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Exact behavior of Jost functions at low energy

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For Schrödinger operators with central potential $q(r)$ and angular momentum l, the behavior of the Jost function $F_l(k)$ as $k\to 0$ is investigated. It is assumed that $\int_0^\infty dr\,(1+r)^\sigma|q(r)|<\infty$, where $\sigma\geqslant 1$. Situations where q is integrable with $1\leqslant\sigma<2$, but not with $\sigma\geqslant 2$ are of particular interest. For potentials satisfying $q(r)\sim q_0 r^{-2-\epsilon}$ $(0<\epsilon\leqslant 1)$ and $l=0$, the leading behavior of $F_0(k)$ and the phase shift $\delta_0(k)$ as $k\to 0$ is derived. Also comments are made on the differentiability properties of the Jost solutions with respect to the variable k at $k=0$. For $\sigma=1$ Levinson's theorem is proved, thereby clarifying some questions raised recently by Newton [J. Math. Phys. **27**, 2720 (1986)].

I. INTRODUCTION

In this paper we study the low-energy behavior of Jost functions and phase shifts of the three-dimensional Schrödinger equation with central potential $q(r)\in L^1_\sigma$, where

$$L^1_\sigma=\left\{q\,\middle|\,\int_0^\infty (1+r)^\sigma|q(r)|dr<\infty,\ \sigma\geqslant 1.\right\} \tag{1.1}$$

Our main concern are potentials that are in L^1_σ with $1\leqslant\sigma<2$, but not necessarily in L^1_2. We were stimulated by a recent paper of Newton[1] on this subject and in particular by one result which we recall here briefly. Let $F_l(k)$ denote the Jost function corresponding to angular momentum l $(l=0,1,2,\ldots)$ and assume that $q\in L^1_\sigma$ with $1\leqslant\sigma<2$ if $l=0$ or $1\leqslant\sigma<3$ if $l\geqslant 1$. Then Newton proved that

$$F_l(k)=F_l(0)+o(k^{\sigma-1}). \tag{1.2}$$

So, if $F_l(0)=0$, then it is consistent with (1.2) if

$$F_l(k)=ak^\alpha+o(k^\alpha),\quad a\neq 0, \tag{1.3}$$

for some $\alpha>\sigma-1$; this implies that Levinson's theorem takes the form

$$\delta_l(0)-\delta_l(\infty)=\pi(n_l+\alpha/2), \tag{1.4}$$

where $\delta_l(k)$ denotes the l th phase shift and n_l is the number of negative eigenvalues for angular momentum l. Thus, if $\alpha\neq 1$ $(l=0)$ or $\alpha\neq 2$ $(l\geqslant 1)$, we would get a modified Levinson's theorem. However, we should be aware of the possibility that if we simply treat (1.3) as a special case of (1.2) we may miss some information that specifically pertains to the case when $F_l(0)=0$. Indeed, it is known that if $q\in L^1_1$ and $F_l(0)=0$, then we always have $\alpha=1$ when $l=0$ and $\alpha=2$ when $l=1$. A proof for $l=0$ (and a hint of how to proceed when $l\geqslant 1$) can be found in the work of Marchenko[2] (for $l=0$ a proof also follows from Ref. 3, Appendix I). Regarding (1.2) this leads to the question of whether the given error estimate is optimal for the class L^1_σ and of how the large-r behavior of $q(r)$ is reflected in the small-k behavior of $F_l(k)$.

The paper is organized as follows. In Sec. II we explain the notation, collect some preliminary material, and state Lemma (2.1), which is needed in the later sections. The proof is given in the Appendix.

Section III is devoted to the special case of inverse power-law potentials satisfying $q(r)\sim q_0r^{-2-\epsilon}$ as $r\to\infty$ with $0<\epsilon\leqslant 1$. For $l=0$ we obtain the leading behaviors of the remainder terms in (1.2) and (1.3) [see Theorem (3.1)]. This entails the leading behavior of the phase shift [see Corollary (3.2)] and extends, for $l=0$, previous results found by Keller and Levy,[4] who assumed $\epsilon>1$. There is a corresponding conjecture in the paper of Keller and Levy (Ref. 4, p. 59), but with the additional restriction that q be repulsive. We shall see that q can have arbitrary sign. Furthermore, we also consider the case when $F_0(0)=0$, which was not done in Ref. 4. It seems conceivable to us that results similar to those of Theorem (3.1) and Corollary (3.2) can be derived for arbitrary l (Keller and Levy also allowed $l>0$), but we have not checked the details. As a by-product of the analysis of power-law potentials we obtain precise information on the differentiability of the Jost solution $f_0(k,r)$ with respect to k at $k=0$ [see Corollary (3.3)]. This result clearly demonstrates why the Jost solutions cause problems in the analysis of $F_0(k)$ if $1\leqslant\sigma<2$, a fact that was also recognized in Ref. 1 (see Appendix C).

In Sec. IV we analyze the small-k behavior of $F_l(k)$ when $F_l(0)=0$ for arbitrary potentials with $1\leqslant\sigma<2$ $(l=0)$ or $1\leqslant\sigma<3$ $(l\geqslant 1)$ and we obtain Levinson's theorem for $\sigma=1$. Our proof makes essential use of Lemma (2.1), which allows us to bypass the differentiability problems associated with the Jost solutions. In fact, if Jost solutions are used (following the basic reference 5), then the stronger condition $q\in L^1_2$ seems to be unavoidable. This may have led to the wrong impression that this condition is actually necessary (Ref. 6, p. 23). Other recent proofs,[7,8] based on Sturmian arguments, also require that $r^2q(r)$ be integrable at infinity[8] or that $r^3q(r)\to 0$ as $r\to\infty$ [in Ref. 8 the special case $q(r)\sim q_0r^{-2}$ is also considered]. For a review of the subject see Bollé.[9] As already mentioned, the proofs in Refs. 2 and 3 work for $\sigma=1$, but they are based on the Marchenko equation (and on an inductive argument with respect to l when $l\geqslant 1$). Our proof is more direct in the sense that it is a refinement of Levinson's original proof,[10] which needed $\sigma=2$ at various places. Moreover, our method allows us to control the error terms.

