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# The Nested Event Tree Model with Application to Combating Terrorism

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In this paper, we model and solve the strategic problem of minimizing the expected loss inflicted by a hostile terrorist organization. An appropriate allocation of certain capability-related, intent-related, vulnerability-related, and consequence-related resources is used to reduce the probabilities of success in the respective attack-related actions and to ameliorate losses in case of a successful attack. We adopt a nested event tree optimization framework and formulate the problem as a specially structured nonconvex factorable program. We develop two branch-and-bound schemes based, respectively, on utilizing a convex nonlinear relaxation and a linear outer approximation, both of which are proven to converge to a global optimal solution. We also design an alternative direct mixed-integer programming model representation for this case, and we investigate a fundamental special-case variant for this scheme that provides a relaxation and affords an optimality gap measure. Several range reduction, partitioning, and branching strategies are proposed, and extensive computational results are presented to study the efficacy of different compositions of these algorithmic ingredients, including comparisons with the commercial software BARON. A sensitivity analysis is also conducted to explore the effect of certain key model parameters.

*Key words:* combating terrorism; outer approximation; branch and bound; global optimization; factorable programs

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## 1. Introduction

The international community applies significant effort and resources to counter the effects of terrorism. Unfortunately, niche agencies with different missions focus on disparate aspects of combating terrorism, and a holistic approach is often lacking. Military and paramilitary agencies work to preemptively degrade terrorists' abilities to mount attacks, intelligence agencies focus on detecting and preventing attacks, law enforcement agencies strive to interdict attacks, and emergency services work to mitigate the consequences of successful attacks. These elements each provide important services; however, the proponent agencies are often based in different executive governmental departments and levels. For example, the National Counterterrorism Center (NCTC) is charged with the mission of "leading the U.S. government in counterterrorism intelligence and strategic operational planning in order to combat the terrorist threat to the U.S. and its interests" (National Counterterrorism Center 2008). However, the NCTC lacks the ability or authority to extend its influence to tactical implementation at the level of state and local

agencies, which encompass nonfederalized National Guard units, state and local police, and local first responders such as medics and firefighters. There is a need, therefore, to better coordinate resources toward an integrated strategy for combating terrorism that envelops both preemptive (counterterrorism) and responsive (antiterrorism) actions.

The present work seeks to model the application of resources for combating terrorism to minimize the overall risk (or expected loss) associated with the contemplation of an attack by a terrorist organization. We represent the basic attack-to-consequence phenomenon as an event tree (Sherali et al. 2008), having tiers of event branches with associated probabilities that are *nested* (or layered) to consider multiple *threats* as determined by combinations of the terrorists' developed *capabilities* and their *intent* to mount attacks using these capabilities against specific sets of targets. Furthermore, we consider the application of resources across four domains of combating terrorism: capabilities, intent, vulnerability, and consequences. These correspond to the three-component paradigm of *threat*, *vulnerability*, and *consequences* that has been

established for terrorist-related risk analysis (Willis et al. 2005), with the threat component further refined by the U.S. Department of Homeland Security into the *capability* of an organization to mount an attack and its *intent* to conduct an attack against a specific target (Masse et al. 2007). *Capability-related resources* degrade the terrorists' ability to mount an attack, *intent-related resources* employ deterrence to reduce the likelihood that such an attack will be conducted against a specific target, *vulnerability-related resources* reduce the level of success of the attack, and *consequence-related resources* reduce the severity of the outcome associated with a successful attack against the target. We refer to the resulting formulation that seeks to apply these resources in order to minimize the overall risk as the *nested event tree optimization (NETO)* model.

