on the base manifold and the fiber. In other words, we have local trivializations,

$$\phi_\alpha : \pi^{-1}(U_\alpha) \to W_\alpha \times F \ . \tag{4.34}$$

Here π is the projection map of the bundle and W_α is an open subset of $\mathbb{C}^{\dim(X)}$ and $F \cong \mathbb{C}^{\mathrm{rk}(V)}$ is the typical fiber. These local trivializations are glued together by transition functions $\phi_{\alpha\beta} := \phi_\alpha \circ \phi_\beta^{-1} : W_\beta \times F \to W_\alpha \times F$ to construct the bundle globally. The transition functions act trivially on the base and as linear maps $T_{\alpha\beta}$ on the fiber F.

In terms of these local trivializations, our real bundle isomorphism f is described by a collection of maps

$$.f_\alpha : W_\alpha \times F \to W_\alpha \times F \ . \tag{4.35}$$

Because we want the isomorphism to act fiber-wise, preserving points on the base, these maps take the form $f_\alpha(z, v) = (z, P_\alpha v)$ where the z are coordinates on the open set W_α in the base and the v are coordinates in the fiber F. In short, the real bundle isomorphism f can be described by a collection of matrices P_α which encodes, for each patch, how the fibers of V' are mapped to those of V. In fact, these are precisely the matrices appearing in (4.30).

The matrices P_α must satisfy several consistency conditions. The first is that, if they are to map V' to V, then they must correctly map the transition functions of the first bundle into those of the second. That is, they must obey the intertwining conditions

$$T_{\alpha\beta} = P_\alpha^{-1} {T'}_{\alpha\beta} P_\beta \tag{4.36}$$

for all patches α, β. All matrices P_α must also be invertible (to define an isomorphism rather than just a morphism) and they must be non-singular (to be well defined). We describe all of these conditions in detail in Appendix C. The non-holomorphic nature of the bundle morphisms we will utilize manifests itself in the fact that the matrices $P_\alpha = P_\alpha(z, \overline{z})$ are, in general, not holomorphic functions of the base coordinates.

Reverting the logic of the discussion, we can say that any collection of matrices P_α, all invertible and non-singular, which satisfy the intertwining conditions (4.36) *define* a bundle morphism f. Thus, in order to compute the holomorphic Chern-Simons invariant $CS_{A_0}(A)$ via (4.33) we need to obtain such a set of matrices P_α.

Let us illustrate this discussion with a concrete example on the simple base manifold $\mathbb{P}^1$. It is known that line bundle sums on $\mathbb{P}^1$ are classified by their total Chern character. In particular, this means that the line bundle sums

$$V' = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \quad \text{and} \quad V = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \tag{4.37}$$

which both have vanishing first Chern class are real isomorphic. How do we write down such an isomorphism in the form we have been discussing? The standard open cover of $\mathbb{P}^1$ has two patches, which we label U_0 and U_1 with affine coordinates z and w, respectively. Relative to those patches, the transition functions for the bundles V' and V in (4.37) are

$$T'_{10} = \text{diag}(z, z^{-1}) \ , \qquad T_{10} = \text{diag}(1, 1) \ . \tag{4.38}$$

Then, two matrices P_α which satisfy the intertwining conditions (4.36) with these transition functions can simply be written as

$$P_0(z, \bar{z}) = \begin{pmatrix} 1 & \frac{\bar{z}}{1+|z|^2} \\ -z & \frac{1}{1+|z|^2} \end{pmatrix} \ , \qquad P_1(w, \bar{w}) = \begin{pmatrix} w & \frac{1}{1+|w|^2} \\ -1 & \frac{\bar{w}}{1+|w|^2} \end{pmatrix} \ . \tag{4.39}$$

These are clearly non-singular and invertible in their respective patches. Note that the matrices in (4.39) depend upon both the complex coordinates and their conjugates. This had to be the case since the bundles (4.37) are not isomorphic as holomorphic objects.

The above construction can be generalized to relate any two rank two line bundle sums on $\mathbb{P}^1$ with the same first Chern class. The resulting bundle morphisms have the following structure

$$f^{(q,p)} \sim \left(P_\alpha^{(q,p)}\right) \quad : \quad \mathcal{O}_{\mathbb{P}^1}(a-p) \oplus \mathcal{O}_{\mathbb{P}^1}(a+p) \xrightarrow{\simeq} \mathcal{O}_{\mathbb{P}^1}(a-q) \oplus \mathcal{O}_{\mathbb{P}^1}(a+q) \tag{4.40}$$

$$g^{(q,p)} \sim \left(Q_\alpha^{(q,p)}\right) \quad : \quad \mathcal{O}_{\mathbb{P}^1}(a-p) \oplus \mathcal{O}_{\mathbb{P}^1}(a+p+1) \xrightarrow{\simeq} \mathcal{O}_{\mathbb{P}^1}(a-q) \oplus \mathcal{O}_{\mathbb{P}^1}(a+q+1), \tag{4.41}$$

for even and odd first Chern classes, respectively. Their explicit form is a generalization of (4.39) and is provided in Appendix C, where we explain this construction in more detail. Of course, these results cannot be applied to our problem directly but, as we will see, they can be used to construct bundle isomorphisms on Calabi-Yau manifolds which are defined in ambient spaces that involve $\mathbb{P}^1$ factors.

4.2.3 An explicit example

In this section we will work on the tetra-quadric Calabi-Yau three-fold, defined as the zero-locus of a polynomial of multi-degree $(2, 2, 2, 2)$ in the ambient space $(\mathbb{P}^1)^4$, and represented by the configuration matrix

$$X \in \left[\begin{array}{c|c} \mathbb{P}^1 & 2 \\ \mathbb{P}^1 & 2 \\ \mathbb{P}^1 & 2 \\ \mathbb{P}^1 & 2 \end{array}\right] . \tag{4.42}$$

An appealing feature of this example is the presence of the $\mathbb{P}^1$ factors which, as we will see, allows us to transfer the results for real bundle equivalence on $\mathbb{P}^1$ to the tetra-quadric.

Before we construct the relevant bundles on this manifold, we introduce the main building blocks

$$\begin{aligned} B &= \mathcal{O}_X(1,0,-1,0) \oplus \mathcal{O}_X(1,1,0,0) \oplus \mathcal{O}_X(-1,1,0,0) \oplus \mathcal{O}_X(1,0,1,0) \\ \tilde{B} &= \mathcal{O}_X(1,0,0,0) \oplus \mathcal{O}_X(1,0,0,0) \oplus \mathcal{O}_X(0,1,0,0) \oplus \mathcal{O}_X(0,1,0,0) \\ R &= \mathcal{O}_X(0,0,1,0) \oplus \mathcal{O}_X(0,0,1,0) \oplus \mathcal{O}_X(0,0,0,1) \oplus \mathcal{O}_X(0,0,0,1) \\ C &= \mathcal{O}_X(2,2,2,2) \end{aligned} \tag{4.43}$$

which underly our construction. Given these line bundle sums, we define the monad bundle V on X by

$$0 \longrightarrow V \longrightarrow B \oplus R \longrightarrow C \longrightarrow 0 \, . \tag{4.44}$$

The Chern connection on V is denoted by A and our goal is to compute the holomorphic Chern-Simons invariant $\mathrm{CS}_\omega(A)$, relative the the spin connection ω on TX. To do this, we first find a way to deform the spin connection ω to the connection A.

Or first step is to introduce a monad representation of the tangent bundle

$$0 \longrightarrow V_0 \longrightarrow \tilde{B} \oplus R \xrightarrow{\mu_0} C \longrightarrow 0 \, . \tag{4.45}$$

Indeed, for a suitable choice of the monad map μ_0 we have $V_0 \cong TX \oplus \mathcal{O}_X^{\oplus 4}$. We denote the Chern connection on V_0 by A_0. However, for different choices of the monad map the sequence (4.45) describes holomorphic deformations away from the tangent bundle. In particular, we can choose μ_0 such that the four line bundles in $\tilde{B}$ split off as a direct sum. This choice leads to a bundle V_1, with Chern connection A_1, which can be written as

$$V_1 = \tilde{B} \oplus U \, , \qquad 0 \longrightarrow U \longrightarrow R \longrightarrow C \longrightarrow 0 \, . \tag{4.46}$$

The next step is crucial. We use real bundle morphisms on $\mathbb{P}^1$, applied to our ambient space and restricted to the Calabi-Yau manifold, to construct a real bundle morphism $\mathcal{F}$ between the line bundle bundle sums $\tilde{B}$ and B. We will explain the procedure in more detail below but for now we continue outlining the structure of the argument.

Thanks to this real bundle morphism, we can relate the above bundle V_1 to the bundle

$$V_2 = B \oplus U \tag{4.47}$$

with Chern connection A_2. Evidently, this bundle is a holomorphic deformation of our gauge bundle V in (4.44).

To summarize, we have now related the tangent bundle TX to our gauge bundle V via a number of deformations which can be schematically written as

$$\begin{array}{ccccccccc} TX \oplus \mathcal{O}_X^{\oplus 4} & \xrightarrow{\text{hol.}} & V_0 & \xrightarrow{\text{hol.}} & V_1 & \xrightarrow{\text{real}} & V_2 & \xrightarrow{\text{hol.}} & V \\ \omega & \longrightarrow & A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A\ . \end{array} \tag{4.48}$$

From (4.14) the holomorphic Chern-Simons invariant $\mathrm{CS}_\omega(A)$ can be computed by summing the holomorphic Chern-Simons invariants of the four steps in the above sequence. However, three of these steps correspond to holomorphic deformations. It is easy to see that these are not only holomorphic deformations at the level of the bundles, but are also holomorphic deformations of the Chern-connections. Hence, from (4.12), the Chern-Simons invariants associated to these three steps vanish. In conclusion, the only contribution arises from the real deformation in the above sequence, so that

$$\mathrm{CS}_\omega(A) = \mathrm{CS}_{A_1}(A_2)\ . \tag{4.49}$$

We know from the general discussion in Section 4.2.1 that $\mathrm{CS}_{A_1}(A_2)$ can be worked out from the bundle isomorphism $\mathcal{F} : \tilde{B} \to B$ so our next task is to construct this isomorphism.

To do this, we recall from the previous subsection (see (4.40)) that we can find explicit real bundle isomorphisms $f^{(q,p)}$ which relate pairs of rank two line bundle sums on $\mathbb{P}^1$ with the same first Chern class. For our present example, we have four $\mathbb{P}^1$ ambient space factors, which we label by $i = 1, 2, 3, 4$, as well as four line bundles in $\tilde{B}$, which we label by $a = 1, 2, 3, 4$. The maps $f^{(q,p)}$ can be applied to any of the four $\mathbb{P}^1$ factors and to any two of the four line bundles, while leaving the other $\mathbb{P}^1$ factors and line bundle undisturbed. We denote the version of $f^{(q,p)}$ which acts on the i^{th} $\mathbb{P}^1$ factors and on the two line bundles a and b by $f^{(q,p)}_{ab,i}$. As is evident, this gives rise to a large number of possibilities and a corresponding web of real bundle isomorphisms for line bundle sums on $(\mathbb{P}^1)^4$ (and, by restriction, on the tetra-quadric). It would be interesting to explore this more systematically.

For present purposes, this formalism can be used to construct a real bundle isomorphism between $\tilde{B}$ and B by the following chain:

$$
\begin{aligned}
\tilde{B} \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
&\xrightarrow{f^{(1,0)}_{12,2}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
\xrightarrow{f^{(1,0)}_{34,1}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \\
&\xrightarrow{f^{(0,1)}_{14,2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
\xrightarrow{f^{(1,0)}_{14,3}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \;=\; B.
\end{aligned}
\tag{4.50}
$$

Here, for ease of notation, we have written the line bundle sums as matrices, with each row representing the multi-degree of one line bundle. Hence, the desired line bundle isomorphism

$\mathcal{F} : \tilde{B} \to B$ can be written as

$$\mathcal{F} = f^{(1,0)}_{14,3} \circ f^{(0,1)}_{14,2} \circ f^{(1,0)}_{34,1} \circ f^{(1,0)}_{12,2} \ , \tag{4.51}$$

suitably restricted to the Calabi-Yau three-fold X. It is straightforward to promote $\mathcal{F}$ to a bundle isomorphism $V_1 \to V_2$ by extending it trivially onto the common summand U.

The direct computation of the holomorphic Chern-Simons invariant (4.49), based on the formula (4.33), is not hard at this stage but simply tedious. We first compute the matrices P_α which represent the local descriptions of the bundle isomorphism (4.51). For the purpose of integration, it is sufficient to carry this out in the standard patch of $(\mathbb{P}^1)^4$ whose affine coordinates we denote by z_i, where $i = 1, 2, 3, 4$. Combining the individual pieces in (4.51), given in (C.21), (C.19) and (C.11), we find the following expression for the local version of $\mathcal{F}$:

$$P = \begin{pmatrix} \frac{(\bar{z}_2+1)(z_2\bar{z}_3+1)}{(|z_2|^2+1)(|z_3|^2+1)} & -\frac{(\bar{z}_2-1)(z_2\bar{z}_3+1)}{2(|z_2|^2+1)(|z_3|^2+1)} & \frac{(z_1-1)(\bar{z}_2-\bar{z}_3)}{(|z_2|^2+1)(|z_3|^2+1)} & \frac{(z_1+1)(\bar{z}_2-\bar{z}_3)}{2(|z_2|^2+1)(|z_3|^2+1)} \\ z_2-1 & \frac{1}{2}(z_2+1) & 0 & 0 \\ 0 & 0 & \frac{\bar{z}_1+1}{|z_1|^2+1} & \frac{1-\bar{z}_1}{2|z_1|^2+2} \\ -\frac{(z_2-z_3)(\bar{z}_2+1)}{(|z_2|^2+1)} & \frac{(z_2-z_3)(\bar{z}_2-1)}{2(|z_2|^2+1)} & \frac{(z_1-1)(z_3\bar{z}_2+1)}{(|z_2|^2+1)} & \frac{(z_1+1)(z_3\bar{z}_2+1)}{2(|z_2|^2+1)} \end{pmatrix} . \tag{4.52}$$

To restrict to the Calabi-Yau three-fold we must pick a defining relation for the configuration (4.42) and then solve it on the given patch for one of the coordinates in terms of the others. For example, the coordinate z_4 on the last $\mathbb{P}^1$ factor in the ambient space is a natural choice given the dependencies appearing in (4.52). Given the defining equation is a quadric in z_4, this yields two disjoint loci describing parts of X inside the open patch, on which we know the matrix P.

Once we have computed P we can then carry out the integral in (4.33) in order to evaluate the Chern-Simons invariant, using the standard expression for the holomorphic $(3,0)$-form on such manifolds [79, 118, 175, 176]. The integral one obtains vanishes.

We have thus completed the first non-trivial computation of a holomorphic Chern-Simons invariant contribution to the heterotic superpotential, for the non-flat bundle (4.44) over the Calabi-Yau manifold (4.42), albeit obtaining a vanishing result. We have repeated such a computation for a large number of different gauge bundles over different Calabi-Yau manifolds. The integrands all exhibit similar structures, not very different from those which arise in period integrals [177, 178], but the resulting Chern-Simons invariant always vanishes. At this stage one is motivated to look for more general reasons as to why a vanishing result might be obtained in many cases, or why fractional Chern-Simons invariants do not appear. This might provide some insight into our results so far and also guidance on how to build non-flat bundles with fractional contributions to the superpotential. We discuss a relevant vanishing theorem in the next section, and an example with a fractional holomorphic Chern-Simons invariant in Section 4.4.

4.3 A vanishing theorem and its consequences

In this section we will consider the consequences of the following theorem due to R. Thomas [165].

Theorem 4.1. *Suppose that the Calabi-Yau three-fold X is a smooth effective anti-canonical divisor in a four-fold Y defined by $s \in H^0(K_Y^{-1})$. If $E \to X$ is a bundle that extends to a bundle $\mathbb{E} \to \mathbb{Y}$, then for a $\overline{\partial}$-operator A on E, let $\mathbb{A}$ be any $\overline{\partial}$-operator on $\mathbb{E}$ extending A. Then we have, modulo periods and for some choice of reference connection,*

$$\mathrm{CS}(A) = \int_Y \mathrm{tr}\,(\mathbb{F}_{0,2} \wedge \mathbb{F}_{0,2}) \wedge s^{-1}\ . \tag{4.53}$$

In terms of the notation in this theorem, the Chern-Simons invariant $\mathrm{CS}_\omega(A)$ of the connection A on the bundle V, relative to the spin connection ω on the tangent bundle, can be written as $\mathrm{CS}_\omega(A) = \mathrm{CS}(A) - \mathrm{CS}(\omega)$. Clearly, this theorem has important implications for the questions being addressed in this chapter. For example, Theorem 4.1 sheds light on the vanishing result obtained in Section 4.2.3. In this instance the manifold is indeed an anti-canonical hypersurface in a smooth ambient space. In addition, the sums of line bundles that appear in the definition of the bundle (4.44) do extend holomorphically to the ambient space. Although it is not guaranteed, it is also not unreasonable to think that the connection on this bundle might also extend holomorphically to the ambient space. Indeed, the ansatze that are used to describe fiber metrics in numerical work [30, 34, 179, 180, 181, 182, 183, 184, 185, 186, 187] are somewhat suggestive of this. Given all of this, Theorem 4.1, together with our discussion of Section 4.2.3 makes it no surprise that $\mathrm{CS}_\omega(A) = 0$.

Despite the discussion of the proceeding paragraph, one might think that Theorem 4.1 has limited applicability in physical contexts. The techniques illustrated in the simple example in Section 4.2.3 clearly generalize to large classes of cases and it might naively appear that the above theorem has a very limited scope in terms of the types of bundles and Calabi-Yau manifolds to which it applies. In fact, Theorem 4.1 is of relevance in a surprisingly wide range of examples.

The first seemingly strong restriction in Theorem 4.1 is the requirement that the Calabi-Yau manifold be described as an anti-canonical hypersurface in an ambient four-fold. This is in fact not much of a restriction at all in many discussions of string compactification. In the case of any complete intersection in a smooth ambient space, for example for any CICY [117, 118, 119, 120] or gCICY [123] (see [188, 189] for related work), one can simply pick Y to be described by $k-1$ of the defining equations where k is the codimension of the three-fold. The final defining equation will then be an anti-canonical hypersurface in that ambient space. Further, for the theorem to apply it is only necessary that the Calabi-Yau manifold under consideration admits *some* description of this type. Calabi-Yau manifolds can be described in a plethora of different manners and even if a three-fold of interest is not described in a manner compatible with Theorem 4.1 that does not mean that such a description does not exist. Indeed, it can be hard in a given case to prove that a description as an anti-canonical hypersurface in an ambient Y does not exist.