At the end of Sec. IV, we remark on how our results tie in with the threshold behavior of the eigenvalues[11,12] when they are born from the continuous spectrum as the coupling constant is increased.

Finally, we mention that the methods and results of this paper have extensions to the Dirac equation[13] and the Schrödinger equation on the line.[14] In the latter case we can, for example, prove continuity of the S matrix at $k=0$ for arbitrary potentials satisfying a L_1^1 condition on the line.

II. PRELIMINARIES

We consider the Schrödinger equation

$$-y'' + [l(l+1)/r^2]y + q(r)y = k^2 y \quad (l=0,1,...). \tag{2.1}$$

We always assume that $k>0$, except in Sec. IV, where we need $\operatorname{Im} k \geqslant 0$ in connection with Levinson's theorem. Let $y_l(k,r)$ denote the solution of (2.1) satisfying the boundary condition

$$y_l(k,r) \sim r^{l+1}, \quad \text{as } r\to 0. \tag{2.2}$$

Then y_l solves the integral equation

$$y_l(k,r) = y_{l0}(k,r) - \int_0^r dt\, g_l(k,r,t) q(t) y_l(k,t), \tag{2.3}$$

where

$$y_{l0}(r) = \Gamma(l+\tfrac{3}{2})(k/2)^{-l-1/2} r^{1/2} J_{l+1/2}(kr), \tag{2.4}$$

$$g_l(k,r,t) = \tfrac{1}{2}\pi (rt)^{1/2}\big(J_{l+1/2}(kr)Y_{l+1/2}(kt) - J_{l+1/2}(kt)Y_{l+1/2}(kr)\big). \tag{2.5}$$

Here, $J_{l+1/2}$ and $Y_{l+1/2}$ are the usual Bessel and Neumann functions and y_{l0} satisfies Eq. (2.1) with $q=0$. The Jost function $F_l(k)$ is defined by

$$F_l(k) = 1 + i\pi 2^{-l-3/2} k^{l+1/2}\Big(\Gamma\Big(l+\frac{3}{2}\Big)\Big)^{-1} \times \int_0^\infty dr\, r^{1/2} q(r) H^{(1)}_{l+1/2}(kr) y_l(k,r), \tag{2.6}$$

where $H^{(1)}_{l+1/2} = J_{l+1/2} + iY_{l+1/2}$ denotes the Hankel functions.

Moreover,

$$\delta_l(k) = -\arg F_l(k). \tag{2.7}$$

The zero-energy solution $y_l(r) \equiv y_l(0,r)$ will play an important role in this paper. Its main properties are the following: $y_l(r)$ is bounded at infinity if and only if $F_l(0)=0$ and in that case it obeys

$$y_l(r) \sim A_l r^{-l}, \quad r\to\infty, \tag{2.8}$$

where

$$A_l = -\frac{1}{2l+1}\int_0^\infty dr\, r^{l+1} q(r) y_l(r) \neq 0. \tag{2.9}$$

Combining (2.2) and (2.8) we see that

$$|y_l(r)| \leqslant C r^{l+1}(1+r)^{-2l-1}. \tag{2.10}$$

Here and subsequently C will denote various constants, although not necessarily the same at each appearance. If $F_l(0)\neq 0$, then $y_l(r)$ is unbounded and

$$y_l(r) \sim D_l r^{l+1}, \quad r\to\infty, \tag{2.11}$$

where

$$D_l = 1 + \frac{1}{2l+1}\int_0^\infty dr\, r^{-l} q(r) y_l(r). \tag{2.12}$$

Moreover,

$$D_l = F_l(0). \tag{2.13}$$

Also, notice that y_l is square integrable precisely when $F_l(0)=0$ and $l\neq 0$. The above properties follow easily from the integral equation (2.3) and from (2.6). See, also, Ref. 1.

For our later proofs we need bounds on the difference $y_l(k,r) - y_l(r)$.

Lemma (2.1): Suppose that $q\in L_1^1$.

(i) If $l=0$, $F_0(0)\neq 0$, then

$$|y_0(k,r) - y_0(r)| \leqslant C_\delta r[kr/(1+kr)]^\delta \tag{2.14}$$

and if $F_0(0)=0$, then

$$|y_0(k,r) - y_0(r)| \leqslant C_\delta [kr/(1+kr)]^\delta, \tag{2.15}$$

where $0\leqslant\delta\leqslant 2$.

(ii) If $l\geqslant 1$, $F_l(0)=0$, then

$$|y_l(k,r) - y_l(r)| \leqslant Ck^2[r/1+kr)]^{l+1}. \tag{2.16}$$

For a proof, see the Appendix. Notice the absence of the factor r in (2.15) as compared to (2.14). In Sec. IV we need to use a second, linearly independent solution $\tilde{y}_l(r)$ of (2.1) for $k=0$ if $l\geqslant 1$, $F_l(0)=0$. We choose $\tilde{y}_l$ such that $y_l\tilde{y}_l' - y_l'\tilde{y}_l = 1$. Then

$$\tilde{y}_l(r) \sim \tilde{A}_l r^{l+1}, \quad r\to\infty, \tag{2.17}$$

where

$$(2l+1)A_l\tilde{A}_l = 1 \tag{2.18}$$

and

$$\tilde{y}_l(r) \sim -[1(2l+1)]r^{-l}, \quad r\to 0. \tag{2.19}$$

Hence

$$|\tilde{y}_l(r)| \leqslant Cr^{-l}(1+r)^{2l+1}. \tag{2.20}$$

Equations (2.17) and (2.19) follow easily from the representation

$$\tilde{y}_l(r) = y_l(r)\int_{r_0}^r dr\, y_l^{-2}(r) + \rho_l y_l(r),$$

where r_0 is at our disposal and ρ_l is a suitable constant (depending on r_0). The asymptotic relations (2.2), (2.8), (2.11), (2.17), and (2.19) may all be differentiated.