Within the context of U.S. efforts to combat terrorism, examples of capability-related resources are military strikes and raids against terrorist training camps, international agency collaborations to seize financial assets, and military or surrogate military interdiction of chemical, biological, radiological, nuclear, or high-yield explosive (CBRNE) material during transport and prior to assembly for an attack. Examples of intent-related resources include overt employment of intelligence assets to detect a planned attack, U.S. Customs inspections at national points of entry, and visible local security measures such as surveillance, monitoring, and trained personnel to protect the target from attack. Vulnerability-related resources include safety measures for containing damage such as automated shutdown procedures for a nuclear plant, armored cars to protect political leaders during transport, and electronic devices installed on civilian aircraft to detonate anti-aircraft missiles prior to impact with fuselage. Some resources have the potential to affect more than one domain of terrorists' ability to attack a target. For example, surveillance resources provide a means of early detection that can be used to reduce capability, intent, or vulnerability. Therefore, we will consider capability-related, intent-related, and vulnerability-related resources under one overall category: *countermeasure resources*. This contrasts with the remaining category of *consequence-related resources*. Resources belonging to the latter category might include trained emergency responders, including medics and weapons of mass destruction civil support teams (WMD-CSTs), to isolate and treat personnel exposed to CBRNE agents, as well as redundancy in computer networks to mitigate a successful electronic attack on a server.

Although previous work does not incorporate the breadth of strategic resource allocations considered herein, several related discussions in the literature have addressed a single tier in this strategic problem. For example, Albores and Shaw (2008) propose

a consequence management model that restricts consideration to the application of constrained resources to *respond* to a catastrophic incident (or incidents) and applied a discrete-event simulation approach to study the effect of resource-usage scenarios. In contrast, Golany et al. (2009) examine an event-focused model within the context of three different objectives, wherein resource application influences the likelihood of a successful terrorist attack against a set of targets via linear relationships and for which the consequences of a successful attack against a given target are fixed. Another notable event-focused model, proposed by Scaparra and Church (2008), considers the application of constrained resources to protect critical infrastructure in the context of a game theoretic approach, which results in a bilevel integer program. Within a game-theoretic context for defending targets from attacks, Zhuang and Bier (2007) develop closed-form solutions for optimal opponent strategies and equilibria for simultaneous and sequential games for a single target, and they extend this investigation to multiple-target games. In related research that considers resource application to influence *both* event probabilities and consequences, Mehr and Tumer (2006) propose a multiobjective model to minimize the expectation and variance for risk of events in a space exploration system but restrict their model to consider linear probability-resource relationships. Stranlund and Field (2006) formulate nonlinear models to apply constrained event- and consequence-related resources and study the effect of uncertainty on the expected loss. Sherali et al. (2008, 2010) examine mixed-integer programming formulations for the application of event- and consequence-related resources to reduce probabilities and outcome costs, respectively, to minimize the overall *risk* (expected loss) within a Bernoulli event tree representing a cascading sequence of occurrences following an initiating hazardous event. They employed nonlinear logit models for the probability-resource relationships and a linear model for the outcome-resource relationships to formulate a nonconvex model that was solved to global optimality by adopting suitable outer-approximating linear programming relaxations. In an earlier work, Beim and Hobbs (1997) also model net risk using an event tree with conditional probabilities but without the context of strategic planning for resource allocation, so that the probabilities in their model are subjectively determined and are fixed. Dillon and Paté-Cornell (2005) explore the application of suitable resources to minimize the expected risk in an information system over three distinct tiers—initial failures, intermediate failures, and total failures—along with the associated cost of failures. Although the form of their objective function and the use of conditional probabilities most closely

align with our approach, Dillon and Paté-Cornell considered a limited set of resource allocation decisions with a discrete probability distribution, which results in a finitely countable set of feasible solutions that can be explicitly enumerated for objective value comparison.

The principal contributions of this paper are threefold. First, we present a novel nested event tree optimization modeling framework that comprehensively addresses capability, intent, vulnerability, and consequence issues in combating terrorism. Second, we design alternative reformulations along with effective specialized global optimization algorithms to solve the challenging nonconvex programming problems that result. Third, we provide insights into the effectiveness of different algorithmic and modeling or reformulation strategies via extensive computational test results, including comparisons with a contemporary commercial software product (BARON).