It is true that many ambient spaces appearing in descriptions of known Calabi-Yau manifolds are singular. This is commonly the case for Calabi-Yau manifolds described as hypersurfaces in toric varieties [127, 128, 190], or quotients of CICYs [129, 191, 192, 193, 194, 195, 196], for example. Even in such cases, however, the ambient spaces which appear in constructions in the literature are frequently resolvable and one can then simply apply Theorem 4.1 to the anti-canonical hypersurface in that resolution. If the initial Calabi-Yau manifold was smooth, then the ambient singularities must have missed the hypersurface and, hence, the three-fold is not changed during the resolution process.

One might have similar reservations about the general applicability of the assumptions made about the bundles and connections as they appear in Theorem 4.1. However, in this case too the structure required is not as restrictive as one might think and the theorem applies to many cases appearing in the physics literature. There are many constructions that are utilized in heterotic compactifications where the bundle does not extend nicely to the ambient space. Bundles constructed as two term monads over CICYs, for example, frequently have the feature that, while they restrict to a bundle over the Calabi-Yau three-fold they are merely a sheaf over the ambient space. As it happens, the bundle (4.44) in our example extends to a bundle on the ambient space, but this is not necessarily the case in other models (see [51, 104] for some explicit cases). Nevertheless, even in these cases it is not clear that Thomas' theorem does not apply. Since the Chern-Simons invariant is unchanged under holomorphic deformations of its argument, we only really require some holomorphic deformation of the bundle under consideration to extend to the ambient space. In addition, as with manifolds, bundles over Calabi-Yau three-folds can typically be described in many ways. Even if some descriptions do not allow for an extension to an ambient space bundle there may well exist others which do.

Using the tools presented in Section 4.2, we have computed the holomorphic Chern-Simons invariants for quite a number of different cases, and each time we have obtained zero. We believe that the above theorem may be one of the culprits behind this conspiracy. In the rest of this chapter we will describe heterotic string compactifications in which non-trivial Chern-Simons invariants can be obtained, culminating in a concrete example of a non-flat gauge bundle giving rise to a fractional invariant.

4.4 Fractional Chern-Simons invariants

4.4.1 General remarks

In this section we will discuss two methods for constructing non-flat bundles in heterotic compactifications which give rise to fractional holomorphic Chern-Simons superpotential contributions. These discussions will focus on manifolds which are freely acting quotients of an initial simply connected Calabi-Yau three-fold (or on three-folds with a non-trivial fundamental group). The technical results that we will need as part of this discussion are presented in Appendix D.

The first argument we wish to give makes use of large gauge transformations on a Calabi-Yau manifold X in order to generate connections with fractional holomorphic Chern-Simons invariants on its quotient $\hat{X}$. Working over the geometry X, it is easy to obtain a vanishing Chern-Simons invariant. Indeed, an example of such a case is given in Section 4.2.3. From such a result one can easily obtain a non-vanishing, but non-fractional holomorphic Chern-Simons invariant, simply by performing a large gauge transformation on the argument of the functional, A. It is not clear what integers one can obtain for the associated ordinary Chern-Simons invariants in such a case, and for a given topology it is not the case that every possible integer will always be obtainable. Furthermore, explicitly writing down such large gauge transformations appears to be prohibitively difficult in many cases. Nevertheless, non-vanishing Chern-Simons invariants can clearly be obtained on X.

Let us denote by Γ the freely acting symmetry on X by which we quotient to obtain $\hat{X}$. Further, a Γ-equivariant bundle V on X descends to a bundle on $\hat{X}$ which we denote by $\hat{V}$. In Appendix D, we introduce the notion of Γ-equivariant connections on Γ-equivariant bundles on X. Suppose we consider such Γ-equivariant connections A and A_0 on V which give rise to a holomorphic Chern-Simons invariant on X with at least one of the numbers a^i and b_i in (4.23) (that is, at least one of the ordinary Chern-Simons terms involved) not being divisible by $|\Gamma|$. In Appendix D, we show that the resulting holomorphic Chern-Simons invariant on $\hat{X}$ is obtained by dividing its counterpart on X by the group order (see (D.19)). Therefore the holomorphic Chern-Simons invariant obtained on the quotient would be fractional.

One might think that such cases are common place, and indeed that may well be true. However, constructing a concrete example as an existence proof is difficult, due to the fact that an explicit expression for, or at least proof of existence of, the large gauge transformation involved is required. Without this one cannot concretely rule out the possibility that all large gauge transformations over X that exist lead to integers divisible by $|\Gamma|$ for all possible symmetries by which the manifold could be quotiented, however unlikely this may seem.

It is interesting to ask how this method for obtaining fractional holomorphic Chern-Simons invariants evades the statement that large gauge transformations on $\hat{X}$ should change those functionals by integer multiples of periods. Suppose we have an equivariant connection A_0 on the (equivariant) bundle $V \to X$, and another connection A on V, related to A_0 by a large gauge transformation, so that $\mathrm{CS}_{A_0}(A)$ is an integer multiple of periods. We prove in Appendix D that the large gauge transform of an equivariant gauge field is always equivariant so that A is equivariant as well. Hence, both A_0 and A descend to the quotient, inducing connections $\hat{A}_0$ and $\hat{A}$ of $\hat{V}$. However, it is not true that the large gauge transformation involved will always descend to $\hat{X}$. In other words, $\hat{A}_0$ and $\hat{A}$ need not be related by a large gauge transformation and, hence, the corresponding holomorphic Chern-Simons invariant $\mathrm{CS}_{\hat{A}_0}(\hat{A})$ on $\hat{X}$ does not have to be an integer multiple of periods.

Given the non-constructive nature of such an argument for the existence of fractional holomorphic Chern-Simons invariants, we will, in the following subsections, present a concrete example of a non-flat bundle exhibiting such a structure. We construct this example in a manner which is presumably less generic, but nevertheless more explicit, than the discussion of the proceeding paragraphs.

4.4.2 Tensor product connections

To discuss the example in the next sub-section we will need a few basic facts and definitions concerning tensor product connections and their Chern-Simons invariants. Given two bundles V_1 and V_2 with connections ∇_1 and ∇_2, the tensor product $V = V_1 \otimes V_2$, can be equipped with the tensor product connection ∇ defined by

$$\nabla(s_1 \otimes s_2) = (\nabla_1 s_1) \otimes s_2 + s_1 \otimes (\nabla_2 s_2)\ . \tag{4.54}$$

Here, s_1 and s_2 are sections of V_1 and V_2, respectively. If we set up local frames s_{1i} and s_{2k} for V_1 and V_2 then these define a local frame $s_{1i}\otimes s_{2k}$ for V. The corresponding gauge fields, introduced in the usual manner as

$$\nabla_1 s_{1\,i} = A_{1i}^{j} s_{1\,j}\ , \qquad \nabla_2 s_{2\,k} = A_{2k}^{l} s_{2\,l} \qquad \nabla(s_{1\,i}\otimes s_{2\,a}) = A_{ia}^{jb}(s_{1\,j}\otimes s_{2\,b})\ , \tag{4.55}$$

are then easily seen to be related by

$$A_{ik}^{jl} = A_{1i}^{j}\delta_k^l + \delta_i^j A_{2k}^{l}\ . \tag{4.56}$$

For the curvature of the tensor product connection, it follows from the definition (4.54) that

$$F(s_1\otimes s_2) = (F_1(s_1))\otimes s_2 + s_1\otimes (F_2(s_2)). \tag{4.57}$$

Hence, if F_1 and F_2 satisfy the Hermitian Yang-Mills equations (4.27) then so does F.

Given this set-up, it is straightforward to compute the Chern-Simons form for the tensor product connection

$$\omega_3(A) = \mathrm{rk}(V_2)\,\omega_3(A_1) + \mathrm{rk}(V_1)\,\omega_3(A_2)\ . \tag{4.58}$$

We introduce reference Chern-connections A_{10} and A_{20} for A_1 and A_2, respectively, along with their tensor product connection A_0 which serves as a reference connection for A. Then, (4.58) combined with the formula (4.9) for the holomorphic Chern-Simons invariant gives

$$\mathrm{CS}_{A_0}(A) = \mathrm{rk}(V_2)\,\mathrm{CS}_{A_{10}}(A_1) + \mathrm{rk}(V_1)\,\mathrm{CS}_{A_{20}}(A_2)\ . \tag{4.59}$$

4.4.3 A non-flat bundle with a fractional invariant

We will now construct an example of a non-flat bundle with a fractional holomorphic Chern-Simons invariant as defined in Section 4.1.2, by using the notion of tensor product connections on a quotient of a CICY three-fold. It should be noted that finding calculable examples of this type is also rather difficult. The structure required, as we will see, is rather specific. In addition, most cases that both exhibit the necessary structure and are calculationally tractable have turned out not to lead to a fractional Chern-Simons invariant. Nevertheless, we find it valuable to provide this example as an existence proof for non-flat bundles in heterotic compactifications with fractional holomorphic Chern-Simons invariants.

Consider the manifold, CICY 5301, in the standard list [118, 197], specified by the configuration matrix

$$X \in \left[\begin{array}{c|cccc} \mathbb{P}^1 & 0 & 1 & 1 & 0 \\ \mathbb{P}^1 & 0 & 1 & 1 & 0 \\ \mathbb{P}^1 & 1 & 0 & 0 & 1 \\ \mathbb{P}^1 & 1 & 0 & 0 & 1 \\ \mathbb{P}^3 & 1 & 1 & 1 & 1 \end{array}\right] . \tag{4.60}$$

We denote the homogeneous ambient space coordinates by $x_{a,i}$, where $a = 1, \ldots, 5$ labels the projective factors and $i = 0, 1, \ldots$ its coordinates.

This manifold admits a freely acting $\mathbb{Z}_4$ symmetry whose generator acts as follows [6]

$$\begin{array}{llllllllllll} x_{1,0} & \mapsto & x_{3,0}, & x_{1,1} & \mapsto & -x_{3,1}, & x_{2,0} & \mapsto & x_{4,0}, & x_{2,1} & \mapsto & x_{4,1}, \\ x_{3,0} & \mapsto & x_{1,0}, & x_{3,1} & \mapsto & x_{1,1}, & x_{4,0} & \mapsto & x_{2,0}, & x_{4,1} & \mapsto & -x_{2,1}, \\ x_{5,0} & \mapsto & x_{5,3}, & x_{5,1} & \mapsto & -x_{5,2}, & x_{5,2} & \mapsto & x_{5,1}, & x_{5,3} & \mapsto & x_{5,0}. \end{array} \quad (4.61)$$

The symmetry also acts non-trivially on the normal bundle as represented by the following action

$$(p_1, p_2, p_3, p_4) \mapsto (p_2, p_1, p_4, -p_3)\,, \quad (4.62)$$

on the defining polynomials. The quotient $\hat{X}$ of X by the symmetry (4.61), (4.62) leads to a transverse variety, and the action is fixed point free, and, hence, $\hat{X}$ is a smooth Calabi-Yau three-fold with fundamental group $\pi_1(\hat{X}) = \mathbb{Z}_4$. It is on $\hat{X}$ that we will construct our example.

To construct our bundle we begin by noting that the following sum of line bundles

$$U = \mathcal{O}_X(-2,-1,0,0,1) \oplus \mathcal{O}_X(0,1,-2,1,0) \oplus \mathcal{O}_X(2,0,2,-1,-1) \quad (4.63)$$

has a second Chern character which is exactly half of that of the tangent bundle TX. In addition, U has a vanishing slope on an appropriate sub-locus of the Kähler cone and admits an equivariant structure with respect to the symmetry (4.61), (4.62), so that it descends to a bundle $\hat{U}$ on $\hat{X}$. Now consider the bundle $V = U \oplus U$ on X. This is an equivariant bundle with a second Chern character which matches that of TX and thus naively gives a good heterotic vacuum on X. Its equivariant nature means that it descends to a bundle

[6] Note we have performed a linear coordinate transformation from the symmetry action as it is usually represented in the standard list [129, 197]. This form of the symmetry preserves a particularly simple SLAG as we will see shortly.

$\hat{V} = \hat{U} \oplus \hat{U}$ on $\hat{X}$. We also introduce the well-know holomorphic deformation of $TX \oplus \mathcal{O}_X^{\oplus 5}$, given by

$$0 \to V_0 \to \mathcal{O}_X(1,0,0,0,0)^{\oplus 2} \oplus \ldots \oplus \mathcal{O}_X(0,0,0,0,1)^{\oplus 4} \to \mathcal{O}_X(0,0,1,1,1)^{\oplus 2} \oplus \mathcal{O}_X(1,1,0,0,1)^{\oplus 2} \to 0.$$

This bundle also has an equivariant structure under the above $\mathbb{Z}_4$ symmetry and descends to a bundle $\hat{V}_0$ on $\hat{X}$. For all these bundles, we introduce Chern connections which satisfy the Hermitian Yang-Mills equations, as indicated in Table 4.1. Note that the connection $\hat{A}$ on $\hat{V}$ is taken as a direct sum connection constructed from two copies of the connection $A_{\hat{U}}$ on $\hat{U}$.

space	X			$\hat{X} = X/\mathbb{Z}_4$				
bundle	U	$V = U^{\oplus 2}$	V_0	$\hat{U}$	$\hat{V} = \hat{U}^{\oplus 2}$	$\hat{V}_0$	$\hat{W}$	$\hat{V}' = \hat{U} \oplus \hat{W}$
connection	A_U	A	A_0	$A_{\hat{U}}$	$\hat{A}$	$\hat{A}_0$	$A_{\hat{W}}$	$\hat{A}'$

Table 4.1: Bundles and associated connections of our construction

We are interested in the holomorphic Chern-Simons invariant $\mathrm{CS}_{\hat{A}_0}(\hat{A})$. If this invariant is fractional then we can stop our search here. If it is not, however, there is a simple modification that allows us to generate a new bundle on the quotient manifold $\hat{X}$ which does have a fractional invariant. More specifically, we can can carry out the following modification

$$\hat{V} = \hat{U} \oplus \hat{U} = \hat{U} \otimes (\mathcal{O}_{\hat{X}} \oplus \mathcal{O}_{\hat{X}}) \quad \longrightarrow \quad \hat{V}' = \hat{U} \otimes \hat{W} \tag{4.64}$$

of the bundle $\hat{V}$, where $\hat{W}$ is a rank two flat bundle (a Wilson line) on $\hat{X}$ which we have used to replace the trivial bundle $\mathcal{O}_{\hat{X}} \oplus \mathcal{O}_{\hat{X}}$. In doing so we do not change the second Chern-character of the bundle. In addition, if we choose the connection $\hat{A}'$ on $\hat{V}'$ to be the tensor product connection of the Hermitian-Yang-Mills connection $A_{\hat{U}}$ on $\hat{U}$ and the flat connection $A_{\hat{W}}$ on $\hat{W}$ then the result will still obey the Hermitian Yang-Mills equations (4.27). Thus the resulting bundle $\hat{V}'$ still gives rise to a good heterotic vacuum before considering Chern-Simons contributions to the superpotential.

We would now like to compare the holomorphic Chern-Simons invariants $\mathrm{CS}_{\hat{A}_0}(\hat{A})$ and $\mathrm{CS}_{\hat{A}_0}(\hat{A}')$ for $\hat{A}$ and $\hat{A}'$, relative to the same reference connection $\hat{A}_0$. From (4.9) we know that

$$\mathrm{CS}_{\hat{A}_0}(\hat{A}) = \int \Big(\omega_3(\hat{A}) - \omega_3(\hat{A}_0)\Big) \wedge \hat{\Omega} = \int \Big(2\omega_3(A_{\hat{U}}) - \omega_3(\hat{A}_0)\Big) \wedge \hat{\Omega}. \tag{4.65}$$

Here $\hat{\Omega}$ is the holomorphic $(3,0)$-form on the quotient $\hat{X}$. Of course, appropriate morphisms must be used to ensure that the two connections in the above integral are written with respect to the same trivialization on the underlying real bundle.

On the other hand, for the connection $\hat{A}'$, we make use of the formula (4.58) which gives

$$\omega_3(\hat{A}') = 2\,\omega_3(A_{\hat{U}}) + 3\,\omega_3(A_{\hat{W}})\ . \tag{4.66}$$

This, in turn, leads to the following relation

$$\mathrm{CS}_{\hat{A}_0}(\hat{A}') = \int \Big(2\,\omega_3(A_{\hat{U}}) + 3\,\omega_3(A_{\hat{W}}) - \omega_3(\hat{A}_0)\Big) \wedge \hat{\Omega} = \mathrm{CS}_{\hat{A}_0}(\hat{A}) + 3\,\mathrm{CS}_0(A_{\hat{W}})\ , \tag{4.67}$$

where $\mathrm{CS}_0(A_{\hat{W}})$ denotes the holomorphic Chern-Simons invariant with a trivial reference connection, which is always available on flat bundles. Thus, under our assumption that $\mathrm{CS}_{\hat{A}_0}(A_{\hat{V}})$ is non-fractional, a flat bundle $\hat{W}$ with a non-integer value for $3\,\mathrm{CS}_0(A_{\hat{W}})$, leads to a fractional Chern-Simons invariant for $\hat{A}'$. The study of Chern-Simons invariants of flat bundles of this type is more advanced in the heterotic literature than for their non-flat cousins [152, 153, 154]. The above construction allows us to use these methods for flat bundles in order to construct non-flat bundles with fractional Chern-Simons invariants.

In order to show that $3\mathrm{CS}_0(A_{\hat{W}})$ can be non-integer, it is enough to find a special Lagrangian three-cycle (SLAG) $\mathcal{C}$ such that the ordinary Chern-Simons invariant $3\,\mathrm{OCS}_0(A_{\hat{W}}, \mathcal{C})$ is non-integer. In the following we follow the analysis and notation of Ref. [154]. It turns out, the Calabi-Yau three-fold X described by the configuration matrix (4.60) admits an A-type SLAG which can be written as the configuration

$$\mathcal{C} \in \left[\begin{array}{c|cccc} \mathbb{RP}^1 & 0 & 1 & 1 & 0 \\ \mathbb{RP}^1 & 0 & 1 & 1 & 0 \\ \mathbb{RP}^1 & 1 & 0 & 0 & 1 \\ \mathbb{RP}^1 & 1 & 0 & 0 & 1 \\ \mathbb{RP}^3 & 1 & 1 & 1 & 1 \end{array}\right] . \tag{4.68}$$

One can always solve the four equations given here to obtain a single point in $\mathbb{RP}^1 \times \mathbb{RP}^1 \times \mathbb{RP}^1 \times \mathbb{RP}^1$. Perhaps the easiest way to see this is to consider (4.60) as a point fibration over $\mathbb{P}^3$. That point fibration does degenerate, of course, but the degeneracy locus misses the SLAG. Hence, the above configuration is simply a description of the Lens space $\mathbb{RP}^3 = S^3/\mathbb{Z}_2$.