III. THE CASE $q(r)\sim q_0 r^{-2-\epsilon}$, $0<\epsilon\leqslant 1$ ($l=0$)

Theorem (3.1): Suppose that, as $r\to\infty$, $q(r)\sim q_0 r^{-2-\epsilon}$, $0<\epsilon\leqslant 1$.

(i) If $0<\epsilon<1$, $F_0(0)\neq 0$, then, as $k\to 0$,

$$F_0(k) = F_0(0) + a_0 e^{-(i/2)\pi\epsilon} F_0(0) k^\epsilon + o(k^\epsilon), \tag{3.1}$$

where

$$a_0 = -q_0 2^\epsilon (\epsilon(\epsilon+1))^{-1}\Gamma(1-\epsilon). \tag{3.2}$$

(ii) If $0<\epsilon<1$, $F_0(0)=0$, then, as $k\to 0$,

$$F_0(k) = -ikA_0 + iA_0 a_0 e^{-(i/2)\pi\epsilon} 2^{-\epsilon} k^{\epsilon+1} + o(k^{\epsilon+1}). \tag{3.3}$$

(iii) If $\epsilon = 1$, then, as $k \to 0$,

$$F_0(k) = \begin{cases} F_0(0) - iq_0F_0(0)k \ln k + o(k \ln k), & F_0(0) \neq 0, \quad (3.4a) \\ -ikA_0 + (A_0/2)q_0k^2 \ln k + o(k^2 \ln k), & F_0(0) = 0. \quad (3.4b) \end{cases}$$

Relation (3.1) shows that in any class L_σ, $1 \leqslant \sigma < 2$, we can find a q such that $F_0(k) - F_0(0)$ vanishes like $k^{\sigma - 1 + \delta}$, with $\delta > 0$ as small as we wish. In this sense, when $F_0(0) \neq 0$, the remainder estimate in (1.2) is optimal. Theorem (3.1) has the following implications about the phase shifts.

Corollary (3.2): Under the assumptions of Theorem (3.1), if $0 < \epsilon < 1$, then

$$\delta_0(k) = \begin{cases} a_0k^\epsilon \sin(\pi\epsilon/2) + o(k^\epsilon), & F_0(0) \neq 0, \quad (3.5a) \\ \pi/2 - a_0 2^{-\epsilon}k^\epsilon \sin(\pi\epsilon/2) + o(k^\epsilon), & F_0(0) = 0, \quad (3.5b) \end{cases}$$

and, if $\epsilon = 1$, then

$$\delta_0(k) = \begin{cases} q_0k \ln k + o(k \ln k), & F_0(0) \neq 0, \quad (3.6a) \\ \pi/2 - (q_0/2)k \ln k + o(k \ln k), & F_0(0) = 0. \quad (3.6b) \end{cases}$$

These relations are all understood to hold mod(π). Corollary (3.2) follows from Theorem (3.1) and (2.7).

Proof of Theorem (3.1): (i) $F_0(0) \neq 0$, $0 < \epsilon < 1$. We break the integral in (2.6) into three parts:

$$F_0(k) = F_0(0) + \int_0^\infty dr(e^{ikr} - 1)q(r)y_0(r) + \int_0^\infty dr\, e^{ikr}q(r)(y_0(k,r) - y_0(r)), \quad (3.7)$$

where we have also used (2.12) and (2.13). We denote the first integral on the rhs by I_1 and the second by I_2. Considering I_1, it is easy to show that the leading behavior of I_1 as $k \to 0$ is determined completely by the asymptotic forms for q and y_0 as $r \to \infty$. On substituting D_0r for y_0 and $q_0r^{-2-\epsilon}$ for $q(r)$ and changing variables, $u = kr$, we obtain

$$I_1 = k^\epsilon D_0q_0 \int_0^\infty du(e^{iu} - 1)u^{-1-\epsilon} + o(k^\epsilon). \quad (3.8)$$

Next we consider I_2. For $l = 0$, (2.3) reads as

$$y_0(k,r) = \frac{\sin kr}{k} + \frac{1}{k}\int_0^r dt \sin k(r-t)q(t)y_0(k,t) \quad (3.9)$$

and

$$y_0(r) = r + \int_0^r dt(r-t)q(t)y_0(t). \quad (3.10)$$

From this we deduce that

$$\begin{aligned} y_0(k,r) - y_0(r) &= r\left(\frac{\sin kr}{kr} - 1\right)\left(1 + \int_0^r dt\, q(t)y_0(t)\right) \\ &+ \frac{\sin kr}{k}\int_0^r dt\, (\cos kt - 1)q(t)y_0(t) \\ &- \frac{1}{k}(\cos kr - 1)\int_0^r dt \sin kt q(t)y_0(t) \\ &- \frac{1}{k}\int_0^r dt(\sin kt - kt)q(t)y_0(t) \\ &+ \frac{1}{k}\int_0^r dt\, q(t)\sin k(r-t)(y_0(k,t) - y_0(t)). \end{aligned} \quad (3.11)$$