The remainder of this paper is organized as follows. In §2, we formulate the proposed NETO model. In §3, we design two relaxation-based branch-and-bound algorithms and also propose a piecewise linear approximation approach that results in a linear mixed-integer programming model. This latter model is further refined to address an important fundamental and special case of the problem. In §4, we propose a series of branching and partitioning strategies as well as certain range reduction techniques to enhance algorithmic performance. In §5, we present extensive computational results along with sensitivity analyses to provide insights into algorithmic performance and the effect of various key modeling parameters on the nature of the solutions produced. We conclude in §6 with further discussion and recommendations for future research.

## 2. Model Formulation

To facilitate our model formulation, we begin by introducing notation for the sets of attack capabilities, targets, and outcomes and for the influencing resource sets that affect related probabilities. Figure 1 depicts the relationship between the defined indices and the associated sets and resource applications. Also displayed are the probabilities associated with tiers of the nested event tree as defined later in this section, which are affected by the application of countermeasure resources and consequence-related resources.

The notation used is defined as follows:

- $n \in N$ : set of possible targets.
- $i \in I$ : set of possible types of terrorist capabilities for attacks, where

$$I_n = \{i \in I: \text{terrorist capability } i \text{ could be used against a target } n\}, n \in N.$$

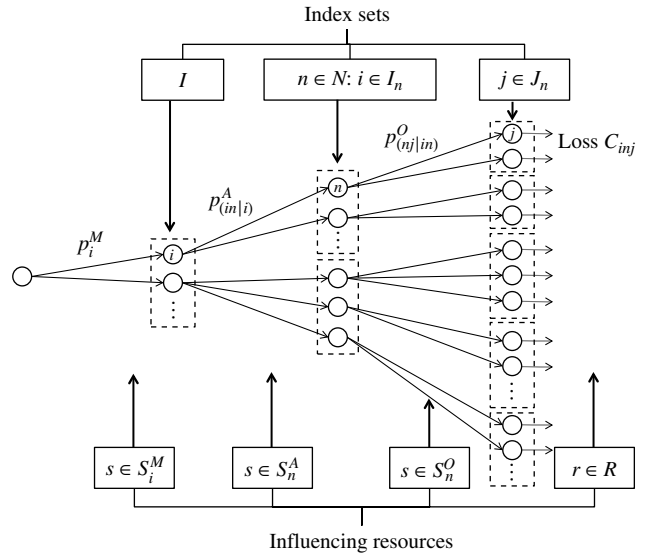


Figure 1 Nested Event Tree Displaying Indices, Resources, and Probabilities

- $j \in J$ : set of possible outcomes of the attacks, where

$$J_n = \{j \in J: \text{outcome } j \text{ is a possible occurrence at target } n \text{ because of some form of attack}\}, n \in N.$$

- $S \equiv$  set of countermeasure resources, where

$$S_i^M = \{s \in S: \text{resource } s \text{ can influence the terrorists' ability to mount an attack using capability type } i\}, i \in I.$$

$$S_n^A = \{s \in S: \text{resource } s \text{ can influence the terrorists' intent to attack target } n\}, n \in N.$$

$$S_n^O = \{s \in S: \text{resource } s \text{ can influence the vulnerability of target } n \text{ with respect to the possible outcomes at that target}\}, n \in N.$$

$$S_n^T \equiv S_n^A \cup S_n^O.$$

- $R \equiv$  set of consequence-related resources.

For reference, examples of types of capabilities, targets, and attack outcomes indexed by  $i$ ,  $n$ , and  $j$ , respectively, are listed in Table 1.

As illustrated in Figure 1, the proposed model formulates the problem of combating terrorism using the framework of a nested event tree and assuming discrete probability distributions. Our focus is

Table 1 Examples of Entities Indexed by  $i$ ,  $n$ , and  $j$

Capabilities ( $i$ )	Targets ( $n$ )	Attack outcomes ( $j$ )
Assassination	Military base	Serious injuries
Dirty bombs	Political leader	Fatalities
Kidnapping	High population density area	Property damages
Chemical attack	Water treatment plant	Water supply contamination
Biological agents	Nuclear power plant	Infectious disease propagation
Nuclear weapons	Economic center	Economic impact

to apply a set of available countermeasure resources ( $s \in S$ ) under a budgetary restriction to safeguard a potential set of targets against a known terrorist group and to mitigate the resulting damage by applying consequence-related resources, thereby minimizing the overall risk as measured by the total expected loss because of attacks.