It is clear that the $\mathbb{Z}_4$ symmetry (4.61) and (4.62) leaves the SLAG $\mathcal{C}$ invariant and, in the quotient $\hat{X}$, it turns into the lense space $\hat{\mathcal{C}} = S^3/\mathbb{Z}_8$. Flat bundles are defined uniquely by a map from the fundamental group of the base space to the structure group. As such, we can easily see how a flat bundle defined on the whole Calabi-Yau three-fold restricts to a SLAG simply by looking at how non-trivial one-cycles embedded in the SLAG descend from the ambient manifold.

For a Lens space with fundamental group is $\mathbb{Z}_p$ we can define an $SU(N)$ flat bundle by specifying the images of the map $\mathbb{Z}_p \to SU(N)$ and we denote these by $\mathrm{diag}(e^{2\pi i k_1/p}, \ldots, e^{2\pi i k_N/p})$, where $k_i \in \mathbb{Z}$. Then the general formula for the Chern-Simons invariant of such a flat bundle, using the trivial connection as the reference, is as follows [170, 198, 199, 200]

$$\mathrm{OCS}_0(A_{\hat{W}}, S^3/\mathbb{Z}_p) = -\sum_i \frac{k_i^2}{2p} \mod \mathbb{Z}. \tag{4.69}$$

For the case at hand, we have $p = 8$ and we choose the images of the defining map $\mathbb{Z}_4 \to SU(2)$ as $\mathrm{diag}(e^{2\pi i2/8}, e^{2\pi i6/8})$. This restricts to the Lens space in an obvious manner. The ordinary Chern-Simons invariant evaluated on this Lens space with the restricted flat bundle is then

$$\mathrm{OCS}_0(A_{\hat{W}}, \hat{\mathcal{C}}) = \frac{5}{2} \mod \mathbb{Z} \ . \tag{4.70}$$

As a result $3\,\mathrm{OCS}_0(A_{\hat{W}}, \mathcal{C})$ is non-integer and, hence, from (4.67), either $\mathrm{CS}_{\hat{A}_0}(\hat{A})$ or $\mathrm{CS}_{\hat{A}_0}(\hat{A}')$ is fractional. In conclusions, we have obtained a contribution to a heterotic superpotential from a fractional holomorphic Chern-Simons invariant associated to a non-flat bundle.

There is an important caveat in the above example that should be mentioned. Although it is true that second Chern-characters of $\hat{V}$ and $\hat{V}'$ match at the level of the image of the Chern-Weil homomorphism, it is not clear they match in torsion. This is also required for a viable heterotic vacuum, and indeed for the holomorphic Chern-Simons invariant (4.8) to be well defined. The Brauer group of the manifold $\hat{X}$ is, to our knowledge, unknown and its computation is beyond the scope of this chapter. This is unfortunately, a common situation in heterotic compactifications. Nevertheless we believe the present example exemplifies well the idea of the construction.

4.5 Conclusions and outlook

In this chapter we have computed the Chern-Simons contribution to the heterotic superpotential arising from the interplay between the gauge and the tangent bundles. To do this we have split the superpotential which originates from the NS fields strength H up into two pieces, one from harmonic flux which is integer quantized and the other from the Chern-Simons invariant. The second contribution is potentially fractional and has been the main focus of the present work. Alternative, we might say that the main purpose of this chapter has been to determine the quantization condition for H. From this point of view, bundles with fractional Chern-Simons invariants do not allow for a vanishing H and, hence, lead to a non-zero flux superpotential.

Chern-Simons invariants in the context of heterotic string compactifications have been considered previously, but only in the context of flat (Wilson line) bundles. However, heterotic compactifications on Calabi-Yau manifolds require non-flat gauge bundles and it is, therefore, essential to analyze Chern-Simons invariants for such cases. In the present chapter, we have presented the first analysis of this kind.

We have developed a number of new methods to carry out our computations. Explicit real bundle isomorphisms between line bundle sums on $\mathbb{P}^1$ have been derived and we have shown how these isomorphisms can be used to construct real bundle isomorphisms between line bundle sums on Calabi-Yau manifolds which are defined in ambient spaces with $\mathbb{P}^1$ factors. These isomorphisms, together with holomorphic deformations, can be combined to isomorphisms between the tangent bundle of the Calabi-Yau manifold and the heterotic gauge bundle. This in turn allows for an explicit calculation of the gauge bundle's Chern-Simons invariant, with the tangent bundle as the reference connection.

Further, we have developed methods for calculating Chern-Simons invariants on Calabi-Yau quotient manifolds, that is, on manifolds with a non-trivial first fundamental group, which apply to non-flat bundles. Since realistic heterotic Calabi-Yau compactifications rely on such a quotient constructions for both the manifold and the bundle, these methods are essential for analyzing the superpotential for phenomenologically relevant models.

Using the methods based on real bundle morphisms, we have calculated the holomorphic Chern-Simons invariants for many examples and have always found a vanishing result. Presumably many of these results can be attributed to the vanishing theorem 4.1. A non-zero Chern-Simons invariant causes a large superpotential contribution which, on it own, destabilizes the model, so the frequent vanishing we have found can be considered good news. However, we have also presented an example of a non-zero and indeed fractional Chern-Simons invariant for a non-flat bundle on a quotient Calabi-Yau.

Knowledge of Chern-Simons superpotentials in heterotic theories is a crucial piece of information, particularly in view of vacuum stability and moduli stabilization. In this chapter, we have presented some progress in calculating such superpotentials but much remains to be done for a systematic understanding of heterotic vacua. To generalize our methods to larger classes of models more general real bundle isomorphisms need to be constructed. So far, our approach is based on rank two line bundle sums on $\mathbb{P}^1$. Explicit knowledge of bundle isomorphisms for higher rank line bundle sums on $\mathbb{P}^1$ and for higher-dimensional projective spaces would significantly expand the scope for applications. Other methods to construct real bundle isomorphisms, for example through deformations to exceptional structure groups such as G_2, might also be of interest (or perhaps methods taking a complementary geometric approach [201]). One long term goal of this work is to derive a general quantization rule for H. Such a rule would be a potentially powerful model-building tool, and would allow us to distinguish marginally stable from unstable heterotic models.

Another obvious extension of the present work would be to include cases where 5-branes are present in the vacuum. This would require new mathematics in that a suitable generalization of Chern-Simons invariants would have to be formulated. This would certainly be interesting to pursue, and is perhaps a case where physics could guide the discovery of new mathematical structures.

It would also be interesting to study the effects which lead to these Chern-Simons superpotentials in dual theories, such as, for example, F-theory models with heterotic duals [103, 104, 105, 106]. The authors of [146], upon which this chapter is based, are planning to explore some of these directions in future publications.

Chapter 5

Conclusions and Outlook

In this dissertation, after giving a short introduction to the general ideas of heterotic string compactification in Chapter 1 and Chapter 2, we studied two important questions in that subject in Chapter 3 and Chapter 4. In Chapter 3, we studied, by using examples in detailed model building, how the two important physical quantities, the family number and Yukawa couplings, can be influenced by subtle effects relating to the mathematics of Calabi-Yau manifolds. In Chapter 4, by developing a method to calculate the real bundle morphisms, we investigated how the Chern-Simons term in heterotic theories can contribute to the superpotential and thus influence model building. In this chapter, we will give a short summary of the last two chapters and point out some possible interesting future directions of research.

In Chapter 3, we first introduced one way to construct standard models in heterotic theory, the heterotic line bundle model construction [49, 50]. Then after this we reviewed how to use a bundle in the hidden sector to help stabilize the complex structure in heterotic models, a mechanism that was introduced in [103, 104]. After this, by comprehensively scanning over all the models constructed in [49, 50], we concluded that building models and stabilizing complex structure moduli should be considered at the same time, due to spectrum jumping caused by moduli stabilization which occurs in a non-negligible number of cases. Even though the models and moduli stabilization we used here are just one of all the possible constructions, this issue should be considered in more general contexts. The reason for this is that the issue here, that different bundle cohomologies, those related to particle spectrum and those related to moduli stabilization, frequently jump in dimension on the same loci in moduli space, can be expected to often generalize beyond the examples considered.

In the second half of Chapter 3, we first reviewed a new way to calculate Yukawa couplings by using differential geometry, which was first introduced in [88, 89]. We then used this method to calculate all the possible Yukawa couplings in models constructed in [49, 50]. The conclusion is that, a large portion of the Yukawa couplings which were formerly thought non-zero actually vanish. This effect could be either good or bad for model building. Either way, it is very important to take care of it, and the rich structure to which it gives rise, in considerations of low energy particle physics. We don't know yet if there is any symmetry constraining this kind of vanishing and uncovering such would be an interesting future research direction. The idea we want to convey from the two pieces of work in this chapter is that detailed aspects of the structure associated to the Calabi-Yau compactification can have an important influence on the study of heterotic string model building.

Compared to Chapter 3's research which is more directly related to detailed model building, in Chapter 4 we studied a more formal question related to Chern-Simons terms in heterotic compactification. The investigation of Chern-Simons term contributions to the heterotic superpotential is crucial since it is related to both vacuum stability and moduli stabilization. In the existing literature [152, 153, 154, 155], there has been work in this direction by studying the Chern-Simons superpotential of Wilson lines (flat bundles). In Chapter 4 we investigated this question in the context of non-flat vector bundles. Of key importance in this investigation is finding a real bundle morphism between the tangent bundle of the Calabi-Yau and the gauge bundle. Based on the real bundle morphism of line bundle sums on $\mathbb{P}^1$, in certain cases, if the ambient space of the Calabi-Yau manifold includes $\mathbb{P}^1$ factors, the aforementioned real bundle morphism could be realized by finding a bundle morphism of line bundle sums on the Calabi-Yau followed by holomorphic vector bundle deformation. By studying examples of this kind, we found vanishing holomorphic Chern-Simons superpotentials, which were consistent with the vanishing theorem 4.1. However, in the latter part of Chapter 4, we showed that on quotients of CICYs, which are widely used in model building, the holomorphic Chern-Simons superpotential could actually be non-zero and indeed fractional.

There are many possible future directions of research that stem from the work presented in this dissertation. The spectrum jumping results and vanishing of Yukawa couplings presented in Chapter 3 all took place in a heterotic context. Given recent work which shows that essentially all known heterotic models should have an F-theory dual [122, 207, 208, 209, 210, 211], it would be interesting to study the analogous effects in that setting. It is possible that such a study could provide more of an insight as to why these effects are occurring. In addition, the physical mechanism which would give rise to this behavior is certainly not completely understood in the F-theory context. In addition, it would be interesting to try to use machine learning technology to study jumping spectra and Yukawa coupling vanishings in heterotic models. To get an overview of string model building it is often necessary to scan over large data sets, and our work here is no exception. Machine Learning techniques could aid in this process greatly, and indeed some related, albeit somewhat different research, already exists (see for example [31, 32, 33, 34, 212, 213, 214]).

In terms of the material presented in Chapter 4, one important future direction could be the study of real morphisms between stable bundles over Calabi-Yau manifolds. In the material of Chapter 4, we made a first attempt in this direction by using real bundle morphisms between line bundle sums on $\mathbb{P}^1$. To generalize the real bundle morphism over $\mathbb{P}^1$ to more general spaces would be helpful in future studies. Another important direction is to include the contribution from 5-branes in the analysis of that chapter. This would require a substantial generalization of the mathematical techniques involved, but at the same time would be more useful both for realistic model building and in inspiring constructions within mathematics.

Appendices

Appendix A

Jumping Spectrum Results

In this appendix we present data on the interplay between $\Sigma_{\rm SM}$ and $\Sigma_{\rm H}$, as defined in Section 3.3, for all of the Line Bundle Standard Models of [49, 50, 139] whose spectra are determined by cohomologies that could potentially jump in dimension. In particular, all cases where a non-trivial map is involved in the sequence chasing used to determine the spectrum are considered. In the tables below, 'CICY No.', 'Symmetry No.' and 'Model No.' refer to the labels for the upstairs manifolds, symmetries and Line Bundle Standard Models that are being considered, relative to the relevant standard lists, [118, 138], [129, 138] and [49, 50, 139] respectively. The entries in the column entitled 'Jump Line' specify the multiplet being considered in that row and the line bundle whose cohomology it is associated to. Finally, the columns 'Jump Standard' and 'Jump Extension' contain information about $\Sigma_{\rm SM}$ and $\Sigma_{\rm H}$ respectively.

For cases where no possible extension bundle of the form (3.21) exists, we place a 'no extension' in the final column and perform no further computations. If the jumping locus for the standard model bundle only jumps on loci in complex structure moduli space where the associated Calabi-Yau manifold becomes singular we place a 'singular' in the penultimate column (or singular* if only a portion of this locus could be determined and that portion exhibited this property). In such cases, there is no need to perform any computations involving the hidden sector bundles and, as such, a 'null' is placed in the final column. If, for a standard model which can indeed jump (indicated by a 'y' in the 'Jump Standard' column) there exists a hidden sector bundle for which we have been able to find an irreducible component to its jumping locus that lies entirely within Σ_{SM} then we put a 'y' in the final column. If all such loci we have been able to find merely intersect the standard model bundle jumping locus we place a 'g' in the final column. A 'singular' in the last column indicates that all of the components of Σ_{H} that exist force the Calabi-Yau manifold to a singular locus in its moduli space. A singular* in the final column means that all of the components of Σ_{H} that we were able to find have this property, but other loci may exist. An 'unknown' in any column simply means that the system was so complicated that we were unable to extract any meaningful data in a reasonable amount of time.

CICY No.	Symmetry No.	Model No.	Jump Line	Jump Standard	Jump Extension
6784	3-6	1-5	$\mathbf{10}_{\mathbf{e}_1},\ \mathcal{O}(3,2,-2,-1)$	singular	null
			$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_3},\ \mathcal{O}(2,2,-1,-1)$	singular	null
			$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_4},\ \mathcal{O}(2,2,-1,-1)$	singular	null
		6	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_5},\ \mathcal{O}(-1,2,2,-1)$	y	y
		7-10	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,2,-3,-1)$	singular	null
		11-50	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,2,-1,-1)$	singular	null
		51	$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-3,2,2,-1)$	y	y
		52	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,2,-1,-1)$	singular	null
		54	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_5},\ \overline{\mathbf{5}}_{\mathbf{e}_2,\mathbf{e}_5},\ \mathcal{O}(-1,2,2,-1)$	y	y
6828	2	1	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,-3,2,-1)$	y	y
		2-5	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,-1,2,-1)$	y	y
		6	$\mathbf{10}_{\mathbf{e}_1},\ \mathcal{O}(2,-2,3,-1)$	singular	null
		7	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,2,-1,-1)$	y	y

CICY No.	Symmetry No.	Model No.	Jump Line	Jump Standard	Jump Extension
7435	2	1	$\mathbf{10}_{\mathbf{e}_5},\ \mathcal{O}(-2,-2,0,1)$	y	y
			$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(4,2,-2,-1)$	y	y
			$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_5},\ \mathcal{O}(-3,-2,1,1)$	y	y
			$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-3,-2,1,1)$	y	y
		2	$\mathbf{10}_{\mathbf{e}_5},\ \mathcal{O}(-2,0,-2,1)$	y	y
			$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(4,-2,2,-1)$	y	y
			$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_5},\ \mathcal{O}(-3,1,-2,1)$	y	y
			$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-3,1,-2,1)$	y	y
		3	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_5},\ \mathcal{O}(-2,4,2,-1)$	y	y
			$\overline{\mathbf{5}}_{\mathbf{e}_2,\mathbf{e}_4},\ \mathcal{O}(1,-3,-2,1)$	y	y
			$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_4},\ \mathcal{O}(1,-3,-2,1)$	y	y
		4	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_5},\ \mathcal{O}(-2,2,4,-1)$	y	y
			$\overline{\mathbf{5}}_{\mathbf{e}_2,\mathbf{e}_4},\ \mathcal{O}(1,-2,-3,1)$	y	y
			$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_4},\ \mathcal{O}(1,-2,-3,1)$	y	y
		5	$\mathbf{10}_{\mathbf{e}_5},\ \mathcal{O}(-2,-2,0,1)$	y	y
			$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,4,-2,-1)$	y	y
			$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_5},\ \mathcal{O}(-2,-3,1,1)$	y	y
			$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-2,-3,1,1)$	y	y
		6	$\mathbf{10}_{\mathbf{e}_5},\ \mathcal{O}(-2,0,-2,1)$	y	y
			$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,-2,4,-1)$	y	y
			$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_5},\ \mathcal{O}(-2,1,-3,1)$	y	y
			$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-2,1,-3,1)$	y	y
7862	3	2	$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-2,3,2,-3)$	y	g
		3-6	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,-2,-2,2)$	y	g
		9-12	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,-2,-2,2)$	y	g
		15-18	$\mathbf{10}_{\mathbf{e}_5},\ \mathcal{O}(-2,2,-2,2)$	y	g

CICY No.	Symmetry No.	Model No.	Jump Line	Jump Standard	Jump Extension
5256	3-6	1-4	$\mathbf{10}_{\mathbf{e}_2},\ \mathcal{O}(0,1,-2,-2,1)$	singular	null
		7-10	$\mathbf{10}_{\mathbf{e}_3},\ \mathcal{O}(0,1,-2,-2,1)$	singular	null
			$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,-1,0,2,-1)$	singular	null
		20	$\mathbf{10}_{\mathbf{e}_5},\ \mathcal{O}(-2,-2,0,1,1)$	singular	null
			$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_4},\ \mathcal{O}(0,2,2,-1,-1)$	y	singular
		21	$\mathbf{10}_{\mathbf{e}_5},\ \mathcal{O}(-2,-2,1,0,1)$	singular	null
		22	$\mathbf{10}_{\mathbf{e}_5},\ \mathcal{O}(-2,-2,1,0,1)$	singular	null
			$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,0,-1,2,-1)$	y	g
		23	$\mathbf{10}_{\mathbf{e}_5},\ \mathcal{O}(-2,-2,0,1,1)$	singular	null
		24	$\mathbf{10}_{\mathbf{e}_5},\ \mathcal{O}(-2,-2,1,0,1)$	singular	null
		26	$\mathbf{10}_{\mathbf{e}_5},\ \mathcal{O}(-2,-2,1,0,1)$	singular	null
5452	7-22	1	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,2,0,0,-2)$	singular	null
			$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-3,0,0,0,1)$	singular	null
			$\mathbf{10}_{\mathbf{e}_3},\ \mathcal{O}(1,-2,0,0,1)$	singular	null
		2	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,2,0,0,-2)$	singular	null
			$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_5},\ \mathcal{O}(-1,-2,-1,2,2)$	singular	null
			$\mathbf{10}_{\mathbf{e}_5},\ \mathcal{O}(-2,0,0,1,1)$	singular	null
		3-6	$\mathbf{10}_{\mathbf{e}_5},\ \mathcal{O}(-2,0,-2,1,1)$	singular	null
		7	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,2,0,0,-2)$	singular	null
			$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-2,-2,1,2,1)$	singular	null
			$\mathbf{10}_{\mathbf{e}_5},\ \mathcal{O}(-2,0,0,1,1)$	singular	null
		8	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,2,0,0,-2)$	singular	null
			$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-2,-2,2,1,1)$	singular	null
			$\mathbf{10}_{\mathbf{e}_4},\ \mathcal{O}(0,-2,1,0,1)$	singular	null
		9-12	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,2,0,-1,-1)$	singular	null
			$\mathbf{10}_{\mathbf{e}_5},\ \mathcal{O}(-2,0,-2,1,1)$	singular	null