We denote the five terms on the rhs by $A_1(k,r),\ldots,A_5(k,r)$, respectively. Upon inserting $A_1(k,r)$ into the expression for I_2, we see that

$$\int_0^\infty dr\, e^{ikr}q(r)A_1(k,r) = k^\epsilon D_0q_0 \int_0^\infty du\, e^{iu}\left(\frac{\sin u}{u} - 1\right)u^{-\epsilon-1} + o(k^\epsilon). \quad (3.12)$$

Next we estimate A_2, A_3, and A_4. Let $\beta \in (\epsilon, 2\epsilon)$. By using elementary estimates we deduce that

$$|A_j(k,r)| \leqslant Ck^\beta r^{1+\beta/2}\int_0^r dt\, |q(t)|t^{\beta/2}|y_0(t)| \leqslant Ck^\beta r^{1+\beta/2}, \quad j = 2,3,4, \quad (3.13)$$

on account of the linear growth of y_0. This, in turn, implies

$$\left|\int_0^\infty dr\, e^{ikr}q(r)A_j(k,r)\right| \leqslant Ck^\beta, \quad j = 2,3,4. \quad (3.14)$$

The term $A_5(k,r)$ is estimated by using (2.14) with $\delta = \beta$, so that

$$|A_5(k,r)| \leqslant Ck^\beta r^{1+\beta/2}\int_0^r dt\, |q(t)|t^{1+\beta/2}$$

and thus

$$\left|\int_0^\infty dr\, e^{ikr}q(r)A_5(k,r)\right| \leqslant Ck^\beta. \quad (3.15)$$

Thus the contributions from A_2 through A_5 to I_2 are $o(k^\epsilon)$. Adding (3.8) and (3.12) and computing the remaining integral yields the second term on the rhs of (3.1).

(ii) $F_0(0) = 0$, $0 < \epsilon < 1$. We again use (3.7). Since now y_0 is bounded at infinity, we obtain, using (2.8) and (2.9),

$$I_1 = -ikA_0 + k^{\epsilon+1}A_0q_0 \cdot \int_0^\infty du(e^{iu} - 1 - iu)u^{-\epsilon-2} + o(k^{\epsilon+1}). \quad (3.16)$$

In this case, however, $A_1(k,r)$ does not contribute to the $k^{\epsilon+1}$ term. In fact, since $D_0 = 0$, we can write

$$A_1(k,r) = -r\left(\frac{\sin kr}{kr} - 1\right)\int_r^\infty dt\, q(t)y_0(t) \quad (3.17)$$

and thus

$$|A_1(k,r)|\leqslant Ck^{\gamma}r^{1/2+\gamma/2}\int_r^\infty dt\,|q(t)|t^{1/2+\gamma/2}|y_0(t)|, \tag{3.18}$$

where $1+\epsilon<\gamma<\min(2,1+2\epsilon)$, so that the contribution of A_1 to I_2 is $O(k^\gamma)$. Also, we have

$$|A_j(k,r)|\leqslant Ck^{\gamma}r^{1/2+\gamma/2}\int_0^r dt\,|q(t)|t^{1/2+\gamma/2}|y_0(t)|,$$
$$j=2,3,4, \tag{3.19}$$

and a similar estimate for A_5 in view of inequality (2.15) with $\delta=\gamma$. Thus the contributions from A_1 through A_5 to I_2 are $O(k^\gamma)=o(k^{\epsilon+1})$. Evaluating the integral in (3.16) yields (3.3).

(iii) If $\epsilon=1$, $F_0(0)\neq 0$, then

$$I_1=-iq_0D_0k\ln k+o(k\ln k). \tag{3.20}$$

Moreover,

$$\int_0^\infty dr\,e^{ikr}q(r)A_1(k,r)$$
$$\sim D_0q_0k\int_0^\infty du\,e^{iu}\frac{\sin u-u}{u^3}=O(k). \tag{3.21}$$

The contributions from A_2 through A_5 are $O(k^\beta)$, $\beta\in(1,2)$. Remembering (2.13), we arrive at (3.4a). If $F_0(0)=0$, then

$$I_1=-ikA_0+(A_0/2)q_0k^2\ln k+o(k^2\ln k). \tag{3.22}$$

In the estimates for A_j $(j=1,\ldots,5)$ we may, of course, choose $\gamma=1+\epsilon=2$ [see (3.18)] and we see that $I_2=O(k^2)$, whence (3.4b). Theorem (3.1) is proved.

Next we turn to the differentiability properties of the Jost solution $f_0(k,r)$ at $k=0$, where $f_0(k,r)$ denotes the solution of (2.1) defined by the boundary condition

$$\lim_{r\to\infty}e^{-ikr}f_0(k,r)=1. \tag{3.23}$$

Corollary (3.3): Assume that $q(r)\sim q_0r^{-2-\epsilon}$ as $r\to\infty$, $0<\epsilon\leqslant 1$ $(q_0\neq 0)$. Then $f_0(k,r)$ is differentiable at $k=0$ if and only if $f_0(0,r)=0$.

Proof: For any $r=r_0\geqslant 0$ we have

$$f_0(k,r_0)=e^{ikr_0}+\int_{r_0}^\infty dt\,e^{ikt}q(t)y(k,t;r_0), \tag{3.24}$$

where $y(k,r;r_0)$ solves (2.1) for $r\geqslant r_0$ with $y(k,r_0;r_0)=0$, $y'(k,r_0;r_0)=1$. The integral (3.24) can be analyzed in the same way as the integral (3.7) and we obtain the analog of Theorem (3.1) with respect to the interval $r\geqslant r_0$. In other words, $f_0(k,r_0)$ is differentiable at $k=0$ if and only if $f_0(0;r_0)=0$, which is the assertion of Corollary (3.3). Under the stronger assumption that $q\in L^1_2$, we know that $f_0(k,r)$ is continuously differentiable with respect to k at $k=0$ for any r.[1]

IV. THE CASE $F_l(0)=0$, LEVINSON'S THEOREM, AND THRESHOLD BEHAVIOR

Here we prove the following theorem.