**Principal Decision Variables**

•  $(x, y) \equiv$  overall decision vector defined as follows:

•  $x = (x^M, x^T)$  for which

$x^M \equiv (x_{si}^M, s \in S_i^M, i \in I) \in \mathbb{R}^{\sum_{i \in I} |S_i^M|}$ , where each component  $x_{si}^M$  represents the amount of resource type  $s$  applied to prevent the terrorists from being able to mount an attack of capability type  $i$ .

$x^T \equiv (x_{sn}^T, s \in S_n^T, n \in N) \in \mathbb{R}^{\sum_{n \in N} |S_n^T|}$ , where each component  $x_{sn}^T$  represents the total amount of resource type  $s$  applied at target  $n$  to ameliorate the potential intent and vulnerability with respect to terrorist attacks.

•  $y \equiv (y_{rj}, r \in R, j \in J) \in \mathbb{R}^{|R||J|}$ , where  $y_{rj}$  represents the amount of consequence-related resource  $r$  applied to mitigate the loss because of an inflicted attack that produces outcome  $j$ .

**Intermediate Variables (Influenced by  $(x, y)$ )**

•  $p = (p^M, p^A, p^O)$ , a vector of probabilities defined as follows (see Figure 1):

$p^M \equiv (p_i^M, i \in I)$ , where  $p_i^M$  represents the probability that the terrorist group can mount an attack using capability type  $i$ .

$p^A \equiv (p_{(in|i)}^A, i \in I, n \in N)$ , where  $p_{(in|i)}^A$  represents the conditional probability that the terrorist group conducts an attack using capability type  $i$  against a potential target  $n$ , given that such an attack can be mounted.

$p^O \equiv (p_{(nj|in)}^O, i \in I, n \in N, j \in J)$ , where  $p_{(nj|in)}^O$  represents the conditional probability that an attack at target  $n$  results in outcome  $j$ , given that a capability type  $i$  attack has been conducted against that target.

•  $C_{inj} \equiv$  loss (cost) incurred when a capability type  $i$  attack has been conducted against target  $n$ , resulting in outcome  $j$ .

**Parameters**

•  $\xi_s \equiv$  amount of countermeasure resource  $s$  available,  $s \in S$ .

•  $\psi_r \equiv$  amount of consequence-related resource  $r$  available,  $r \in R$ .

•  $a \equiv (a_{si}, s \in S_i^M, i \in I)$ , where  $a_{si}$  represents the per-unit cost of applying countermeasure resource  $s$  to reduce the ability of the terrorist group to mount a capability type  $i$  attack.

•  $b \equiv (b_{sn}, s \in S_n^T, n \in N)$ , where  $b_{sn}$  represents the per-unit cost of applying countermeasure resource  $s$  at target  $n$  to ameliorate the potential intent and vulnerability with respect to terrorist attacks.

•  $c \equiv (c_{rj}, r \in R, j \in J)$ , where  $c_{rj}$  represents the per-unit cost of applying consequence-related resource  $r$  to mitigate the effects of outcome  $j$  because of an inflicted attack.

•  $B \equiv$  total amount of budget available.

•  $\alpha \equiv (\alpha_{si}, s \in S_i^M, i \in I)$  a vector of nonnegative constants for defining the logit choice models for the probabilities  $p_i^M$  (see Equation (5)). Moreover, we have

$\alpha^0 \equiv (\alpha_i^0, i \in I)$ , where  $\alpha_i^0$  provides an upper bound on  $p_i^M$  through the logit choice model.

•  $\beta \equiv (\beta_{sn}^i, s \in S_n^T, i \in I, n \in N)$ , a vector of nonnegative constants for defining the logit choice models for the probabilities  $p_{(in|i)}^A$  (see Equation (6)). Moreover, we have

$\beta^0 \equiv (\beta_n^{0i}, i \in I, n \in N)$ , where  $\beta_n^{0i}$  provides an upper bound on  $p_{(in|i)}^A$  through the logit choice model.