CICY No.	Symmetry No.	Model No.	Jump Line	Jump Standard	Jump Extension
5452	7-22	13	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,2,0,0,-2)$	singular	null
			$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_4},\ \mathcal{O}(0,-3,0,0,1)$	singular	null
			$\mathbf{10}_{\mathbf{e}_5},\ \mathcal{O}(-2,1,0,0,1)$	singular	null
		14	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,2,0,0,-2)$	singular	null
			$\mathbf{10}_{\mathbf{e}_4},\ \mathcal{O}(0,-2,1,0,1)$	singular	null
		18-25	$\mathbf{10}_{\mathbf{e}_5},\ \mathcal{O}(-2,0,-2,1,1)$	singular	null
		30-33	$\mathbf{10}_{\mathbf{e}_5},\ \mathcal{O}(-2,0,-2,1,1)$	singular	null
		39-42	$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_4},\ \mathcal{O}(0,2,2,-1,-1)$	singular	null
			$\mathbf{10}_{\mathbf{e}_5},\ \mathcal{O}(-2,0,-2,1,1)$	singular	null
		43-46	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,2,-1,0,-1)$	singular	null
			$\mathbf{10}_{\mathbf{e}_3},\ \mathcal{O}(0,-2,1,-2,1)$	singular	null
		47-50	$\mathbf{10}_{\mathbf{e}_3},\ \mathcal{O}(0,-2,1,-2,1)$	singular	null
		52-55	$\mathbf{10}_{\mathbf{e}_3},\ \mathcal{O}(0,-2,1,-2,1)$	singular	null
		58-61	$\mathbf{10}_{\mathbf{e}_3},\ \mathcal{O}(0,-2,1,-2,1)$	singular	null
			$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,0,-1,2,-1)$	singular	null
		63-66	$\mathbf{10}_{\mathbf{e}_4},\ \mathcal{O}(0,-2,1,-2,1)$	singular	null
		67-70	$\mathbf{10}_{\mathbf{e}_4},\ \mathcal{O}(0,-2,1,-2,1)$	singular	null
6732	1-2	1	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_4},\ \mathcal{O}(0,2,2,-1,-1)$	y	y
		2	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_4},\ \mathcal{O}(0,2,2,-1,-1)$	y	y
		3-4	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,0,2,-1,-1)$	y	y
			$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-3,1,-2,1,1)$	singular*	null
		5-8	$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-2,-2,2,-2,2)$	unknown	unknown
		9	$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-2,-2,0,1,1)$	singular	null
		10-13	$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-2,-2,1,0,1)$	singular	null
		15-17	$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_5},\ \mathcal{O}(-2,0,-2,1,1)$	y	y
		19	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_5},\ \mathcal{O}(-1,2,0,2,-1)$	y	y
			$\overline{\mathbf{5}}_{\mathbf{e}_2,\mathbf{e}_4},\ \mathcal{O}(1,-2,1,-3,1)$	y	y

CICY No.	Symmetry No.	Model No.	Jump Line	Jump Standard	Jump Extension
6732	1-2	20	$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-2,-2,0,1,1)$	singular	null
		21-24	$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-2,-2,1,0,1)$	singular	null
		26-28	$\overline{\mathbf{5}}_{\mathbf{e}_2,\mathbf{e}_5},\ \mathcal{O}(0,-2,-2,1,1)$	y	y
		30-31	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,0,2,-1,-1)$	y	y
		32	$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-2,0,-2,2,1)$	y	y
		33	$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-2,1,1,-3,1)$	y	y
			$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,-1,0,2,-1)$	y	y
		34	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_4},\ \mathcal{O}(0,2,2,-1,-1)$	y	y
			$\overline{\mathbf{5}}_{\mathbf{e}_2,\mathbf{e}_3},\ \mathcal{O}(1,-3,-2,1,1)$	singular*	null
		35	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_4},\ \mathcal{O}(0,2,2,-1,-1)$	y	y
			$\overline{\mathbf{5}}_{\mathbf{e}_2,\mathbf{e}_3},\ \mathcal{O}(1,-3,-2,1,1)$	singular*	null
		36	$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_4},\ \mathcal{O}(0,-2,-2,2,1)$	y	y
6770	1-2	13	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(1,1,-2,-2,0)$	y	y
		14	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(1,1,-2,1,-2)$	y	y
6890	1-2	1-2	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_4},\ \mathcal{O}(0,2,2,-1,-1)$	y	y
		4	$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_5},\ \mathcal{O}(-2,0,-2,1,1)$	singular	null
		5	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_3},\ \mathcal{O}(1,1,-2,-3,1)$	y	y
			$\overline{\mathbf{5}}_{\mathbf{e}_2,\mathbf{e}_5},\ \mathcal{O}(-1,0,2,2,-1)$	singular	null
		6-9	$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_5},\ \mathcal{O}(-2,2,-2,-2,2)$	unknown	unknown
		10-13	$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_5},\ \mathcal{O}(-2,1,-2,0,1)$	singular	null
		16-17	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_4},\ \mathcal{O}(0,2,2,-1,-1)$	y	y
			$\overline{\mathbf{5}}_{\mathbf{e}_2,\mathbf{e}_3},\ \mathcal{O}(1,-2,-3,1,1)$	singular*	null
		18-19	$\overline{\mathbf{5}}_{\mathbf{e}_2,\mathbf{e}_5},\ \mathcal{O}(0,-2,-2,1,1)$	y	y
		20-21	$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-2,1,1,-3,1)$	y	y
		22	$\overline{\mathbf{5}}_{\mathbf{e}_2,\mathbf{e}_5},\ \mathcal{O}(0,-2,-2,1,1)$	y	y
		24-27	$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-2,1,-2,0,1)$	singular	null
			$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_4},\ \mathcal{O}(0,-2,-2,2,1)$	y	y

CICY No.	Symmetry No.	Model No.	Jump Line	Jump Standard	Jump Extension
6777	1-4	1-4	$\overline{\mathbf{5}}_{\mathbf{e}_2,\mathbf{e}_3},\ \mathcal{O}(2,-2,-2,-2,2)$	unknown	unknown
		5-12	$\overline{\mathbf{5}}_{\mathbf{e}_2,\mathbf{e}_4},\ \mathcal{O}(1,-2,-2,0,1)$	singular	null
		16	$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_4},\ \mathcal{O}(0,-2,-2,1,1)$	singular	null
		17	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_3},\ \mathcal{O}(1,1,-2,-3,1)$	y	y
		19	$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_4},\ \mathcal{O}(0,-2,-2,1,1)$	singular	null
		20	$\overline{\mathbf{5}}_{\mathbf{e}_2,\mathbf{e}_3},\ \mathcal{O}(0,2,-1,2,-1)$	y	y
7447	2	3	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_3},\ \mathcal{O}(1,-2,1,-2,2)$	y	singular*
7487	3-6	11-20	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_3},\ \mathcal{O}(1,-2,1,-2,2)$	singular	null
		22	$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_5},\ \mathcal{O}(-2,2,1,1,-2)$	y	singular*
		23	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,-2,-2,1,1)$		no extension
			$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_5},\ \mathcal{O}(-2,2,1,-2,1)$		no extension
			$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-2,1,2,1,-2)$		no extension
		24	$\overline{\mathbf{5}}_{\mathbf{e}_4,\mathbf{e}_5},\ \mathcal{O}(-2,1,1,2,-2)$	y	singular*
		26	$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_5},\ \mathcal{O}(-2,2,1,1,-2)$		no extension
		27	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,-2,-2,1,1)$	y	singular*
		28	$\overline{\mathbf{5}}_{\mathbf{e}_1,\mathbf{e}_2},\ \mathcal{O}(2,-2,-2,1,1)$	y	singular*
		36-39	$\overline{\mathbf{5}}_{\mathbf{e}_2,\mathbf{e}_4},\ \mathcal{O}(1,-2,2,1,-2)$	singular	null
		61	$\overline{\mathbf{5}}_{\mathbf{e}_3,\mathbf{e}_5},\ \mathcal{O}(-2,2,1,1,-2)$	y	singular*
		72-81	$\overline{\mathbf{5}}_{\mathbf{e}_2,\mathbf{e}_3},\ \mathcal{O}(2,-2,-2,1,1)$	y	singular*

Appendix B

Vanishing Coupling Results

In this appendix we give in detail, for every heterotic Line Bundle Standard Model in the data set of [49, 50, 139], which couplings vanish due to the topological considerations discussed in Section 3.4. In these tables, 'CICY No.', 'Sym. No.' and 'Model No.' refer to the labels for the upstairs manifolds, symmetries and Line Bundle Standard Models that are being considered, relative to the relevant standard lists, [118, 138], [129, 138] and [49, 50, 139] respectively. The column 'Yukawa Pattern' lists the couplings that are consistent with the obvious symmetries of these models in each case. Finally the column 'Top. Van.' details whether these couplings are affected by the topological vanishing condition (3.56) of [88, 89].

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
6784	1	$\mathbf{10_{e_1}\bar{5}_{e_2,e_5}\bar{5}_{e_3,e_4}}$	n	1,3
	2	$\mathbf{10_{e_1}\bar{5}_{e_2,e_5}\bar{5}_{e_3,e_4}}$	n	1-4
	3	$\mathbf{10_{e_1}\bar{5}_{e_2,e_5}\bar{5}_{e_3,e_4}}$	n	1-4
	4	$\mathbf{10_{e_1}\bar{5}_{e_2,e_5}\bar{5}_{e_3,e_4}}$	n	1-4
	5	$\mathbf{10_{e_1}\bar{5}_{e_2,e_5}\bar{5}_{e_3,e_4}}$	n	2,4
	6	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_4}\bar{5}_{e_1,e_5}\bar{5}_{e_2,e_3}}$	n	1-4
	7	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	1-4
	8	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	1-4
	9	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	1-4

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
6784	10	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	1-4
	11	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	12	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	13	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	14	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	15	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	16	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	17	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	18	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	19	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	20	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	2, 4
	21	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1, 3
	22	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	23	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	24	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	25	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	26	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	27	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	28	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	29	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	30	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	2, 4
	31	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	2, 4
	32	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	33	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	34	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
6784	35	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	36	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	37	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	38	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	39	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	40	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1,3
	41	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	2,4
	42	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	43	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	44	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	45	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	46	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	47	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	48	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	49	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	50	$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1,3
	51	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	1-4
	52	$\mathbf{10_{e_1}\bar{5}_{e_2,e_5}\bar{5}_{e_3,e_4}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	1-4

On CICY 6784, a total of 188 models have Yukawa couplings consistent with the gauge symmetries of the models, with 264 Yukawa couplings being permitted in total. There are no allowed couplings of the form $\mathbf{5_{-e_i,-e_j}10_{e_i}10_{e_j}}$, 120 of the form $\mathbf{10_{e_i}\bar{5}_{e_j,e_k}\bar{5}_{e_l,e_m}}$, and 144 of the form $\mathbf{1_{e_i,-e_j}5_{-e_i,-e_k}\bar{5}_{e_j,e_k}}$. None of these couplings exhibit the topological vanishings we have studied here.

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
6828	1	$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	2
	7	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	n	2
		$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	2

On CICY 6828, a total of 2 models have Yukawa couplings consistent with the gauge symmetries of the models, with 5 Yukawa couplings being permitted in total. All of these couplings are of the form $\mathbf{10_{e_i}\bar{5}_{e_j,e_k}\bar{5}_{e_l,e_m}}$. None of these couplings exhibit the topological vanishings we have studied here.

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
7862	2	$\mathbf{5_{-e_2,-e_4}10_{e_2}10_{e_4}}$	n	3
	9	$\mathbf{1_{e_1,-e_4}5_{-e_1,-e_5}\bar{5}_{e_4,e_5}}$	n	3
		$\mathbf{1_{e_2,-e_4}5_{-e_2,-e_5}\bar{5}_{e_4,e_5}}$	n	3
		$\mathbf{10_{e_1}\bar{5}_{e_2,e_5}\bar{5}_{e_3,e_4}}$	n	3
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_5}\bar{5}_{e_3,e_4}}$	n	3
	10	$\mathbf{1_{e_1,-e_4}5_{-e_1,-e_5}\bar{5}_{e_4,e_5}}$	n	3
		$\mathbf{1_{e_2,-e_4}5_{-e_2,-e_5}\bar{5}_{e_4,e_5}}$	n	3
		$\mathbf{10_{e_1}\bar{5}_{e_2,e_5}\bar{5}_{e_3,e_4}}$	n	3
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_5}\bar{5}_{e_3,e_4}}$	n	3
	11	$\mathbf{1_{e_1,-e_4}5_{-e_1,-e_5}\bar{5}_{e_4,e_5}}$	n	3
		$\mathbf{1_{e_2,-e_4}5_{-e_2,-e_5}\bar{5}_{e_4,e_5}}$	n	3
		$\mathbf{10_{e_1}\bar{5}_{e_2,e_5}\bar{5}_{e_3,e_4}}$	n	3
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_5}\bar{5}_{e_3,e_4}}$	n	3

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
7862	12	$\mathbf{1}_{\mathbf{e_1,-e_4}}\mathbf{5}_{\mathbf{-e_1,-e_5}}\bar{\mathbf{5}}_{\mathbf{e_4,e_5}}$	n	3
		$\mathbf{1}_{\mathbf{e_2,-e_4}}\mathbf{5}_{\mathbf{-e_2,-e_5}}\bar{\mathbf{5}}_{\mathbf{e_4,e_5}}$	n	3
		$\mathbf{10}_{\mathbf{e_1}}\bar{\mathbf{5}}_{\mathbf{e_2,e_5}}\bar{\mathbf{5}}_{\mathbf{e_3,e_4}}$	n	3
		$\mathbf{10}_{\mathbf{e_2}}\bar{\mathbf{5}}_{\mathbf{e_1,e_5}}\bar{\mathbf{5}}_{\mathbf{e_3,e_4}}$	n	3
	13	$\mathbf{1}_{\mathbf{e_1,-e_2}}\mathbf{5}_{\mathbf{-e_1,-e_5}}\bar{\mathbf{5}}_{\mathbf{e_2,e_5}}$	y	3
		$\mathbf{10}_{\mathbf{e_3}}\bar{\mathbf{5}}_{\mathbf{e_1,e_4}}\bar{\mathbf{5}}_{\mathbf{e_2,e_5}}$	y	3
	15	$\mathbf{10}_{\mathbf{e_2}}\bar{\mathbf{5}}_{\mathbf{e_1,e_5}}\bar{\mathbf{5}}_{\mathbf{e_3,e_4}}$	n	3
		$\mathbf{5}_{\mathbf{-e_1,-e_2}}\mathbf{10}_{\mathbf{e_1}}\mathbf{10}_{\mathbf{e_2}}$	y	3
	16	$\mathbf{10}_{\mathbf{e_2}}\bar{\mathbf{5}}_{\mathbf{e_1,e_5}}\bar{\mathbf{5}}_{\mathbf{e_3,e_4}}$	n	3
		$\mathbf{5}_{\mathbf{-e_1,-e_2}}\mathbf{10}_{\mathbf{e_1}}\mathbf{10}_{\mathbf{e_2}}$	y	3
	17	$\mathbf{10}_{\mathbf{e_2}}\bar{\mathbf{5}}_{\mathbf{e_1,e_5}}\bar{\mathbf{5}}_{\mathbf{e_3,e_4}}$	n	3
		$\mathbf{5}_{\mathbf{-e_1,-e_2}}\mathbf{10}_{\mathbf{e_1}}\mathbf{10}_{\mathbf{e_2}}$	y	3
	18	$\mathbf{10}_{\mathbf{e_2}}\bar{\mathbf{5}}_{\mathbf{e_1,e_5}}\bar{\mathbf{5}}_{\mathbf{e_3,e_4}}$	n	3
		$\mathbf{5}_{\mathbf{-e_1,-e_2}}\mathbf{10}_{\mathbf{e_1}}\mathbf{10}_{\mathbf{e_2}}$	y	3
	19	$\mathbf{10}_{\mathbf{e_3}}\bar{\mathbf{5}}_{\mathbf{e_1,e_4}}\bar{\mathbf{5}}_{\mathbf{e_2,e_5}}$	y	3
		$\mathbf{10}_{\mathbf{e_5}}\bar{\mathbf{5}}_{\mathbf{e_1,e_3}}\bar{\mathbf{5}}_{\mathbf{e_2,e_4}}$	n	3
		$\mathbf{5}_{\mathbf{-e_3,-e_5}}\mathbf{10}_{\mathbf{e_3}}\mathbf{10}_{\mathbf{e_5}}$	n	3
	20	$\mathbf{10}_{\mathbf{e_3}}\bar{\mathbf{5}}_{\mathbf{e_1,e_4}}\bar{\mathbf{5}}_{\mathbf{e_2,e_5}}$	y	3
		$\mathbf{10}_{\mathbf{e_5}}\bar{\mathbf{5}}_{\mathbf{e_1,e_3}}\bar{\mathbf{5}}_{\mathbf{e_2,e_4}}$	n	3
		$\mathbf{5}_{\mathbf{-e_3,-e_5}}\mathbf{10}_{\mathbf{e_3}}\mathbf{10}_{\mathbf{e_5}}$	n	3
	21	$\mathbf{10}_{\mathbf{e_3}}\bar{\mathbf{5}}_{\mathbf{e_1,e_4}}\bar{\mathbf{5}}_{\mathbf{e_2,e_5}}$	y	3
		$\mathbf{10}_{\mathbf{e_5}}\bar{\mathbf{5}}_{\mathbf{e_1,e_3}}\bar{\mathbf{5}}_{\mathbf{e_2,e_4}}$	n	3
		$\mathbf{5}_{\mathbf{-e_3,-e_5}}\mathbf{10}_{\mathbf{e_3}}\mathbf{10}_{\mathbf{e_5}}$	n	3
	22	$\mathbf{10}_{\mathbf{e_3}}\bar{\mathbf{5}}_{\mathbf{e_1,e_4}}\bar{\mathbf{5}}_{\mathbf{e_2,e_5}}$	y	3
		$\mathbf{10}_{\mathbf{e_5}}\bar{\mathbf{5}}_{\mathbf{e_1,e_3}}\bar{\mathbf{5}}_{\mathbf{e_2,e_4}}$	n	3
		$\mathbf{5}_{\mathbf{-e_3,-e_5}}\mathbf{10}_{\mathbf{e_3}}\mathbf{10}_{\mathbf{e_5}}$	n	3

On CICY 7862, a total of 14 models have Yukawa couplings consistent with the gauge symmetries of the models, with 90 Yukawa couplings being permitted in total. Of these, 53 are of the form $\mathbf{10_{e_i}\bar{5}_{e_j,e_k}\bar{5}_{e_l,e_m}}$, 19 are of the form $\mathbf{1_{e_i,-e_j}5_{-e_i,-e_k}\bar{5}_{e_j,e_k}}$ and 18 are of the form $\mathbf{5_{-e_i,-e_j}10_{e_i}10_{e_j}}$. A total of 16 couplings exhibit the topological vanishing we have been studying in this paper, 3 of the form $\mathbf{1_{e_i,-e_j}5_{-e_i,-e_k}\bar{5}_{e_j,e_k}}$, 5 of the form $\mathbf{10_{e_i}\bar{5}_{e_j,e_k}\bar{5}_{e_l,e_m}}$, and 8 of the form $\mathbf{5_{-e_i,-e_j}10_{e_i}10_{e_j}}$. A total of 9 out of the 14 models have at least one topologically vanishing coupling.