Theorem (4.1): Suppose that $q\in L^1_\sigma$ and that (i) $l=0$, $F_0(0)=0$, $1\leqslant\sigma<2$, then, as $k\to 0$,

$$F_0(k)=-iA_0k+o(k^\sigma); \tag{4.1}$$

or (ii) $l\geqslant 1$, $F_l(0)=0$, $1\leqslant\sigma<3$, then, as $k\to 0$,

$$F_l(k)=c_lk^2+\begin{cases}o(k^{\sigma+1}), & 1\leqslant\sigma<3,\ l\geqslant 2 \text{ or } 1\leqslant\sigma<2,\ l=1,\\ O(k^3), & 2\leqslant\sigma<3,\ l=1,\end{cases} \tag{4.2}$$

where

$$c_l=-\frac{\|y_l\|^2}{(2l+1)A_l}=-\frac{(\partial F_l/\partial\lambda)(0;1)\|y_l\|^2}{(qy_l,y_l)}. \tag{4.3}$$

Here and subsequently (,) denotes the L^2 inner product and $\|\ \|$ denotes the L^2 norm. In (4.3) $F_l(k;\lambda)$ is the Jost function for (2.1) with q replaced by λq $(\lambda\in\mathbb{R})$. The second equation in (4.3) allows us to establish the connection with the threshold coupling constant behavior of the eigenvalues (see below).

Proof: (i) For $l=0$ we need only look back at the proof of Theorem (3.1), Eq. (3.7). We have

$$I_1=iA_0k+\int_0^\infty dr(e^{ikr}-1-ikr)q(r)y_0(r). \tag{4.4}$$

The integrand is $O(k^2)$ and dominated by $Ck^\sigma r^\sigma|q(r)|\,|y_0(r)|$; hence by dominated convergence the integral is $o(k^\sigma)$. Thus

$$I_1=-iA_0k+o(k^\sigma). \tag{4.5}$$

From (2.15) with $\delta=\sigma$,

$$|I_2|\leqslant Ck^\sigma\int_0^\infty dr|q(r)|\left(\frac{r}{1+kr}\right)^\sigma,$$

so again by dominated convergence [since $y_0(k,r)-y_0(r)=O(k^2)$ for fixed r],

$$I_2=o(k^\sigma). \tag{4.6}$$

This establishes (4.1).

(ii) The case when $l\geqslant 1$ is complicated by the fact that part of the leading contribution comes from the analog of I_2. We begin by splitting $F_l(k)$ as

$$\begin{aligned}F_l(k)=&\ \alpha_lk^{l+1/2}\int_0^\infty dr\,r^{1/2}q(r)(H^{(1)}_{l+1/2}(kr)\\&-\widetilde{H}^{(1)}_{l+1/2}(kr))y_l(r)\\&+\alpha_lk^{l+1/2}\int_0^\infty dr\,r^{1/2}q(r)(H^{(1)}_{l+1/2}(kr)\\&-\widetilde{H}^{(1)}_{l+1/2}(kr))(y_l(k,r)-y_l(r))\\&+\alpha_lk^{l+1/2}\int_0^\infty dr\,r^{1/2}q(r)\\&\times\widetilde{H}^{(1)}_{l+1/2}(kr)(y_l(k,r)-y_l(r))\end{aligned} \tag{4.7}$$

and we denote the three terms on the rhs by B_1, B_2, and B_3, respectively. Here

$$\alpha_l=i\pi 2^{-l-3/2}(\Gamma(l+\tfrac{3}{2}))^{-1} \tag{4.8}$$

and $\widetilde{H}^{(1)}_{l+1/2}(kr)$ is the leading term of $H^{(1)}_{l+1/2}(kr)$ as $r\to 0$, i.e.,

$$\widetilde{H}^{(1)}_{l+1/2}(kr)=\beta_l(kr)^{-l-1/2}, \tag{4.9}$$

where

$$\beta_l = 1/(2l+1)\alpha_l. \quad (4.10)$$

Then

$$|H^{(1)}_{l+1/2}(kr) - \widetilde{H}^{(1)}_{l+1/2}(kr)| \leqslant c(kr)^{-l+3/2}(1+kr)^{l-2}. \quad (4.11)$$

We consider B_2 first. Upon inserting (4.11) and (2.16) into B_2 we see that

$$|B_2| \leqslant Ck^4 \int_0^\infty dr \frac{r^3|q(r)|}{(1+kr)^3} \leqslant Ck^{\sigma+1}\int_0^\infty dr\, r^\sigma |q(r)| \frac{(kr)^{3-\sigma}}{(1+kr)^3} = o(k^{\sigma+1}). \quad (4.12)$$