•  $\chi \equiv (\chi_{sj}^{in}, s \in S_n^O, i \in I, n \in N, j \in J)$ , a vector of nonnegative constants for defining the logit choice models for the probabilities  $p_{(nj|in)}^O$  (see Equation (7)). Moreover, we have

$\chi^0 \equiv (\chi_j^{0in}, i \in I, n \in N, j \in J)$ , where  $\chi_j^{0in}$  provides an upper bound on  $p_{(nj|in)}^O$  through the logit choice model.

•  $p_i^{Ml} > 0$  and  $p_i^{Mu} < 1 \equiv$  lower and upper bounds, respectively, for  $p_i^M$ . Considering the case for the logit choice model (see Equation (5)) in which no countermeasure resources are applied, we have  $p_i^{Mu} \leq e^{\alpha_i^0} / (1 + e^{\alpha_i^0}) \forall i \in I$ .

•  $p_{(in|i)}^{Al} > 0$  and  $p_{(in|i)}^{Au} < 1 \equiv$  lower and upper bounds, respectively, for  $p_{(in|i)}^A$ . Considering the case for the logit choice model (see Equation (6)) in which no countermeasure resources are applied, we have  $p_{(in|i)}^{Au} \leq e^{\beta_n^{0i}} / (1 + e^{\beta_n^{0i}}) \forall (i \in I, n \in N)$ .

•  $p_{(nj|in)}^{Ol} > 0$  and  $p_{(nj|in)}^{Ou} < 1 \equiv$  lower and upper bounds, respectively, for  $p_{(nj|in)}^O$ . Considering the case for the logit choice model (see Equation (7)) in which no countermeasure resources are applied, we have  $p_{(nj|in)}^{Ou} \leq e^{\chi_j^{0in}} / (1 + e^{\chi_j^{0in}}) \forall (i \in I, n \in N, j \in J)$ .

•  $\delta \equiv (\delta_{rj}^{in}, r \in R, i \in I, n \in N, j \in J)$ , a vector of nonnegative constants for defining the negative exponential model that represents the effect of applying the consequence-related resource  $r$  to reduce the cost  $C_{inj}$  resulting from outcome  $j$  under an attack using capability type  $i$  against target  $n$  (see Equation (12)). In addition, we have

$\delta^0 \equiv (\delta_j^{0in}, i \in I, n \in N, j \in J)$ , where  $\delta_j^{0in}$  represents the unmitigated consequence or cost

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for outcome  $j$  because of an inflicted attack using capability type  $i$  against target  $n$ .

•  $h \equiv (h_j^{in}, i \in I_n, j \in J_n, n \in N)$ , a vector of nonnegative constants that determine the (exponentially decaying) rate at which the consequence is decreased with respect to the combined resource-effect  $\sum_{r \in R} \delta_{rj}^{in} y_{rj}$  (see Equation (12)).

The nested event tree optimization model can then be formulated as follows:

$$\text{(NETO): } \min \text{ Risk} \equiv \sum_{n \in N} \sum_{i \in I_n} \sum_{j \in J_n} p_i^M p_{(in|i)}^A p_{(nj|in)}^O C_{inj} \quad (1)$$

$$\text{subject to } \sum_{i \in I: s \in S_i^M} x_{si}^M + \sum_{n \in N: s \in S_n^T} x_{sn}^T \leq \xi_s \quad \forall s \in S, \quad (2)$$

$$\sum_{j \in J} y_{rj} \leq \psi_r \quad \forall r \in R, \quad (3)$$

$$\sum_{i \in I} \sum_{s \in S_i^M} a_{si} x_{si}^M + \sum_{n \in N} \sum_{s \in S_n^T} b_{sn} x_{sn}^T + \sum_{j \in J} \sum_{r \in R} c_{rj} y_{rj} \leq B, \quad (4)$$

$$\ln \left[ \frac{p_i^M}{1 - p_i^M} \right] = \alpha_i^0 - \sum_{s \in S_i^M} \alpha_{si} x_{si}^M \quad \forall i \in I, \quad (5)$$