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
5256	2	$\mathbf{1_{e_3,-e_4}5_{-e_3,-e_5}\bar{5}_{e_4,e_5}}$	n	1-2
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_4}\bar{5}_{e_3,e_5}}$	n	1-2
	3	$\mathbf{10_{e_5}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_4}}$	n	1-2
	5	$\mathbf{1_{e_1,-e_2}5_{-e_1,-e_5}\bar{5}_{e_2,e_5}}$	y	3-6
		$\mathbf{10_{e_3}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_5}}$	y	3-6
	6	$\mathbf{1_{e_1,-e_2}5_{-e_1,-e_5}\bar{5}_{e_2,e_5}}$	y	3-6
		$\mathbf{10_{e_3}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_5}}$	y	3-6
	7	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	3-6
	8	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	3-6
	9	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	3-6
	10	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	3-6
	11	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	3-6
	12	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	3-6
	13	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	3-6
	14	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	3-6
	15	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	y	3-6

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
5256	16	$\mathbf{10_{e_5}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_4}}$	y	3-6
	17	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	y	3-6
	18	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	y	3-6
	19	$\mathbf{10_{e_5}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_3}}$	y	3-6
	20	$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	3-6
	21	$\mathbf{10_{e_5}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_3}}$	n	3-6
	22	$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	3-6
	23	$\mathbf{10_{e_5}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_4}}$	n	3-6
	25	$\mathbf{10_{e_2}\bar{5}_{e_1,e_4}\bar{5}_{e_3,e_5}}$	y	3-6

On CICY 5256, a total of 84 models have Yukawa couplings consistent with the gauge symmetries of the models, with 158 Yukawa couplings being permitted in total. Of these, 126 are of the form $\mathbf{10_{e_i}\bar{5}_{e_j,e_k}\bar{5}_{e_l,e_m}}$, 32 are of the form $\mathbf{1_{e_i,-e_j}5_{-e_i,-e_k}\bar{5}_{e_j,e_k}}$. A total of 56 couplings exhibit the topological vanishing we have been studying in this paper, 24 of the form $\mathbf{1_{e_i,-e_j}5_{-e_i,-e_k}\bar{5}_{e_j,e_k}}$, 32 of the form $\mathbf{10_{e_i}\bar{5}_{e_j,e_k}\bar{5}_{e_l,e_m}}$. A total of 32 out of the 84 models have at least one topologically vanishing coupling.

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
5452	1	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	7-22
	2	$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	7-22
	7	$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	7-22
	8	$\mathbf{10_{e_4}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_5}}$	n	7-22
	9	$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	7-22
	10	$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	7-22
	11	$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	7-22
	12	$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	7-22

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
	13	$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	7-22
	14	$\mathbf{10_{e_4}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_5}}$	n	7-22
	15	$\mathbf{10_{e_1}\bar{5}_{e_2,e_4}\bar{5}_{e_3,e_5}}$	y	7-22
	16	$\mathbf{10_{e_1}\bar{5}_{e_2,e_4}\bar{5}_{e_3,e_5}}$	y	7-22
	17	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	y	7-22
	18	$\mathbf{10_{e_5}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_4}}$	n	7-22
	19	$\mathbf{10_{e_5}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_4}}$	n	7-22
	20	$\mathbf{10_{e_5}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_4}}$	n	7-22
	21	$\mathbf{10_{e_5}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_4}}$	n	7-22
	22	$\mathbf{10_{e_5}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_4}}$	n	7-22
	23	$\mathbf{10_{e_5}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_4}}$	n	7-22
	24	$\mathbf{10_{e_5}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_4}}$	n	7-22
5452	25	$\mathbf{10_{e_5}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_4}}$	n	7-22
	26	$\mathbf{10_{e_5}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_3}}$	y	7-22
	27	$\mathbf{10_{e_5}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_3}}$	y	7-22
	28	$\mathbf{10_{e_1}\bar{5}_{e_2,e_4}\bar{5}_{e_3,e_5}}$	y	7-22
	29	$\mathbf{10_{e_1}\bar{5}_{e_2,e_4}\bar{5}_{e_3,e_5}}$	y	7-22
	34	$\mathbf{10_{e_5}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_3}}$	y	7-22
	35	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	y	7-22
	36	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	y	7-22
	37	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	y	7-22
	38	$\mathbf{10_{e_5}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_4}}$	y	7-22
	39	$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	7-22
	40	$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	7-22
	41	$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	7-22

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
5452	42	$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	7-22
	43	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	7-22
	44	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	7-22
	45	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	7-22
	46	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	7-22
	47	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	7-22
	48	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	7-22
	49	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	7-22
	50	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	7-22
	51	$\mathbf{1_{e_1,-e_2}5_{-e_1,-e_4}\bar{5}_{e_2,e_4}}$	y	7-22
		$\mathbf{10_{e_3}\bar{5}_{e_1,e_5}\bar{5}_{e_2,e_4}}$	y	7-22
	52	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	7-22
	53	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	7-22
	54	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	7-22
	55	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	7-22
	56	$\mathbf{1_{e_1,-e_2}5_{-e_1,-e_4}\bar{5}_{e_2,e_4}}$	y	7-22
		$\mathbf{10_{e_3}\bar{5}_{e_1,e_5}\bar{5}_{e_2,e_4}}$	y	7-22
	57	$\mathbf{10_{e_3}\bar{5}_{e_1,e_5}\bar{5}_{e_2,e_4}}$	n	7-22
		$\mathbf{1_{e_2,-e_1}5_{-e_2,-e_5}\bar{5}_{e_1,e_5}}$	y	7-22
	58	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	7-22
	59	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	7-22
	60	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	7-22
	61	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	7-22
	62	$\mathbf{1_{e_1,-e_2}5_{-e_1,-e_5}\bar{5}_{e_2,e_5}}$	y	7-22
		$\mathbf{10_{e_3}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_5}}$	y	7-22

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
5452	1	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_3}}$	n	1-4
		$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	2	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_3}}$	n	1-4
		$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	3	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_3}}$	n	1-4
		$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	4	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_3}}$	n	1-4
		$\mathbf{1_{e_4,-e_3}5_{-e_4,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
	6	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_4}\bar{5}_{e_1,e_5}\bar{5}_{e_2,e_3}}$	n	1-4
	7	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_4}\bar{5}_{e_1,e_5}\bar{5}_{e_2,e_3}}$	n	1-4
	8	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_4}\bar{5}_{e_1,e_5}\bar{5}_{e_2,e_3}}$	n	1-4
	9	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_4}\bar{5}_{e_1,e_5}\bar{5}_{e_2,e_3}}$	n	1-4

On CICY 5452, a total of 800 models have Yukawa couplings consistent with the gauge symmetries of the models, with 1584 Yukawa couplings being permitted in total. Of these, 1376 are of the form $\mathbf{10_{e_i}\bar{5}_{e_j,e_k}\bar{5}_{e_l,e_m}}$ and 208 are of the form $\mathbf{1_{e_i,-e_j}5_{-e_i,-e_k}\bar{5}_{e_j,e_k}}$. A total of 432 couplings exhibit the topological vanishing we have been studying in this paper, 192 of the form $\mathbf{1_{e_i,-e_j}5_{-e_i,-e_k}\bar{5}_{e_j,e_k}}$ and 240 of the form $\mathbf{10_{e_i}\bar{5}_{e_j,e_k}\bar{5}_{e_l,e_m}}$. A total of 256 out of the 800 models have at least one topologically vanishing coupling.

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
6947	1	$\mathbf{10_{e_1}\bar{5}_{e_2,e_4}\bar{5}_{e_3,e_5}}$	y	3
	2	$\mathbf{10_{e_1}\bar{5}_{e_2,e_4}\bar{5}_{e_3,e_5}}$	y	3
	3	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	y	3
	4	$\mathbf{10_{e_2}\bar{5}_{e_1,e_4}\bar{5}_{e_3,e_5}}$	y	3
	5	$\mathbf{10_{e_5}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_3}}$	y	3
	6	$\mathbf{10_{e_5}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_3}}$	y	3
	7	$\mathbf{10_{e_1}\bar{5}_{e_2,e_4}\bar{5}_{e_3,e_5}}$	y	3
	8	$\mathbf{10_{e_1}\bar{5}_{e_2,e_4}\bar{5}_{e_3,e_5}}$	y	3
	9	$\mathbf{10_{e_5}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_4}}$	y	3
	10	$\mathbf{10_{e_5}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_4}}$	y	3
	11	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	y	3
	12	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	y	3
	13	$\mathbf{10_{e_2}\bar{5}_{e_1,e_4}\bar{5}_{e_3,e_5}}$	y	3
	14	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	y	3
	15	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	y	3
	16	$\mathbf{10_{e_5}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_4}}$	y	3
	17	$\mathbf{10_{e_5}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_4}}$	y	3
	18	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	y	3
	19	$\mathbf{1_{e_1,-e_2}5_{-e_1,-e_4}\bar{5}_{e_2,e_4}}$	y	3
		$\mathbf{10_{e_3}\bar{5}_{e_1,e_5}\bar{5}_{e_2,e_4}}$	y	3
	20	$\mathbf{1_{e_1,-e_2}5_{-e_1,-e_4}\bar{5}_{e_2,e_4}}$	y	3
		$\mathbf{10_{e_3}\bar{5}_{e_1,e_5}\bar{5}_{e_2,e_4}}$	y	3

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
6947	21	$\mathbf{10_{e_3}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_5}}$	y	3
	22	$\mathbf{10_{e_3}\bar{5}_{e_1,e_5}\bar{5}_{e_2,e_4}}$	y	3
	23	$\mathbf{1_{e_1,-e_2}5_{-e_1,-e_5}\bar{5}_{e_2,e_5}}$	y	3
		$\mathbf{10_{e_3}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_5}}$	y	3
	24	$\mathbf{1_{e_1,-e_2}5_{-e_1,-e_5}\bar{5}_{e_2,e_5}}$	y	3
		$\mathbf{10_{e_3}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_5}}$	y	3

On CICY 6947, a total of 24 models have Yukawa couplings consistent with the gauge symmetries of the models, with 36 couplings being permitted in total. Of these, 24 are of the form $\mathbf{10_{e_i}\bar{5}_{e_j,e_k}\bar{5}_{e_l,e_m}}$ and 12 are of the form $\mathbf{1_{e_i,-e_j}5_{-e_i,-e_k}\bar{5}_{e_j,e_k}}$. All of these couplings exhibit topological vanishing.

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
6732	1	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-2
	2	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-2
	5	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	1-2
		$\mathbf{10_{e_4}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_5}}$	n	1-2
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	1-2
		$\mathbf{5_{-e_3,-e_4}10_{e_3}10_{e_4}}$	n	1-2
		$\mathbf{5_{-e_3,-e_5}10_{e_3}10_{e_5}}$	n	1-2
	6	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	1-2
		$\mathbf{10_{e_4}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_5}}$	n	1-2
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	1-2
		$\mathbf{5_{-e_3,-e_4}10_{e_3}10_{e_4}}$	n	1-2
		$\mathbf{5_{-e_3,-e_5}10_{e_3}10_{e_5}}$	n	1-2

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
6732	7	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	1-2
		$\mathbf{10_{e_4}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_5}}$	n	1-2
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	1-2
		$\mathbf{5_{-e_3,-e_4}10_{e_3}10_{e_4}}$	n	1-2
		$\mathbf{5_{-e_3,-e_5}10_{e_3}10_{e_5}}$	n	1-2
	8	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	1-2
		$\mathbf{10_{e_4}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_5}}$	n	1-2
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	1-2
		$\mathbf{5_{-e_3,-e_4}10_{e_3}10_{e_4}}$	n	1-2
		$\mathbf{5_{-e_3,-e_5}10_{e_3}10_{e_5}}$	n	1-2
	18	$\mathbf{5_{-e_3,-e_5}10_{e_3}10_{e_5}}$	n	1-2
	29	$\mathbf{5_{-e_2,-e_5}10_{e_2}10_{e_5}}$	n	1-2
	30	$\mathbf{10_{e_3}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_5}}$	n	1-2
	31	$\mathbf{10_{e_3}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_5}}$	n	1-2
	32	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	1-2
	34	$\mathbf{1_{e_1,-e_3}5_{-e_1,-e_5}\bar{5}_{e_3,e_5}}$	n	1-2
	35	$\mathbf{1_{e_1,-e_3}5_{-e_1,-e_5}\bar{5}_{e_3,e_5}}$	n	1-2
	36	$\mathbf{10_{e_2}\bar{5}_{e_1,e_5}\bar{5}_{e_3,e_4}}$	n	1-2

On CICY 6732, a total of 28 models have Yukawa couplings consistent with the gauge symmetries of the models, with 104 couplings being permitted in total. Of these, 68 are of the form $\mathbf{10_{e_i}\bar{5}_{e_j,e_k}\bar{5}_{e_l,e_m}}$, 12 are of the form $\mathbf{1_{e_i,-e_j}5_{-e_i,-e_k}\bar{5}_{e_j,e_k}}$ and 24 are of the form $\mathbf{5_{-e_i,-e_j}10_{e_i}10_{e_j}}$. None of these couplings exhibit the topological vanishings we have studied here.

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
6770	1	$\mathbf{5_{-e_1,-e_2}10_{e_1}10_{e_2}}$	y	1-2
	2	$\mathbf{5_{-e_1,-e_2}10_{e_1}10_{e_2}}$	y	1-2
	5	$\mathbf{5_{-e_1,-e_2}10_{e_1}10_{e_2}}$	y	1-2
	6	$\mathbf{5_{-e_1,-e_2}10_{e_1}10_{e_2}}$	y	1-2
	7	$\mathbf{10_{e_2}\bar{5}_{e_1,e_4}\bar{5}_{e_3,e_5}}$	n	1-2
	8	$\mathbf{10_{e_2}\bar{5}_{e_1,e_4}\bar{5}_{e_3,e_5}}$	n	1-2
	11	$\mathbf{10_{e_1}\bar{5}_{e_2,e_5}\bar{5}_{e_3,e_4}}$	n	1-2
	12	$\mathbf{10_{e_1}\bar{5}_{e_2,e_5}\bar{5}_{e_3,e_4}}$	n	1-2

On CICY 6770, a total of 16 models have Yukawa couplings consistent with the gauge symmetries of the models, with 48 couplings being permitted in total. Of these, 32 are of the form $\mathbf{10_{e_i}\bar{5}_{e_j,e_k}\bar{5}_{e_l,e_m}}$ and 16 are of the form $\mathbf{5_{-e_i,-e_j}10_{e_i}10_{e_j}}$. There are a total of 16 couplings that exhibit the topological vanishing we have been studying here. All of these are of the form $\mathbf{5_{-e_i,-e_j}10_{e_i}10_{e_j}}$. A total of 8 models have at least one coupling which exhibits this topological vanishing.

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
6777	1	$\mathbf{1_{e_2,-e_3}5_{-e_2,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
		$\mathbf{1_{e_3,-e_2}5_{-e_3,-e_5}\bar{5}_{e_2,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_4}\bar{5}_{e_3,e_5}}$	n	1-4
		$\mathbf{10_{e_3}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_5}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_3}}$	n	1-4
		$\mathbf{5_{-e_2,-e_5}10_{e_2}10_{e_5}}$	n	1-4
		$\mathbf{5_{-e_3,-e_5}10_{e_3}10_{e_5}}$	n	1-4

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
6777	2	$\mathbf{1_{e_2,-e_3}5_{-e_2,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
		$\mathbf{1_{e_3,-e_2}5_{-e_3,-e_5}\bar{5}_{e_2,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_4}\bar{5}_{e_3,e_5}}$	n	1-4
		$\mathbf{10_{e_3}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_5}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_3}}$	n	1-4
		$\mathbf{5_{-e_2,-e_5}10_{e_2}10_{e_5}}$	n	1-4
		$\mathbf{5_{-e_3,-e_5}10_{e_3}10_{e_5}}$	n	1-4
	3	$\mathbf{1_{e_2,-e_3}5_{-e_2,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
		$\mathbf{1_{e_3,-e_2}5_{-e_3,-e_5}\bar{5}_{e_2,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_4}\bar{5}_{e_3,e_5}}$	n	1-4
		$\mathbf{10_{e_3}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_5}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_3}}$	n	1-4
		$\mathbf{5_{-e_2,-e_5}10_{e_2}10_{e_5}}$	n	1-4
		$\mathbf{5_{-e_3,-e_5}10_{e_3}10_{e_5}}$	n	1-4
	4	$\mathbf{1_{e_2,-e_3}5_{-e_2,-e_5}\bar{5}_{e_3,e_5}}$	n	1-4
		$\mathbf{1_{e_3,-e_2}5_{-e_3,-e_5}\bar{5}_{e_2,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_4}\bar{5}_{e_3,e_5}}$	n	1-4
		$\mathbf{10_{e_3}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_5}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_3}}$	n	1-4
		$\mathbf{5_{-e_2,-e_5}10_{e_2}10_{e_5}}$	n	1-4
		$\mathbf{5_{-e_3,-e_5}10_{e_3}10_{e_5}}$	n	1-4
	13	$\mathbf{1_{e_2,-e_4}5_{-e_2,-e_5}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{5_{-e_2,-e_5}10_{e_2}10_{e_5}}$	n	1-4
	14	$\mathbf{1_{e_2,-e_4}5_{-e_2,-e_5}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{5_{-e_2,-e_5}10_{e_2}10_{e_5}}$	n	1-4

On CICY 6777, a total of 24 models have Yukawa couplings consistent with the gauge symmetries of the models, with 192 Yukawa couplings being permitted in total. Of these, 64 are of the form $\mathbf{10_{e_i}\bar{5}_{e_j,e_k}\bar{5}_{e_l,e_m}}$, 80 are of the form $\mathbf{1_{e_i,-e_j}5_{-e_i,-e_k}\bar{5}_{e_j,e_k}}$ and 48 are of the form $\mathbf{5_{-e_i,-e_j}10_{e_i}10_{e_j}}$. None of these couplings exhibit the topological vanishings we have studied here.