To analyze B_1, we expand $H^{(1)}_{l+1/2}$ one term further:

$$H^{(1)}_{l+1/2}(kr) = \beta_l (kr)^{-l-1/2} + \gamma_l (kr)^{-l+3/2} + R_l(kr), \quad (4.13)$$

where

$$\gamma_l = \beta_l/2(2l-1). \quad (4.14)$$

Now, $R_l(k,r)$ obeys the following estimates. For $l=1$,

$$|R_1(k,r)| \leqslant C(kr)^{3/2}(1+kr)^{-1} \quad (4.15)$$

and, for $l \geqslant 2$,

$$|R_l(k,r)| \leqslant C(kr)^{7/2-l}(1+kr)^{l-4}. \quad (4.16)$$

By using (2.10) it is easy to see that the contribution from $R_l(k,r)$ to B_1 is $O(k^3)$ if $l=1$ and $O(k^4)$ if $l \geqslant 2$, provided only $\sigma = 1$. Splitting off the leading term, we obtain

$$B_1 = \alpha_l \gamma_l k^2 \int_0^\infty dr\, r^{2-l} q(r) y_l(r) + \begin{cases} O(k^3), & l=1, \\ O(k^4), & l \geqslant 2. \end{cases} \quad (4.17)$$

It remains for us to consider B_3. We make use of another representation for $y_l(k,r)$, which we obtain by applying the variation of parameter formula to (2.1), namely

$$y_l(k,r) = y_l(r) + k^2 u_l(r) + T_l(k,r), \quad (4.18)$$

where

$$u_l(r) = \int_0^r dt\, h_l(r,t) y_l(t), \quad (4.19)$$

$$h_l(r,t) = y_l(r)\tilde{y}_l(t) - \tilde{y}_l(r) y_l(t), \quad (4.20)$$

and

$$T_l(k,r) = k^2 \int_0^r dt\, h_l(r,t)(y_l(k,t) - y_l(t)). \quad (4.21)$$

Here $\tilde{y}_l$ is the solution discussed in Sec. II, (2.17)–(2.20). From the properties of y_l and $\tilde{y}_l$ we infer that

$$|h_l(r,t)| \leqslant Cr^{l+1}t^{-l}, \quad t \leqslant r, \quad (4.22)$$

$$u_l(r) \sim -\widetilde{A}_l \|y_l\|^2 r^{l+1}, \quad r \to \infty, \quad (4.23)$$

$$u_l(r) \sim \text{const}\, r^{l+3}, \quad r \to 0. \quad (4.24)$$

Moreover, by (4.22) and (2.16) we obtain the bound

$$|T_l(k,r)| \leqslant Ck^4 r^{l+3}. \quad (4.25)$$

Now we put $R = 1/k$ and write

$$B_3 = \alpha_l k^{l+1/2} \int_R^\infty dr\, r^{1/2} q(r) \widetilde{H}_{l+1/2}(kr) \times (y_l(k,r) - y_l(r)) + \alpha_l k^{l+1/2} \int_0^R dr\, r^{1/2} q(r) \times \widetilde{H}_{l+1/2}(kr)(y_l(k,r) - y_l(r)) = J_1 + J_2. \quad (4.26)$$

The term J_1 is estimated by means of (2.16) and (4.9):

$$|J_1| \leqslant Ck^2 \int_R^\infty dr\, r|q(r)| \leqslant Ck^{1+\sigma}\int_R^\infty dr\, r^\sigma |q(r)| = o(k^{\sigma+1}). \quad (4.27)$$

Now we write J_2 [using (4.9)] as

$$J_2 = \alpha_l \beta_l k^2 \int_0^R dr\, q(r) r^{-l} u_l(r) + \alpha_l \beta_l \int_0^R dr\, q(r) r^{-l} T_l(k,r). \quad (4.28)$$

We split the second integral in Eq. (4.28) into two, with one going from 0 to $R^{1/2}$ and the other from $R^{1/2}$ to R and estimate them by using (4.25):

$$\left| \int_0^R dr\, q(r) r^{-l} T_l(k,r) \right| \leqslant Ck^{(5+\sigma)/2} \int_0^{\sqrt{R}} dr |q(r)| r^\sigma + Ck^{\sigma+1} \int_{\sqrt{R}}^R dr |q(r)| r^\sigma = o(k^{\sigma+1}). \quad (4.29)$$

We write the first term on the rhs of (4.28) as

$$\alpha_l \beta_l k^2 \int_0^\infty dr\, q(r) r^{-l} u_l(r) - \alpha_l \beta_l k^2 \int_R^\infty dr\, q(r) r^{-l} u_l(r) \quad (4.30)$$

and observe that here the second term is bounded by [use (4.23)]

$$Ck^{\sigma+1} \int_R^\infty dr |q(r)| r^\sigma = o(k^{\sigma+1}). \quad (4.31)$$

Thus

$$B_3 = \alpha_l \beta_l k^2 \int_0^\infty dr\, q(r) r^{-l} u_l(r) + o(k^{\sigma+1}). \quad (4.32)$$

Thus from (4.12), (4.17), and (4.32) we obtain (4.2) with

$$c_l = \alpha_l \gamma_l \int_0^\infty dr\, r^{2-l} q(r) y_l(r) + \alpha_l \beta_l \int_0^\infty dr\, r^{-l} q(r) u_l(r). \quad (4.33)$$

We must still transform c_l into (4.3). To this end, we observe that

$$-u_l'' + qu_l + [l(l+1)/r^2] u_l = y_l. \quad (4.34)$$

Upon multiplying Eq. (4.34) by r^{-l} and integrating by parts twice, we obtain the relation

$$\int_0^\infty dr\, r^{-l} y_l(r) = (2l+1)\widetilde{A}_l \|y_l\|^2 + \int_0^\infty dr\, q(r) r^{-l} u_l(r). \quad (4.35)$$

The first term on the rhs comes from $r = \infty$ because of (4.23) and (2.8). In a similar manner we deduce from Eq. (2.1) with $k = 0$ that

$$2(1-2l)\int_0^\infty dr\, y_l(r) r^{-l} = \int_0^\infty dr\, q(r) y_l(r) r^{2-l}. \quad (4.36)$$