$$\ln \left[ \frac{p_{(in|i)}^A}{1 - p_{(in|i)}^A} \right] = \beta_n^{0i} - \sum_{s \in S_n^A} \beta_{sn}^i x_{sn}^T \quad \forall (i \in I_n, n \in N), \quad (6)$$

$$\ln \left[ \frac{p_{(nj|in)}^O}{1 - p_{(nj|in)}^O} \right] = \chi_j^{0in} - \sum_{s \in S_n^O} \chi_{sj}^{in} x_{sn}^T \quad \forall (i \in I_n, j \in J_n, n \in N), \quad (7)$$

$$p_i^M \leq p_i^M \leq p_i^{Mu} \quad \forall i \in I, \quad (8)$$

$$p_{(in|i)}^A \leq p_{(in|i)}^A \leq p_{(in|i)}^{Au} \quad \forall (i \in I_n, n \in N), \quad (9)$$

$$p_{(nj|in)}^O \leq p_{(nj|in)}^O \leq p_{(nj|in)}^{Ou} \quad \forall (i \in I_n, j \in J_n, n \in N), \quad (10)$$

$$p \in P, \quad (11)$$

$$C_{inj} = \delta_j^{0in} e^{-h_j^{in} \sum_{r \in R} \delta_{rj}^{in} y_{rj}} \quad \forall (i \in I_n, j \in J_n, n \in N), \quad (12)$$

$$(x, y) \geq 0. \quad (13)$$

The objective function (1) computes the overall risk, i.e., the expected loss, as given by the sum over all possible events of incurring a cost  $C_{inj}$  because of an outcome  $j$  resulting from a capability type  $i$  attack inflicted against a target  $n$  times the probability  $p_{inj} \equiv p_i^M p_{(in|i)}^A p_{(nj|in)}^O$  that this event will occur. Although the

National Strategy for Combating Terrorism (2006) and the National Strategy for Homeland Security (2007) provide a conceptual framework for the taxonomy of the  $(x, y)$  variables, the relational effect of these principal decision variables on the intermediate variables for computing the expected loss is specific to our model. Note that to analyze the model solution (or to formulate a multiobjective model), we can use partial sums over appropriate terms in (1) to assess the risk associated with a particular outcome  $j$  or the risk pertaining to a particular type of loss having at least some specified severity level over all possible types of attacks on all targets.

Constraints (2) and (3) represent the restrictions on each type of resource. Constraint (4) asserts that the overall expenditure should not exceed the available budget. Constraints (5)–(7) represent the logit models for relating the pertinent probabilities to the applied resources. By the choice of the functional relationships in (5)–(7), note that the governed probabilities decrease at an increasing rate initially with respect to additional committed resources and then continue to decrease at a diminishing marginal rate, asymptotically approaching zero. There exist several alternative approaches to representing such resource-probability relationships. For example, Bier et al. (2008) assume a simple exponential decay function for assessing the success probability of an attack but adopt a more sophisticated representation for the probability of launching an attack based on determining the maximum expected valuation (following a Rayleigh distribution) of the different targets by the terrorist. A very different and even more intricate approach would be to model the problem as a simultaneous or sequential two-person (defender–attacker) game as in Zhuang and Bier (2007), where the effect of the defender’s resource investments is only implicitly defined via the response of the attacker. Whereas we model in greater detail the defender’s strategies in this work, we assume for the sake of computational tractability that the logit models described above can be adequately calibrated to capture the different resource-probability relationships. Constraints (8)–(10) bind the probabilities to certain imposed intervals within  $(0, 1)$ . Note that, as indicated earlier, the imposed upper bounds in (8)–(10) are at least as tight as the corresponding implied bounds when no countermeasure resources are applied. For example, we have  $p_i^{Mu} \leq e^{\alpha_i^0} / (1 + e^{\alpha_i^0}) \quad \forall i \in I$  based on (5) and (13). Constraint (11) enforces any additional applicable restrictions on the  $p$ -variables, for example, based on suitable natural properties of the sample space at each node in the event tree. For example, the capabilities  $i \in I$  for some terrorist organization might be mutually exclusive and collectively exhaustive where (11) would inherit a constraint of the