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
6890	1	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	1-2
	2	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	1-2
	6	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	1-2
		$\mathbf{10_{e_4}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_5}}$	n	1-2
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	1-2
		$\mathbf{5_{-e_3,-e_4}10_{e_3}10_{e_4}}$	n	1-2
		$\mathbf{5_{-e_4,-e_5}10_{e_4}10_{e_5}}$	n	1-2
	7	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	1-2
		$\mathbf{10_{e_4}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_5}}$	n	1-2
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	1-2
		$\mathbf{5_{-e_3,-e_4}10_{e_3}10_{e_4}}$	n	1-2
		$\mathbf{5_{-e_4,-e_5}10_{e_4}10_{e_5}}$	n	1-2
	8	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	1-2
		$\mathbf{10_{e_4}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_5}}$	n	1-2
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	1-2
		$\mathbf{5_{-e_3,-e_4}10_{e_3}10_{e_4}}$	n	1-2
		$\mathbf{5_{-e_4,-e_5}10_{e_4}10_{e_5}}$	n	1-2

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
6890	9	$\mathbf{10_{e_3}\bar{5}_{e_1,e_2}\bar{5}_{e_4,e_5}}$	n	1-2
		$\mathbf{10_{e_4}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_5}}$	n	1-2
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	1-2
		$\mathbf{5_{-e_3,-e_4}10_{e_3}10_{e_4}}$	n	1-2
		$\mathbf{5_{-e_4,-e_5}10_{e_4}10_{e_5}}$	n	1-2
	14	$\mathbf{5_{-e_4,-e_5}10_{e_4}10_{e_5}}$	n	1-2
	16	$\mathbf{1_{e_1,-e_3}5_{-e_1,-e_5}\bar{5}_{e_3,e_5}}$	n	1-2
	17	$\mathbf{1_{e_1,-e_3}5_{-e_1,-e_5}\bar{5}_{e_3,e_5}}$	n	1-2
	23	$\mathbf{5_{-e_2,-e_4}10_{e_2}10_{e_4}}$	n	1-2
	28	$\mathbf{10_{e_2}\bar{5}_{e_1,e_5}\bar{5}_{e_3,e_4}}$	n	1-2

On CICY 6890, a total of 22 models have Yukawa couplings consistent with the gauge symmetries of the models, with 86 Yukawa couplings being permitted in total. Of these, 50 are of the form $\mathbf{10_{e_i}\bar{5}_{e_j,e_k}\bar{5}_{e_l,e_m}}$, 12 are of the form $\mathbf{1_{e_i,-e_j}5_{-e_i,-e_k}\bar{5}_{e_j,e_k}}$ and 24 are of the form $\mathbf{5_{-e_i,-e_j}10_{e_i}10_{e_j}}$. None of these couplings exhibit the topological vanishings we have studied here.

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
7447	1	$\mathbf{1_{e_1,-e_2}5_{-e_1,-e_5}\bar{5}_{e_2,e_5}}$	y	2
		$\mathbf{10_{e_3}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_5}}$	y	2
	2	$\mathbf{10_{e_2}\bar{5}_{e_1,e_4}\bar{5}_{e_3,e_5}}$	n	2
		$\mathbf{1_{e_3,-e_1}5_{-e_3,-e_4}\bar{5}_{e_1,e_4}}$	n	2
	3	$\mathbf{10_{e_1}\bar{5}_{e_2,e_5}\bar{5}_{e_3,e_4}}$	n	2
	4	$\mathbf{10_{e_1}\bar{5}_{e_2,e_5}\bar{5}_{e_3,e_4}}$	n	2
		$\mathbf{10_{e_4}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_5}}$	n	2

On CICY 7447, a total of 4 models have Yukawa couplings consistent with the gauge symmetries of the models, with 9 couplings being permitted in total. Of these, 5 are of the form $\mathbf{10_{e_i}\bar{5}_{e_j,e_k}\bar{5}_{e_l,e_m}}$ and 4 are of the form $\mathbf{1_{e_i,-e_j}5_{-e_i,-e_k}\bar{5}_{e_j,e_k}}$. There are a total of 4 couplings that exhibit the topological vanishing we have been studying here, 3 of the form $\mathbf{1_{e_i,-e_j}5_{-e_i,-e_k}\bar{5}_{e_j,e_k}}$ and 1 of the form $\mathbf{10_{e_i}\bar{5}_{e_j,e_k}\bar{5}_{e_l,e_m}}$. All of the topologically vanishing couplings occur in 1 model.

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
7487	1	$\mathbf{1_{e_2,-e_3}5_{-e_2,-e_4}\bar{5}_{e_3,e_4}}$	n	1,3
		$\mathbf{1_{e_3,-e_1}5_{-e_3,-e_4}\bar{5}_{e_1,e_4}}$	n	1-4
		$\mathbf{1_{e_3,-e_2}5_{-e_3,-e_4}\bar{5}_{e_2,e_4}}$	n	1,3
		$\mathbf{10_{e_1}\bar{5}_{e_2,e_4}\bar{5}_{e_3,e_5}}$	n	1,3
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_4}\bar{5}_{e_3,e_5}}$	n	1-4
		$\mathbf{5_{-e_2,-e_3}10_{e_2}10_{e_3}}$	y	1,3
	2	$\mathbf{1_{e_2,-e_3}5_{-e_2,-e_4}\bar{5}_{e_3,e_4}}$	n	1-4
		$\mathbf{1_{e_3,-e_1}5_{-e_3,-e_4}\bar{5}_{e_1,e_4}}$	n	1-4
		$\mathbf{1_{e_3,-e_2}5_{-e_3,-e_4}\bar{5}_{e_2,e_4}}$	n	1-4
		$\mathbf{10_{e_1}\bar{5}_{e_2,e_4}\bar{5}_{e_3,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_4}\bar{5}_{e_3,e_5}}$	n	1-4
		$\mathbf{5_{-e_2,-e_3}10_{e_2}10_{e_3}}$	y	1-4
	3	$\mathbf{1_{e_2,-e_3}5_{-e_2,-e_4}\bar{5}_{e_3,e_4}}$	n	1-4
		$\mathbf{1_{e_3,-e_1}5_{-e_3,-e_4}\bar{5}_{e_1,e_4}}$	n	1-4
		$\mathbf{1_{e_3,-e_2}5_{-e_3,-e_4}\bar{5}_{e_2,e_4}}$	n	1-4
		$\mathbf{10_{e_1}\bar{5}_{e_2,e_4}\bar{5}_{e_3,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_4}\bar{5}_{e_3,e_5}}$	n	1-4
		$\mathbf{5_{-e_2,-e_3}10_{e_2}10_{e_3}}$	y	1-4

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
7487	4	$\mathbf{1}_{e_2,-e_3}\mathbf{5}_{-e_2,-e_4}\bar{\mathbf{5}}_{e_3,e_4}$	n	1-4
		$\mathbf{1}_{e_3,-e_1}\mathbf{5}_{-e_3,-e_4}\bar{\mathbf{5}}_{e_1,e_4}$	n	1-4
		$\mathbf{1}_{e_3,-e_2}\mathbf{5}_{-e_3,-e_4}\bar{\mathbf{5}}_{e_2,e_4}$	n	1-4
		$\mathbf{10}_{e_1}\bar{\mathbf{5}}_{e_2,e_4}\bar{\mathbf{5}}_{e_3,e_5}$	n	1-4
		$\mathbf{10}_{e_2}\bar{\mathbf{5}}_{e_1,e_4}\bar{\mathbf{5}}_{e_3,e_5}$	n	1-4
		$\mathbf{5}_{-e_2,-e_3}\mathbf{10}_{e_2}\mathbf{10}_{e_3}$	y	1-4
	5	$\mathbf{1}_{e_2,-e_3}\mathbf{5}_{-e_2,-e_4}\bar{\mathbf{5}}_{e_3,e_4}$	n	1-4
		$\mathbf{1}_{e_3,-e_1}\mathbf{5}_{-e_3,-e_4}\bar{\mathbf{5}}_{e_1,e_4}$	n	1-4
		$\mathbf{1}_{e_3,-e_2}\mathbf{5}_{-e_3,-e_4}\bar{\mathbf{5}}_{e_2,e_4}$	n	1-4
		$\mathbf{10}_{e_1}\bar{\mathbf{5}}_{e_2,e_4}\bar{\mathbf{5}}_{e_3,e_5}$	n	1-4
		$\mathbf{10}_{e_2}\bar{\mathbf{5}}_{e_1,e_4}\bar{\mathbf{5}}_{e_3,e_5}$	n	1-4
		$\mathbf{5}_{-e_2,-e_3}\mathbf{10}_{e_2}\mathbf{10}_{e_3}$	y	1-4
	6	$\mathbf{1}_{e_2,-e_3}\mathbf{5}_{-e_2,-e_4}\bar{\mathbf{5}}_{e_3,e_4}$	n	1-4
		$\mathbf{1}_{e_3,-e_1}\mathbf{5}_{-e_3,-e_4}\bar{\mathbf{5}}_{e_1,e_4}$	n	1-4
		$\mathbf{1}_{e_3,-e_2}\mathbf{5}_{-e_3,-e_4}\bar{\mathbf{5}}_{e_2,e_4}$	n	1-4
		$\mathbf{10}_{e_1}\bar{\mathbf{5}}_{e_2,e_4}\bar{\mathbf{5}}_{e_3,e_5}$	n	1-4
		$\mathbf{10}_{e_2}\bar{\mathbf{5}}_{e_1,e_4}\bar{\mathbf{5}}_{e_3,e_5}$	n	1-4
		$\mathbf{5}_{-e_2,-e_3}\mathbf{10}_{e_2}\mathbf{10}_{e_3}$	y	1-4
	7	$\mathbf{1}_{e_2,-e_3}\mathbf{5}_{-e_2,-e_4}\bar{\mathbf{5}}_{e_3,e_4}$	n	1-4
		$\mathbf{1}_{e_3,-e_1}\mathbf{5}_{-e_3,-e_4}\bar{\mathbf{5}}_{e_1,e_4}$	n	1-4
		$\mathbf{1}_{e_3,-e_2}\mathbf{5}_{-e_3,-e_4}\bar{\mathbf{5}}_{e_2,e_4}$	n	1-4
		$\mathbf{10}_{e_1}\bar{\mathbf{5}}_{e_2,e_4}\bar{\mathbf{5}}_{e_3,e_5}$	n	1-4
		$\mathbf{10}_{e_2}\bar{\mathbf{5}}_{e_1,e_4}\bar{\mathbf{5}}_{e_3,e_5}$	n	1-4
		$\mathbf{5}_{-e_2,-e_3}\mathbf{10}_{e_2}\mathbf{10}_{e_3}$	y	1-4

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
7487	8	$\mathbf{1_{e_2,-e_3}5_{-e_2,-e_4}\bar{5}_{e_3,e_4}}$	n	1-4
		$\mathbf{1_{e_3,-e_1}5_{-e_3,-e_4}\bar{5}_{e_1,e_4}}$	n	1-4
		$\mathbf{1_{e_3,-e_2}5_{-e_3,-e_4}\bar{5}_{e_2,e_4}}$	n	1-4
		$\mathbf{10_{e_1}\bar{5}_{e_2,e_4}\bar{5}_{e_3,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_4}\bar{5}_{e_3,e_5}}$	n	1-4
		$\mathbf{5_{-e_2,-e_3}10_{e_2}10_{e_3}}$	y	1-4
	9	$\mathbf{1_{e_2,-e_3}5_{-e_2,-e_4}\bar{5}_{e_3,e_4}}$	n	1-4
		$\mathbf{1_{e_3,-e_1}5_{-e_3,-e_4}\bar{5}_{e_1,e_4}}$	n	1-4
		$\mathbf{1_{e_3,-e_2}5_{-e_3,-e_4}\bar{5}_{e_2,e_4}}$	n	1-4
		$\mathbf{10_{e_1}\bar{5}_{e_2,e_4}\bar{5}_{e_3,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_4}\bar{5}_{e_3,e_5}}$	n	1-4
		$\mathbf{5_{-e_2,-e_3}10_{e_2}10_{e_3}}$	y	1-4
	10	$\mathbf{1_{e_2,-e_3}5_{-e_2,-e_4}\bar{5}_{e_3,e_4}}$	n	2,4
		$\mathbf{1_{e_3,-e_1}5_{-e_3,-e_4}\bar{5}_{e_1,e_4}}$	n	1-4
		$\mathbf{1_{e_3,-e_2}5_{-e_3,-e_4}\bar{5}_{e_2,e_4}}$	n	2,4
		$\mathbf{10_{e_1}\bar{5}_{e_2,e_4}\bar{5}_{e_3,e_5}}$	n	2,4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_4}\bar{5}_{e_3,e_5}}$	n	1-4
		$\mathbf{5_{-e_2,-e_3}10_{e_2}10_{e_3}}$	y	2,4
	11	$\mathbf{1_{e_2,-e_3}5_{-e_2,-e_4}\bar{5}_{e_3,e_4}}$	n	1,3
		$\mathbf{1_{e_3,-e_2}5_{-e_3,-e_4}\bar{5}_{e_2,e_4}}$	n	1,3
		$\mathbf{10_{e_1}\bar{5}_{e_2,e_5}\bar{5}_{e_3,e_4}}$	n	1-4
		$\mathbf{5_{-e_2,-e_3}10_{e_2}10_{e_3}}$	y	1,3

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
7487	12	$\mathbf{1_{e_2,-e_3}5_{-e_2,-e_4}\bar{5}_{e_3,e_4}}$	n	1-4
		$\mathbf{1_{e_3,-e_2}5_{-e_3,-e_4}\bar{5}_{e_2,e_4}}$	n	1-4
		$\mathbf{10_{e_1}\bar{5}_{e_2,e_5}\bar{5}_{e_3,e_4}}$	n	1-4
		$\mathbf{5_{-e_2,-e_3}10_{e_2}10_{e_3}}$	y	1-4
	13	$\mathbf{1_{e_2,-e_3}5_{-e_2,-e_4}\bar{5}_{e_3,e_4}}$	n	1-4
		$\mathbf{1_{e_3,-e_2}5_{-e_3,-e_4}\bar{5}_{e_2,e_4}}$	n	1-4
		$\mathbf{10_{e_1}\bar{5}_{e_2,e_5}\bar{5}_{e_3,e_4}}$	n	1-4
		$\mathbf{5_{-e_2,-e_3}10_{e_2}10_{e_3}}$	y	1-4
	14	$\mathbf{1_{e_2,-e_3}5_{-e_2,-e_4}\bar{5}_{e_3,e_4}}$	n	1-4
		$\mathbf{1_{e_3,-e_2}5_{-e_3,-e_4}\bar{5}_{e_2,e_4}}$	n	1-4
		$\mathbf{10_{e_1}\bar{5}_{e_2,e_5}\bar{5}_{e_3,e_4}}$	n	1-4
		$\mathbf{5_{-e_2,-e_3}10_{e_2}10_{e_3}}$	y	1-4
	15	$\mathbf{1_{e_2,-e_3}5_{-e_2,-e_4}\bar{5}_{e_3,e_4}}$	n	1-4
		$\mathbf{1_{e_3,-e_2}5_{-e_3,-e_4}\bar{5}_{e_2,e_4}}$	n	1-4
		$\mathbf{10_{e_1}\bar{5}_{e_2,e_5}\bar{5}_{e_3,e_4}}$	n	1-4
		$\mathbf{5_{-e_2,-e_3}10_{e_2}10_{e_3}}$	y	1-4
	16	$\mathbf{1_{e_2,-e_3}5_{-e_2,-e_4}\bar{5}_{e_3,e_4}}$	n	1-4
		$\mathbf{1_{e_3,-e_2}5_{-e_3,-e_4}\bar{5}_{e_2,e_4}}$	n	1-4
		$\mathbf{10_{e_1}\bar{5}_{e_2,e_5}\bar{5}_{e_3,e_4}}$	n	1-4
		$\mathbf{5_{-e_2,-e_3}10_{e_2}10_{e_3}}$	y	1-4
	17	$\mathbf{1_{e_2,-e_3}5_{-e_2,-e_4}\bar{5}_{e_3,e_4}}$	n	1-4
		$\mathbf{1_{e_3,-e_2}5_{-e_3,-e_4}\bar{5}_{e_2,e_4}}$	n	1-4
		$\mathbf{10_{e_1}\bar{5}_{e_2,e_5}\bar{5}_{e_3,e_4}}$	n	1-4
		$\mathbf{5_{-e_2,-e_3}10_{e_2}10_{e_3}}$	y	1-4