By using (4.36), (4.35), (4.8), (4.10), (4.14), and (2.18) we obtain

$$c_l = -\widetilde{A}_l \|y_l\|^2 = -\|y_l\|^2/(2l+1)A_l, \quad (4.37)$$

which is the first relation in (4.3). To establish the second relation in (4.3) we proceed as follows. Let G_l denote the integral operator having kernel $g_l(0,r,r')$. Then (2.3) for $k = 0$ becomes

$$y_l = r^{l+1} - G_l q y_l. \quad (4.38)$$

Since $F_l(0) = 0$, i.e., $D_l = 0$, we also have

$$y_l = A_l r^{-l} - G_l^* q y_l, \quad (4.39)$$

where G_l^* is the adjoint of G_l. Now we introduce a coupling constant λ [i.e., we replace q by λq in (2.1)] and denote the corresponding zero-energy solution and Jost function by $y_l(r;\lambda)$ and $F_l(k;\lambda)$, respectively. We have that $y_l(r;1) = y_l(r)$ and $F_l(k;1) = F_l(k)$, $F_l(0;1) = 0$. We also put

$$y_{l;\lambda}(r) = \frac{\partial y_l}{\partial z}(r;z)\bigg|_{z=\lambda}, \quad (4.40)$$

$$F_{l;\lambda}(k) = \frac{\partial F_l}{\partial z}(k;z)\bigg|_{z=\lambda}. \quad (4.41)$$

Then by (2.6) and (2.12),

$$F_{l;1}(0) = \frac{1}{2l+1}\int_0^\infty dr\, r^{-l} q(r) y_l(r) + \frac{1}{2l+1}\int_0^\infty dr\, r^{-l} q(r) y_{l;1}(r) = -1 + \frac{1}{2+1}\int_0^\infty dr\, r^{-l} q(r) y_{l;1}(r), \quad (4.42)$$

$$y_{l;1} = -G_l q y_l - G_l q y_{l;1}. \quad (4.43)$$

From Eq. (4.43) $y_{l;1}$ can be obtained by iteration. To simplify the notation in the following calculations, we freely use the notation (f,g) even if f and g are not in L^2, but fg is in L^1. Then, by (4.38), (4.39), (4.43), and (2.9),

$$\begin{aligned} A_l(qr^{-l}, y_{l;1}) &= (q(1+G_l^* q) y_l, y_{l;1}) \\ &= ((1+qG_l^*) q y_l, y_{l;1}) \\ &= (qy_l (1+G_l q) y_{l;1}) \\ &= -(qy_l, G_l q y_l) \\ &= (qy_l, y_l) - (qy_l, r^{l+1}) \\ &= (qy_l, y_l) + (2l+1)A_l. \end{aligned} \quad (4.44)$$

Thus

$$F_{l;1}(0) = (qy_l, y_l)/(2l+1)A_l \quad (4.45)$$

or

$$c_l = -F_{l;1}(0)\|y_l\|^2/(qy_l, y_l), \quad (4.46)$$

which is the desired second form for c_l. The proof of Theorem (4.1) is complete.

Relation (4.45) also holds when $l = 0$ [recall that y_0 is bounded so that (qy_0, y_0) exists].

The proof of part (i) given here is a simpler version of a proof that appears in Ref. 15.

A. Levinson's theorem

Since if $F_l(0) = 0$, then $k^{-1}F_0(k)$ or $k^{-2}F_l(k)$ $(l \geqslant 1)$ tends to a finite limit as $k \downarrow 0$ and $F_l(-k) = \overline{F_l(k)}$; the same limits are approached as $k \uparrow 0$. Moreover, $F_l(k)$ is analytic for Im $k > 0$ and continuous for Im $k \geqslant 0$. Then by a Phragmen–Lindelöf theorem[16] (the required exponential bound is established easily) these same limits are approached as $k \to 0$ from the upper half-plane. Hence by the usual contour argument, the contribution from the point $k = 0$ leads to $\alpha = 1$ $(l = 0)$ or $\alpha = 2$ $(l \geqslant 1)$ in (1.4) [when $F_l(0) = 0$] and the ordinary Levinson theorem holds.

B. Threshold behavior

Suppose that $q \in L_1^1$ and $F_l(0;1) = 0$. Upon expanding $F_l(k;\lambda)$ near $k = 0$ and $\lambda = 1$ (F_l is analytic in λ) and using (4.1), (4.2), and (4.46) we conclude that there is a function $k(\lambda)$ obeying

$$F_l(k(\lambda),\lambda) = 0 \quad (4.47)$$

and

$$k(\lambda) = -iA_0^{-2}(qy_0, y_0)(\lambda - 1) + o(\lambda - 1), \quad l = 0, \quad (4.48)$$

$$k(\lambda) = i[|(qy_l, y_l)|^{1/2}/\|y_l\|](\lambda - 1)^{1/2} + o((\lambda - 1)^{1/2}), \quad l \geqslant 1. \quad (4.49)$$

Since $(qy_l, y_l) < 0$ (4.47) means that $k^2(\lambda)$ is a negative eigenvalue of Eq. (2.1), converging to 0 as $\lambda \downarrow 1$. The manner in which it does so is, for $l \geqslant 1$, in agreement with a general theorem found by Simon[12] and, for $l = 0$, consistent with related results[11] but at least $\sigma = 2$ was required in Ref. 11.

Note added in proof: We were unaware of the book by V. V. Babikov [*The Variable Phase Method in Quantum Mechanics* (Nauka, Moscow, 1968) (in Russian)]. It contains some results about the low-energy behavior of the phase shift for inverse power-law potentials (p. 121). We wish to thank D. Bollé and F. Gesztesy for pointing this reference out to us.