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
7487	18	$\mathbf{1}_{\mathbf{e_2},-\mathbf{e_3}}\mathbf{5}_{-\mathbf{e_2},-\mathbf{e_4}}\bar{\mathbf{5}}_{\mathbf{e_3},\mathbf{e_4}}$	n	1-4
		$\mathbf{1}_{\mathbf{e_3},-\mathbf{e_2}}\mathbf{5}_{-\mathbf{e_3},-\mathbf{e_4}}\bar{\mathbf{5}}_{\mathbf{e_2},\mathbf{e_4}}$	n	1-4
		$\mathbf{10}_{\mathbf{e_1}}\bar{\mathbf{5}}_{\mathbf{e_2},\mathbf{e_5}}\bar{\mathbf{5}}_{\mathbf{e_3},\mathbf{e_4}}$	n	1-4
		$\mathbf{5}_{-\mathbf{e_2},-\mathbf{e_3}}\mathbf{10}_{\mathbf{e_2}}\mathbf{10}_{\mathbf{e_3}}$	y	1-4
	19	$\mathbf{1}_{\mathbf{e_2},-\mathbf{e_3}}\mathbf{5}_{-\mathbf{e_2},-\mathbf{e_4}}\bar{\mathbf{5}}_{\mathbf{e_3},\mathbf{e_4}}$	n	1-4
		$\mathbf{1}_{\mathbf{e_3},-\mathbf{e_2}}\mathbf{5}_{-\mathbf{e_3},-\mathbf{e_4}}\bar{\mathbf{5}}_{\mathbf{e_2},\mathbf{e_4}}$	n	1-4
		$\mathbf{10}_{\mathbf{e_1}}\bar{\mathbf{5}}_{\mathbf{e_2},\mathbf{e_5}}\bar{\mathbf{5}}_{\mathbf{e_3},\mathbf{e_4}}$	n	1-4
		$\mathbf{5}_{-\mathbf{e_2},-\mathbf{e_3}}\mathbf{10}_{\mathbf{e_2}}\mathbf{10}_{\mathbf{e_3}}$	y	1-4
	20	$\mathbf{1}_{\mathbf{e_2},-\mathbf{e_3}}\mathbf{5}_{-\mathbf{e_2},-\mathbf{e_4}}\bar{\mathbf{5}}_{\mathbf{e_3},\mathbf{e_4}}$	n	2,4
		$\mathbf{1}_{\mathbf{e_3},-\mathbf{e_2}}\mathbf{5}_{-\mathbf{e_3},-\mathbf{e_4}}\bar{\mathbf{5}}_{\mathbf{e_2},\mathbf{e_4}}$	n	2,4
		$\mathbf{10}_{\mathbf{e_1}}\bar{\mathbf{5}}_{\mathbf{e_2},\mathbf{e_5}}\bar{\mathbf{5}}_{\mathbf{e_3},\mathbf{e_4}}$	n	1-4
		$\mathbf{5}_{-\mathbf{e_2},-\mathbf{e_3}}\mathbf{10}_{\mathbf{e_2}}\mathbf{10}_{\mathbf{e_3}}$	y	2,4
	21	$\mathbf{1}_{\mathbf{e_1},-\mathbf{e_2}}\mathbf{5}_{-\mathbf{e_1},-\mathbf{e_5}}\bar{\mathbf{5}}_{\mathbf{e_2},\mathbf{e_5}}$	n	1-4
		$\mathbf{10}_{\mathbf{e_3}}\bar{\mathbf{5}}_{\mathbf{e_1},\mathbf{e_4}}\bar{\mathbf{5}}_{\mathbf{e_2},\mathbf{e_5}}$	y	1-4
	22	$\mathbf{10}_{\mathbf{e_5}}\bar{\mathbf{5}}_{\mathbf{e_1},\mathbf{e_2}}\bar{\mathbf{5}}_{\mathbf{e_3},\mathbf{e_4}}$	n	1-4
	23	$\mathbf{10}_{\mathbf{e_3}}\bar{\mathbf{5}}_{\mathbf{e_1},\mathbf{e_2}}\bar{\mathbf{5}}_{\mathbf{e_4},\mathbf{e_5}}$	n	1-4
		$\mathbf{10}_{\mathbf{e_4}}\bar{\mathbf{5}}_{\mathbf{e_1},\mathbf{e_2}}\bar{\mathbf{5}}_{\mathbf{e_3},\mathbf{e_5}}$	n	1-4
		$\mathbf{10}_{\mathbf{e_5}}\bar{\mathbf{5}}_{\mathbf{e_1},\mathbf{e_2}}\bar{\mathbf{5}}_{\mathbf{e_3},\mathbf{e_4}}$	n	1-4
		$\mathbf{5}_{-\mathbf{e_3},-\mathbf{e_4}}\mathbf{10}_{\mathbf{e_3}}\mathbf{10}_{\mathbf{e_4}}$	y	1-4
	24	$\mathbf{5}_{-\mathbf{e_3},-\mathbf{e_4}}\mathbf{10}_{\mathbf{e_3}}\mathbf{10}_{\mathbf{e_4}}$	y	1-4
	26	$\mathbf{10}_{\mathbf{e_2}}\bar{\mathbf{5}}_{\mathbf{e_1},\mathbf{e_4}}\bar{\mathbf{5}}_{\mathbf{e_3},\mathbf{e_5}}$	n	1-4
	27	$\mathbf{10}_{\mathbf{e_5}}\bar{\mathbf{5}}_{\mathbf{e_1},\mathbf{e_2}}\bar{\mathbf{5}}_{\mathbf{e_3},\mathbf{e_4}}$	n	1-4
		$\mathbf{10}_{\mathbf{e_5}}\bar{\mathbf{5}}_{\mathbf{e_1},\mathbf{e_3}}\bar{\mathbf{5}}_{\mathbf{e_2},\mathbf{e_4}}$	n	1-4

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
7487	28	$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_4}}$	y	1-4
	29	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_5}\bar{5}_{e_3,e_4}}$	n	1,3
	30	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_5}\bar{5}_{e_3,e_4}}$	n	1-4
	31	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_5}\bar{5}_{e_3,e_4}}$	n	1-4
	32	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_5}\bar{5}_{e_3,e_4}}$	n	1-4
	33	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_5}\bar{5}_{e_3,e_4}}$	n	2,4
	34	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	y	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_5}\bar{5}_{e_3,e_4}}$	n	1-4
	35	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	y	1-4
	36	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_4}}$	n	1-4
	37	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_4}}$	n	1-4
	38	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_4}}$	n	1-4
	39	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_4}}$	n	1-4
	40	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
7487	41	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
	42	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
	43	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
	44	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
	45	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
	46	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
	47	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
	48	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
	49	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
	50	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
	51	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
	52	$\mathbf{10_{e_1}\bar{5}_{e_2,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
7487	53	$\mathbf{10}_{e_1}\bar{\mathbf{5}}_{e_2,e_3}\bar{\mathbf{5}}_{e_4,e_5}$	n	1-4
		$\mathbf{10}_{e_2}\bar{\mathbf{5}}_{e_1,e_3}\bar{\mathbf{5}}_{e_4,e_5}$	n	1-4
	54	$\mathbf{10}_{e_1}\bar{\mathbf{5}}_{e_2,e_3}\bar{\mathbf{5}}_{e_4,e_5}$	n	1-4
		$\mathbf{10}_{e_2}\bar{\mathbf{5}}_{e_1,e_3}\bar{\mathbf{5}}_{e_4,e_5}$	n	1-4
	55	$\mathbf{10}_{e_1}\bar{\mathbf{5}}_{e_2,e_3}\bar{\mathbf{5}}_{e_4,e_5}$	n	1-4
		$\mathbf{10}_{e_2}\bar{\mathbf{5}}_{e_1,e_3}\bar{\mathbf{5}}_{e_4,e_5}$	n	1-4
	56	$\mathbf{10}_{e_1}\bar{\mathbf{5}}_{e_2,e_3}\bar{\mathbf{5}}_{e_4,e_5}$	n	1-4
		$\mathbf{10}_{e_2}\bar{\mathbf{5}}_{e_1,e_3}\bar{\mathbf{5}}_{e_4,e_5}$	n	1-4
	57	$\mathbf{10}_{e_1}\bar{\mathbf{5}}_{e_2,e_3}\bar{\mathbf{5}}_{e_4,e_5}$	n	1-4
		$\mathbf{10}_{e_2}\bar{\mathbf{5}}_{e_1,e_3}\bar{\mathbf{5}}_{e_4,e_5}$	n	1-4
	58	$\mathbf{10}_{e_1}\bar{\mathbf{5}}_{e_2,e_3}\bar{\mathbf{5}}_{e_4,e_5}$	n	1-4
		$\mathbf{10}_{e_2}\bar{\mathbf{5}}_{e_1,e_3}\bar{\mathbf{5}}_{e_4,e_5}$	n	1-4
	59	$\mathbf{10}_{e_1}\bar{\mathbf{5}}_{e_2,e_3}\bar{\mathbf{5}}_{e_4,e_5}$	n	1-4
		$\mathbf{10}_{e_2}\bar{\mathbf{5}}_{e_1,e_3}\bar{\mathbf{5}}_{e_4,e_5}$	n	1-4
	60	$\mathbf{10}_{e_5}\bar{\mathbf{5}}_{e_1,e_3}\bar{\mathbf{5}}_{e_2,e_4}$	y	1-4
	61	$\mathbf{10}_{e_2}\bar{\mathbf{5}}_{e_1,e_3}\bar{\mathbf{5}}_{e_4,e_5}$	y	1-4
	62	$\mathbf{10}_{e_5}\bar{\mathbf{5}}_{e_1,e_3}\bar{\mathbf{5}}_{e_2,e_4}$	n	1-4
	63	$\mathbf{10}_{e_5}\bar{\mathbf{5}}_{e_1,e_3}\bar{\mathbf{5}}_{e_2,e_4}$	y	1-4
	64	$\mathbf{10}_{e_2}\bar{\mathbf{5}}_{e_1,e_5}\bar{\mathbf{5}}_{e_3,e_4}$	n	1,3
		$\mathbf{10}_{e_3}\bar{\mathbf{5}}_{e_1,e_5}\bar{\mathbf{5}}_{e_2,e_4}$	n	1-4
		$\mathbf{10}_{e_5}\bar{\mathbf{5}}_{e_1,e_2}\bar{\mathbf{5}}_{e_3,e_4}$	n	1,3
		$\mathbf{10}_{e_5}\bar{\mathbf{5}}_{e_1,e_3}\bar{\mathbf{5}}_{e_2,e_4}$	n	1-4
	65	$\mathbf{10}_{e_2}\bar{\mathbf{5}}_{e_1,e_5}\bar{\mathbf{5}}_{e_3,e_4}$	n	1-4
		$\mathbf{10}_{e_3}\bar{\mathbf{5}}_{e_1,e_5}\bar{\mathbf{5}}_{e_2,e_4}$	n	1-4
		$\mathbf{10}_{e_5}\bar{\mathbf{5}}_{e_1,e_2}\bar{\mathbf{5}}_{e_3,e_4}$	n	1-4
		$\mathbf{10}_{e_5}\bar{\mathbf{5}}_{e_1,e_3}\bar{\mathbf{5}}_{e_2,e_4}$	n	1-4

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
7487	66	$\mathbf{10_{e_2}\bar{5}_{e_1,e_5}\bar{5}_{e_3,e_4}}$	n	1-4
		$\mathbf{10_{e_3}\bar{5}_{e_1,e_5}\bar{5}_{e_2,e_4}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_4}}$	n	1-4
	67	$\mathbf{10_{e_2}\bar{5}_{e_1,e_5}\bar{5}_{e_3,e_4}}$	n	1-4
		$\mathbf{10_{e_3}\bar{5}_{e_1,e_5}\bar{5}_{e_2,e_4}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_4}}$	n	1-4
	68	$\mathbf{10_{e_2}\bar{5}_{e_1,e_5}\bar{5}_{e_3,e_4}}$	n	2,4
		$\mathbf{10_{e_3}\bar{5}_{e_1,e_5}\bar{5}_{e_2,e_4}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_2}\bar{5}_{e_3,e_4}}$	n	2,4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_4}}$	n	1-4
	69	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
	70	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	y	1-4
	71	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_3}\bar{5}_{e_2,e_4}}$	n	1-4
	72	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_3}}$	n	1,3
	73	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_3}}$	n	1-4
	74	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_3}}$	n	1-4
	75	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_3}}$	n	1-4
	76	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_3}}$	n	1-4

CICY No.	Model No.	Yukawa Pattern	Top. Van.	Sym. No.
7487	77	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_3}}$	n	1-4
	78	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_3}}$	n	1-4
	79	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_3}}$	n	1-4
	80	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_3}}$	n	1-4
	81	$\mathbf{10_{e_2}\bar{5}_{e_1,e_3}\bar{5}_{e_4,e_5}}$	n	1-4
		$\mathbf{10_{e_5}\bar{5}_{e_1,e_4}\bar{5}_{e_2,e_3}}$	n	2,4

On CICY 7487, a total of 276 models have Yukawa couplings consistent with the gauge symmetries of the models, with 1188 Yukawa couplings being permitted in total. Of these, 580 are of the form $\mathbf{10_{e_i}\bar{5}_{e_j,e_k}\bar{5}_{e_l,e_m}}$, 444 are of the form $\mathbf{1_{e_i,-e_j}5_{-e_i,-e_k}\bar{5}_{e_j,e_k}}$ and 164 are of the form $\mathbf{5_{-e_i,-e_j}10_{e_i}10_{e_j}}$. A total of 112 couplings exhibit the topological vanishing we have been studying in this paper, 32 of the form $\mathbf{10_{e_i}\bar{5}_{e_j,e_k}\bar{5}_{e_l,e_m}}$ and 80 of the form $\mathbf{5_{-e_i,-e_j}10_{e_i}10_{e_j}}$. A total of 112 out of the 276 models have at least one topologically vanishing coupling.

Appendix C

Real bundle morphisms

In this appendix, we review some standard mathematics concerning bundles and their morphisms and we construct explicitly the real bundle isomorphisms between rank two line bundle sums on $\mathbb{P}^1$ which are used in the main part of the chapter.

We start by recalling some general facts about bundle morphisms. Suppose we have two bundles

$$V \xrightarrow{\pi} X\ , \qquad \tilde{V} \xrightarrow{\tilde{\pi}} X\ , \tag{C.1}$$

with typical fiber F over a manifold X with cover U_α and charts $\varphi_\alpha : U_\alpha \to W_\alpha \subset \mathbb{C}^n$. A bundle morphism is a map $f : V \to \tilde{V}$, for which the diagram

$$\begin{array}{ccccc} & V & \xrightarrow{f} & \tilde{V} & \\ \pi & \downarrow & & \downarrow & \tilde{\pi} \\ & X & \xrightarrow{\text{id}} & X & \end{array} \tag{C.2}$$

commutes. We are looking for a practical way to construct such bundle morphisms and to this end we introduce local trivializations and their associated transition functions

$$\begin{array}{rclrcl} \phi_\alpha & : & \pi^{-1}(U_\alpha) \to W_\alpha \times F, & \tilde{\phi}_\alpha & : & \tilde{\pi}^{-1}(U_\alpha) \to W_\alpha \times F, \\ \phi_{\alpha\beta} := \phi_\alpha \circ \phi_\beta^{-1} & : & W_\beta \times F \to W_\alpha \times F, & \tilde{\phi}_{\alpha\beta} := \tilde{\phi}_\alpha \circ \tilde{\phi}_\beta^{-1} & : & W_\beta \times F \to W_\alpha \times F. \end{array} \tag{C.3}$$

Given this set-up, we can define local versions of the bundle morphism f by

$$f_\alpha := \tilde{\phi}_\alpha \circ f \circ \phi_\alpha^{-1} : W_\alpha \times F \to W_\alpha \times F \ , \tag{C.4}$$

and a simple calculation shows that these local morphisms have to satisfy the intertwining rules

$$f_\alpha \circ \phi_{\alpha\beta} = \tilde{\phi}_{\alpha\beta} \circ f_\beta \ , \tag{C.5}$$

on the overlaps $(W_\alpha \cap W_\beta) \times F$. Conversely, any collection of local morphisms f_α which satisfies the conditions (C.5) defines a bundle morphism f. To be more explicit, we introduce coordinates $(z, v) \in W_\beta \times F$ and write the transition functions and local morphisms as

$$\phi_{\alpha\beta}(z, v) = (z, T_{\alpha\beta}(z)v) \ , \qquad \tilde{\phi}_{\alpha\beta}(z, v) = (z, \tilde{T}_{\alpha\beta}(z)v) \ , \qquad f_\beta(z, v) = (z, P_\alpha(z, \bar{z})v) \ , \tag{C.6}$$

where $T_{\alpha\beta}$, $\tilde{T}_{\alpha\beta}$ and P_α are z-dependent matrices which act on the fiber. Using this notation, the intertwining conditions (C.5) translate into the matrix equations

$$P_\alpha T_{\alpha\beta} = \tilde{T}_{\alpha\beta} P_\beta \ . \tag{C.7}$$

These conditions point to a practical way of finding bundle morphisms. Suppose we are given the transition functions $T_{\alpha\beta}$ and $\tilde{T}_{\alpha\beta}$ for the two bundles V and $\tilde{V}$. Then, the task is to find matrices P_α which contain smooth functions on W_α, are invertible for all $z \in W_\alpha$ and satisfy the matrix relations (C.7). These matrices then define a bundle isomorphism $f \sim (P_\alpha) : V \to \tilde{V}$ which establishes the equivalence of the two bundles.

We will now apply this method to find isomorphisms between line bundle sums on $X = \mathbb{P}^1$. The two standard patches on $\mathbb{P}^1$ are denoted by $U_0 \cong \mathbb{C}$ and $U_1 \cong \mathbb{C}$, with affine coordinates $z \in U_0$ and $w = 1/z \in U_1$. Line bundles on $\mathbb{P}^1$ are denoted by $\mathcal{O}_{\mathbb{P}^1}(k)$, as usual.

It is known that two line bundle sums on $\mathbb{P}^1$ with the same rank are (real) isomorphic if their first Chern classes match. Our task is to construct this isomorphism explicitly for the case of rank two line bundle sums

$$V(k,l) := \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(l) \ , \qquad T_{10}^{(k,l)} = \mathrm{diag}(z^{-k}, z^{-l}) \ , \tag{C.8}$$

with transition functions $T_{10}^{(k,l)}$. We start by considering the two bundles $V = V(-p,p)$ and $\tilde{V} = V(0,0)$ where $p > 0$. Evidently, they are both rank two bundles with vanishing first Chern class so they must be a real isomorphism $f^{(p)} \sim (P_0^{(p)}, P_1^{(p)}) : V(-p,p) \to V(0,0)$. To find this isomorphism explicitly, we write down the transition functions

$$T_{10} = T_{10}^{(-p,p)}(z) = \mathrm{diag}(z^p, z^{-p}) \ , \qquad \tilde{T}_{10} = T_{10}^{(0,0)}(z) = \mathrm{diag}(1,1) \ , \tag{C.9}$$

and we try to find non-singular matrices $P_\alpha^{(p)}$ which satisfy the intertwining conditions (C.7). For the present case, we have only two patches so there is only one such condition which reads

$$P_1^{(p)} T_{10}^{(-p,p)} = T_{10}^{(0,0)} P_0^{(p)} \ . \tag{C.10}$$

Here $P_0^{(p)}$ contains smooth functions in $z \in U_0 \cong \mathbb{C}$ and is invertible everywhere in its domain and $P_1^{(p)}$ contains smooth functions in $w \in U_1 \cong \mathbb{C}$ and is also invertible everywhere in its domain. Starting with a guess for $P_1^{(p)}$, (C.10) then determines $P_0^{(p)}$ and this leads to

$$P_1^{(p)}(w,\bar{w}) = \begin{pmatrix} w^p & \frac{1}{1+|w|^{2p}} \\ -1 & \frac{\bar{w}^p}{1+|w|^{2p}} \end{pmatrix} \quad \Longrightarrow \quad P_0^{(p)}(z,\bar{z}) = \begin{pmatrix} 1 & \frac{\bar{z}^p}{1+|z|^{2p}} \\ -z^p & \frac{1}{1+|z|^{2p}} \end{pmatrix} \ . \tag{C.11}$$

Evidently, both matrices are smooth in their respective domain, they are invertible since $\det(P_0^{(p)}) = \det(P_1^{(p)}) = 1$ for all $z, w \in \mathbb{C}$ and they satisfy (C.10) by construction. So, in conclusion, this defines real bundle isomorphisms

$$f^{(p)} \sim \left(P_\alpha^{(p)}\right) : V(-p,p) \xrightarrow{\simeq} V(0,0)\ . \tag{C.12}$$

As mentioned above, it is a well-known fact that these bundles are real isomorphic [202] (see [203] for a discussion in the physics literature). However, their isomorphy is normally established in a somewhat different manner and we are not aware of the explicit real isomorphism being written down in this form in the literature. It is this kind of construction that we will need in the rest of the chapter, hence the above discussion.

The above construction can easily be generalized by twisting up with another line bundle. The transitions function for the bundle $V(a-p,a+p) = V(-p,p) \otimes \mathcal{O}_{\mathbb{P}^1}(a)$ satisfies

$$T_{10}^{(a-p,a+p)} = z^{-a} T_{10}^{(-p,p)}\ . \tag{C.13}$$

Hence, multiplying (C.10) with z^{-a} it follows easily that

$$P_1^{(p)} T_{10}^{(a-p,a+p)} = T_{10}^{(a,a)} P_0^{(p)}\ , \tag{C.14}$$

for the same matrices $P_0^{(p)}$ and $P_1^{(p)}$ as given in (C.11). Hence, we have explicitly constructed the real bundle isomorphisms

$$f^{(p)} \sim \left(P_\alpha^{(p)}\right) : V(a-p,a+p) \xrightarrow{\simeq} V(a,a)\ . \tag{C.15}$$

between two rank two line bundle sums on $\mathbb{P}^1$ with the same *even* first Chern class.

What about the case of two rank two line bundle sums with the same *odd* first Chern class? Define the matrix $D = \mathrm{diag}(1,z)$ and multiply (C.14) with this matrix from the right. This leads to

$$Q_1^{(p)} T_{10}^{(a-p,a+p+1)} = T_{10}^{(a,a+1)} Q_0^{(p)} \; , \tag{C.16}$$

where

$$Q_1^{(p)} = P_1^{(p)} \; , \quad Q_0^{(p)} = D^{-1} P_0^{(p)} D = \begin{pmatrix} 1 & \frac{z\bar{z}^p}{1+|z|^{2p}} \\ -z^{p-1} & \frac{1}{1+|z|^{2p}} \end{pmatrix} \; . \tag{C.17}$$

Note that $Q_0^{(p)}$ and $Q_1^{(p)}$ are still smooth in their respective coordinates and $\det(Q_0^{(p)}) = \det(Q_1^{(p)}) = 1$ for all $z, w \in \mathbb{C}$. This means we have the real bundle isomorphisms

$$g^{(p)} \sim \left(Q_\alpha^{(p)}\right) : V(a-p, a+p+1) \xrightarrow{\simeq} V(a, a+1) \; , \tag{C.18}$$

between two rank two line bundle sums on $\mathbb{P}^1$ with the same odd first Chern.

So far, we have constructed isomorphism to somewhat special bundles of the form $V(a,a)$ or $V(a,a+1)$. This limitation is easily removed by introducing the matrices

$$P_\alpha^{(q,p)}(z,\bar{z}) := P_\alpha^{(q)}(z,\bar{z})^{-1} P_\alpha^{(p)}(z,\bar{z}) \; , \quad Q_\alpha^{(q,p)}(z,\bar{z}) := Q_\alpha^{(q)}(z,\bar{z})^{-1} Q_\alpha^{(p)}(z,\bar{z}) \; . \tag{C.19}$$

Note that these matrices are still well-defined on their respective patches - since we are dealing with $\mathrm{SL}(2,\mathbb{C})$ matrices the inverse does not introduce any singularities. By transitivity, these matrices satisfy

$$P_\alpha^{(q,p)} T_{\alpha\beta}^{(a-p,a+p)} = T_{\alpha\beta}^{(a-q,a+q)} P_\beta^{(q,p)} \; , \qquad Q_\alpha^{(q,p)} T_{\alpha\beta}^{(a-p,a+p+1)} = T_{\alpha\beta}^{(a-q,a+q+1)} Q_\beta^{(q,p)} \tag{C.20}$$

and, hence, they define bundle isomorphisms

$$f^{(q,p)} \sim \left(P_\alpha^{(q,p)}\right) : V(a-p, a+p) \xrightarrow{\simeq} V(a-q, a+q) \tag{C.21}$$

$$g^{(q,p)} \sim \left(Q_\alpha^{(q,p)}\right) : V(a-p, a+p+1) \xrightarrow{\simeq} V(a-q, a+q+1) \tag{C.22}$$

between two arbitrary rank two line bundle sums on $\mathbb{P}^1$ with even and odd first Chern class, respectively.

In conclusion, we have shown explicitly, by writing down the relevant real bundle isomorphisms, the well known fact that the first Chern class really does classify rank two line bundle sums on $\mathbb{P}^1$ as topological bundles. Crucially we have explicit forms for the relevant isomorphisms which will be important to us in the main part of chapter.

Appendix D

Quotients and equivariant structures

We require a small amount of mathematical formalism in order to describe the relationship between holomorphic Chern-Simons invariants on Calabi-Yau three-folds X and their quotients by freely acting symmetries $\hat{X}$.

Calabi-Yau quotients and equivariant bundles: Let us introduce a finite group $\Gamma = \{g_0 = e, g_1, \ldots, g_n\}$, where $n = |\Gamma| - 1$, which acts freely on the Calabi-Yau three-fold X. The quotient by this symmetry is denoted $\hat{X} = X/\Gamma$ and $p : X \to \hat{X}$ is the natural projection to the quotient. We need to define the notion of a Γ-equivariant vector bundle $V \to X$. A bundle is equivariant if there exists bundle morphisms Φ_g, for all $g \in \Gamma$, such that the diagrams

$$\begin{array}{ccc} V & \xrightarrow{\Phi_g} & V \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{g} & X \end{array} \tag{D.1}$$

commute and the bundle morphisms satisfy the group law $\Phi_{gh} = \Phi_g \circ \Phi_h$ for all $g, h \in \Gamma$. Such a Γ-equivariant bundle descends to a bundle $\hat{V} \to \hat{X}$ on the quotient such that $V = p^* \hat{V}$. More constructively, the downstairs bundle can be defined as $\hat{V} = V/\sim$, with the equivalence relation defined by $v' \sim v \Leftrightarrow v' = \Phi_g(v)$ for a $g \in \Gamma$. The downstairs projection can be defined as $\hat{\pi}([v]) := [\pi(v)]$.

We also recall that the bundle morphisms Φ_g can be used to defined maps $\Psi_g : \Gamma(X, V) \to$

$\Gamma(X, V)$ between sections of V by

$$\Psi_g(s) := \Phi_g \circ s \circ g^{-1} \,. \tag{D.2}$$

Interplay of equivariant structure and real bundle morphisms: The possible choices of equivariant structure on holomorphic bundles V on X are in 1-1 correspondence with holomorphic bundles $\hat{V}$ on $\hat{X}$. Because of this there is a compatibility requirement between the possible real isomorphisms f between two bundles V and $\tilde{V}$ and choices of equivariant structure on each, if a real bundle isomorphism is to descend to the quotient. If two equivariant structures are to give rise to real isomorphic bundles on $\hat{X}$ then there should exist *some* real bundle isomorphism f for the choices of equivariant structure Φ_g and $\tilde{\Phi}_g$ such that the following diagram commutes for all $g \in \Gamma$:

$$\begin{array}{cccc} & V & \xrightarrow{f} & \tilde{V} & \\ \Phi_g & \downarrow & & \downarrow & \tilde{\Phi}_g \\ & V & \xrightarrow{f} & \tilde{V} & \end{array} \,. \tag{D.3}$$

Note that any given real isomorphism $f : V \to \tilde{V}$ may not satisfy this condition. The requirement, if the two bundles on $\hat{X}$ are to be real isomorphic, is simply that there exists some bundle morphism that does.

Pullback of a connection: In the context of this dissertation we need to go beyond the bundles themselves and consider also connections on them. In order to consider how maps such as Φ_g or f induce a mapping on connections we will need to recall how pullbacks of such objects can be defined. Consider an invertable differentiable map $f : X \to \hat{X}$ between two manifolds X, $\hat{X}$ and a vector bundle $\hat{V} \to \hat{X}$ with connection $\hat{\nabla}$. Then a connection

$f^*\hat{\nabla}$ on the pullback bundle $V = f^*\hat{V} \to X$ can be defined locally as follows [204]

$$\nabla_i^{(0)} := f^*\hat{\nabla}|_{U_i} := d + f^*\hat{A}_i \ . \tag{D.4}$$

Here, $\hat{A}_i$ is the gauge field on $\hat{U}_i$ relative to the frames $\hat{s}_{i,a}$ on the cover $\hat{U}_i \subset \hat{X}$ and $U_i = f^{-1}(\hat{U}_i)$. There are natural frames on $U_i \subset X$ defined by $s_{i,a} = \hat{s}_{i,a} \circ f$, and we can glue the local connections $\nabla_i^{(0)}$ together to a global connection $\nabla^{(0)}$ relative to these frames. To see how this works we first note that gauge transformations

$$A_g := g^{-1}Ag + g^{-1}dg \ , \tag{D.5}$$

and pull-backs f^*A commute, that is,

$$f^*A_g = (f^*A)_{f^*g} \ . \tag{D.6}$$

This means, if the gauge fields $\hat{A}_j$ and $\hat{A}_i$ on $\hat{U}_j$ and $\hat{U}_i$ are related by the gauge transformation $\hat{A}_j = \hat{A}_{i,\hat{g}_{(ij)}}$, then

$$f^*\hat{A}_j = (f^*\hat{A}_i)_{g_{(ij)}} \ , \tag{D.7}$$

so the pulled-back gauge fields $f^*\hat{A}_j$ and $f^*\hat{A}_i$ are glued together by the pull-backs $g_{(ij)} = f^*\hat{g}_{(ij)}$ of the original gauge transformations.

The above is natural but is not the most general way we can define the pullback. Suppose that, instead of (D.4), we define local connections ∇_i on U_i by

$$\nabla_i = d + A_i \ , \qquad f^*\hat{A}_i = A_{i,P_i} = P_i^{-1}A_iP_i + P_i^{-1}dP_i \ , \tag{D.8}$$

where P_i are gauge transformations which can be chosen and the second equation defines what we mean by A_i. These local connections glue together to a global connection ∇ by virtue of

$$A_j = A_{i, P_i g_{(ij)} P_j^{-1}} \tag{D.9}$$

(which follows immediately by combining (D.7) and (D.8)), so the glueing gauge transformations in this case are given by $P_i g_{(ij)} P_j^{-1}$. Of course the so-defined gauge field A_i depends on the choice of the local gauge transformations P_i. These different gauge fields describe the same pullback connection for different choices of local trivializations.

Equivariant connections: We will say that a connection ∇ on $V \to X$ is Γ-equivariant[1] (with respect to the equivariant structure on V defined by bundle morphisms Φ_g) iff

$$\nabla(\Psi_g(s)) = \Psi'_g(\nabla(s)) \ , \tag{D.10}$$

for all $g \in \Gamma$. Here, s is a section of V and the maps Ψ_g have been defined in (D.2). The prime on the right-hand-side of (D.10) indicates that an action of the induced equivariant structure on the co-tangent bundle should be included (since $\nabla(s)$ is a one-form).

What do the gauge fields associated to such an equivariant connection look like? To see this consider an open set U_0 which is sufficiently small that no two points inside it are mapped in to each other under the symmetry action Γ. We define the open sets $U_i = g_i(U_0)$ where $i = 0, 1, \ldots, n$ which are the image of U_0 under the elements of the finite group. On each such patch U_i we choose a frame $s_{i,a}$. This choice does not necessarily have to be aligned with the "natural" choice $\Psi_{g_i}(s_{0,a})$ (which is swept out by the equivariant structure once a frame $s_{0,a}$ on U_0 has been fixed). Let us parametrize the difference between those two frames by

$$\Psi_{g_i}(s_{0,a}) = P^b_{g_i,a} s_{i,b} \ . \tag{D.11}$$

[1] See [205] for related definitions in a different geometric context.

It follows from the group law for Ψ_g that the P_g are matrices which must satisfy $P_{gh} = P_g P_h$. Consider a situation where we have sets of such matrices P_g, one set for all possible choices of initial open set U_0 within an open cover composed of such objects. Then, if we fix the frames $s_{i,a}$ once and for all, the sets of matrices P_g encode the choice of equivariant structure on V. Call A_i the gauge field for ∇ on U_i and relative to the frame $s_{i,a}$, that is,

$$\nabla(s_{i,a}) = A^c_{ia} s_{i,c} \ . \tag{D.12}$$

Then, a short calculation shows that the equivariance condition (D.10) translates to the conditions

$$(g_i^{-1})^* A_0 = P_i^{-1} A_i P_i + P_i^{-1} dP_i = A_{i,P_i} \ . \tag{D.13}$$

on the local gauge fields. This should be compared with (D.8). In the special case when all $P_i = \mathbb{I}$ (which corresponds to a particular choice of equivariant structure), these conditions simplify to

$$(g_i^{-1})^* A_0 = A_i \ . \tag{D.14}$$

In short, for two points on X related by the symmetry, the corresponding gauge fields relate by a pull-back combined with a gauge transformation. This is certainly an intuitively reasonable definition for a gauge field we would expect to descend to a quotient. We discuss this further next.

For later discussion it will be useful to note that it is obvious given the above definitions that any globally defined gauge transformation, $t : X \to G$ where G is the gauge group, acting on an equivariant connection gives rise to another equivariant connection.

Descent of connections: Suppose we have a (local) section s of V which is invariant, that is, $\Psi_g(s) = s$ and which descends to a section $\tilde{p}(s)$ of $\hat{V}$. For an equivariant connection ∇

on V such an invariant section satisfies, from (D.10), that

$$\Psi'_g(\nabla(s)) = \nabla(s)\ . \tag{D.15}$$

We therefore see that $\nabla(s)$ is an invariant section of $V \otimes TX^\vee$ and thus descends to a well-defined section of $\hat{V} \otimes T\hat{X}^\vee$ on the quotient. Thus we can define an associated downstairs connection by $\hat{\nabla}(\tilde{p}(s)) := \tilde{p}(\nabla(s))$, where by a slight abuse of notation we are using $\tilde{p}$ to indicate the descent of sections for two different sets of bundles.

To show this structure from a different perspective, let us start with a connection on the quotient and show that the pullback of the associated gauge field under the projection map, as defined in (D.8), is an equivariant connection on X as defined in (D.10). Let us denote by $\hat{\nabla}$ a connection on $\hat{X}$, and, given the projections $p_i := p|_{U_i} : U_i \to \hat{U}_i \subset \hat{X}$ of the open sets defined above (D.11), we can define the local pull-back connections $\nabla_i := p_i^* \hat{\nabla} = d + A_i$. Here the associated gauge field A_i is defined as the generalized pull-back of the gauge field on $\hat{X}$ following the discussion above, so that, according to (D.8)

$$p_i^* \hat{A}_i = P_i^{-1} A_i P_i + P_i^{-1} dP_i\ , \tag{D.16}$$

for some P_i. In this equation A_i is the pullback of $\hat{A}_i$ under p_i as an ordinary one-form. Note that on any given patch we are free to choose the gauge transformation appearing in this expression to be $P = \mathbb{I}$, simply by performing a globally defined gauge transformation on each of the patches which agrees on all overlaps. If, for example, we choose P_0 to be the identity, then the gauge field $A_0 = p_0^* \hat{A}$ on that patch is the straightforward pull-back of the downstairs gauge field. For a set of projections that are consistent with the group action we would have that $p_0 = p_i \circ g_i$ so that, in terms of ordinary pullbacks of one forms, $p_0^* = g_i^* \circ p_i^*$. Thus $p_0^* \hat{A} = g^*(p_i^*(\hat{A}))$ so that $p_i^* \hat{A} = (g^{-1})^* A_0$. Given this and equation (D.16), we find the following relationship between the generalized pullbacks of the gauge fields to patches U_0 and U_i:

$$(g_i^{-1})^* A_0 = P_i^{-1} A_i P_i + P_i^{-1} dP_i. \tag{D.17}$$

This is precisely the relationship we saw in (D.13) for an equivariant connection.

Pullbacks and integration: Let $f : X \to \hat{X}$ be a (smooth, surjective) map between manifolds, as before and ν a top form on $\hat{X}$. Then

$$\int_X f^* \nu = \deg(f) \int_{\hat{X}} \nu \ , \tag{D.18}$$

where deg denotes the degree of a map. The degree is the integer which arises in the pullback of a top form. It is $+1$ for orientation-preserving diffeomorphisms and -1 for orientation-reversing diffeomorphisms. For non-injective maps which corresponds to an N-fold cover the degree is $\pm N$, with the sign is determined by what happens to the orientation [206].

Note that this can immediately be applied to see how the holomorphic Chern-Simons invariant (4.8) descends to a quotient in some cases. We would define the Chern-Simons invariant in a heterotic string context by using a real isomorphism f as in Section 4.2. We would choose this real isomorphism and equivariant structures on the two bundles involved to obey the commutativity condition (D.3). In addition, it is necessary, if the integral (4.8) is to be well defined, that both connections are written with respect to the same local trivialization. With (D.3) effectively saying that the two equivariant structures are the same for the mapped connection A and reference connection A_0, and with the choice of trivializations being the same, we then see from (D.11) that the gauge transformations appearing in the equivariance conditions for the two connections would be identical. Because the integrand in (4.8) is a gauge invariant if we transform both A and A_0 simultaneously, we see that the group Γ simply acts upon the integrand of the Chern-Simons invariant as though it were an ordinary differential form. Thus (D.18) applies and we have that,

$$\mathrm{CS}_{\hat{A}_0}(\hat{A}) = \frac{1}{\deg(p)}\mathrm{CS}_{A_0}(A) \ . \tag{D.19}$$

Here, A and A_0 are the equivariant connections for the bundle $V \to X$ that are pullbacks of the connections $\hat{A}$ and $\hat{A}_0$ on $\hat{V} \to \hat{X}$, in the sense we have described above.

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