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APPENDIX: PROOF OF LEMMA (2.1)

(i) Clearly, it suffices to prove (2.14) and (2.15) when $\delta = 2$. Since a proof of (2.14) is given in Ref. 1 we omit it here. Thus we turn to the case $F_0(0) = 0$. Here we use the decomposition (3.11) along with (2.12) and (2.13) (i.e., $D_0=0$) and rewrite the term $A_1(k,r)$ as

$$A_1(k,r) = -r\left(\frac{\sin kr}{kr} - 1\right)\int_r^\infty dt\, q(t)y_0(t). \quad \text{(A1)}$$

Thus

$$|A_1(k,r)| \leqslant C\,[kr/(1+kr)]^2. \quad \text{(A2)}$$

Similarly, we easily see by means of elementary estimates such as $|\sin z - z| \leqslant cz^3/(1+z)^2$ and by using the monotonicity of $z/1+z$ that

$$|A_j(k,r)| \leqslant C\,[kr/(1+kr)]^2, \quad j=2,3,4. \quad \text{(A3)}$$

Also,

$$|A_5(k,r)| \leqslant \frac{Cr}{1+kr}\int_0^r dt\,|q(t)|\,|y_0(k,t) - y_0(t)|. \quad \text{(A4)}$$

Thus, letting $u(k,r) = y_0(k,r) - y_0(t)$, we have

$$|u(k,r)| \leqslant c\left(\frac{kr}{1+kr}\right)^2 + c\,\frac{r}{1+kr}\int_0^r dt\,|q(t)|\,|u(k,t)|. \quad \text{(A5)}$$

By applying Gronwall's lemma, we obtain

$$|u(k,r)| \leqslant C\,[kr/(1+kr)]^2, \quad \text{(A6)}$$

whence (2.15).

(ii) The proof is similar in spirit to case (i). By using $D_l=0$, we may write

$$y_l(k,r) - y_l(r) = I_1 + \cdots + I_5, \quad \text{(A7)}$$

where

$$I_1 = -2^{l-1/2}\Gamma(l+1/2)k^{-l-1/2}r^{1/2}(J_{l+1/2}(kr) - \widetilde{J}_{l+1/2}(kr))\int_r^\infty dt\, t^{-l}q(t)y_l(t), \quad \text{(A8)}$$

$$I_2 = -\frac{\pi}{2}\int_0^r dt(rt)^{1/2}J_{l+1/2}(kr)(Y_{l+1/2}(kt) - \widetilde{Y}_{l+1/2}(kt))q(t)y_l(t), \quad \text{(A9)}$$

$$I_3 = \frac{\pi}{2}\int_0^r dt(rt)^{1/2}(Y_{l+1/2}(kr) - \widetilde{Y}_{l+1/2}(kr))J_{l+1/2}(kt)q(t)y_l(t), \quad \text{(A10)}$$

$$I_4 = \frac{\pi}{2}\int_0^r dt(rt)^{1/2}\widetilde{Y}_{l+1/2}(kr)(J_{l+1/2}(kt) - \widetilde{J}_{l+1/2}(kt))q(t)y_l(t), \quad \text{(A11)}$$

$$I_5 = -\int_0^r dt\, g_l(k,r,t)q(t)(y_l(k,t) - y_l(t)), \quad \text{(A12)}$$

and where $\widetilde{J}_{l+1/2}$, $\widetilde{Y}_{l+1/2}$ denote the leading parts of $J_{l+1/2}$, $Y_{l+1/2}$ as $r\to 0$, respectively. Explicitly,

$$\widetilde{J}_{l+1/2}(kr) = (\Gamma(l+\tfrac{3}{2}))^{-1}(kr/2)^{l+1/2}, \quad \text{(A13)}$$

$$\widetilde{Y}_{l+1/2}(kr) = -2^{l+1/2}\pi^{-1}\Gamma(l+1/2)(kr)^{-l-1/2}. \quad \text{(A14)}$$

We note the estimates

$$|J_{l+1/2}(z)| \leqslant Cz^{l+1/2}(1+z)^{-l-1}, \quad z>0, \quad \text{(A15)}$$

$$|J_{l+1/2}(z) - \widetilde{J}_{l+1/2}(z)| \leqslant Cz^{l+5/2}(1+z)^{-2}, \quad \text{(A16)}$$

$$|Y_{l+1/2}(z) - \widetilde{Y}_{l+1/2}(z)| \leqslant Cz^{-l+3/2}(1+z)^{l-2}, \quad \text{(A17)}$$

$$|g_l(k,r,t)| \leqslant Ck^{-1}[kr/(1+kr)]^{l+1} \times [kt/(1+kt)]^{-l}, \quad t<r. \quad \text{(A18)}$$

By using these estimates and (2.10) we can check that

$$|I_i| \leqslant Ck^2[r/(1+kr)]^{l+1}, \quad i=1,\ldots,4. \quad \text{(A19)}$$

Using (A18) to estimate I_5 and letting $u_l(k,r) = y_l(k,t) - y_l(t)$ we obtain

$$|u_l(k,r)| \leqslant Ck^2\left(\frac{r}{1+kr}\right)^{l+1} + Ck^{-1}\left(\frac{kr}{1+kr}\right)^{l+1} \times \int_0^r dt\,|q(t)|\left(\frac{kt}{1+kt}\right)^{-l}|u_l(k,t)|. \quad \text{(A20)}$$

Hence by Gronwall's lemma,

$$|u_l(k,r)| \leqslant Ck^2[r/(1+kr)]^{l+1}. \quad \text{(A21)}$$

This proves Lemma (2.1).

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