type  $\sum_{i \in I} p_i^M = 1$ . Constraint (12) computes the consequences based on the resources applied to mitigate the effect of the inflicted attacks, and constraint (13) requires nonnegative resource allocations. Note that some of the decision variables  $(x, y)$  may also be logically restricted to integer values, for example, when representing trained personnel allocations. However, we shall assume in such cases that an appropriate rounding of such variable values based on an optimal solution derived for problem NETO provides an acceptable strategy. Assuming that the specified logit parameters are such that we satisfy constraint (11) by substituting  $x = 0$  in constraints (5)–(7), the application of no resources yields a feasible solution for problem NETO.

### 3. Algorithmic Development

In this section, we propose three reformulations and algorithmic approaches for determining a global optimal solution to problem NETO, including the analysis of a special fundamental case of the model in which the generic side constraint (11) is absent; i.e.,  $P \equiv \mathbb{R}^{\bar{n}}$ , where  $\bar{n}$  is the number of  $p$  variables.

#### 3.1. Algorithm 1: Convex Relaxations

We begin by reformulating NETO to isolate the inherent nonlinearities into a set of two specially structured nonconvex constraint sets having common form and identifying a hyperrectangle to bound the auxiliary decision variables that are used to define these nonlinear constraint sets. Although similar in concept to Sherali et al. (2008, 2010), in which the authors reformulate the original problem as per Sherali and Wang (2001), our solution methodology exploits the particular structure of the NETO problem, and we also design different tailored procedures to solve a fundamental special case of our model. We then construct a polyhedral outer approximation to these isolated nonconvex sets and embed this in a branch-and-bound algorithm, which ensures that a sequence of convex programs solved over recursively partitioned hyperrectangles will yield an optimal solution to NETO (within any prescribed  $\epsilon$ -optimality tolerance).

**3.1.1. Reformulation.** Consider the following transformations to simplify the nonlinearities in problem NETO:

$$q_{1i}^M = \ln(p_i^M) \quad \forall i \in I, \quad (14)$$

$$q_{2i}^M = \ln(1 - p_i^M) \quad \forall i \in I, \quad (15)$$

$$q_{1(in|i)}^A = \ln(p_{(in|i)}^A) \quad \forall (i \in I_n, n \in N), \quad (16)$$

$$q_{2(in|i)}^A = \ln(1 - p_{(in|i)}^A) \quad \forall (i \in I_n, n \in N), \quad (17)$$

$$q_{1(nj|in)}^O = \ln(p_{(nj|in)}^O) \quad \forall (i \in I_n, j \in J_n, n \in N), \quad (18)$$

$$q_{2(nj|in)}^O = \ln(1 - p_{(nj|in)}^O) \quad \forall (i \in I_n, j \in J_n, n \in N). \quad (19)$$

We reference the above original and auxiliary variables in generic terms as  $p_k^L$  and  $q_{vk}^L$ , where  $L \in \{M, A, O\}$ ,  $v \in \{1, 2\}$ , and  $k \in \{i, (in|i), (nj|in)\}$ . These latter indices correspond, respectively, to  $\{M, A, O\}$  and comprise the sets  $K_L$ ,  $L \in \{M, A, O\}$ . Based on the upper and lower bounds for each intermediate variable  $p_k^L$ , we will inherit upper and lower bounds for the corresponding auxiliary variable  $q_{vk}^L$  over the appropriately indexed sets, which results in

$$q_{vk}^{Ll} \leq q_{vk}^L \leq q_{vk}^{Lu} < 0 \quad \forall L, v, (k \in K_L),$$

where

$$\begin{aligned} q_{1k}^{Ll} &= \ln(p_k^{Ll}), & q_{1k}^{Lu} &= \ln(p_k^{Lu}), \\ q_{2k}^{Ll} &= \ln(1 - p_k^{Lu}), & \text{and} & \\ q_{2k}^{Lu} &= \ln(1 - p_k^{Ll}) \quad \forall L, (k \in K_L). \end{aligned} \quad (20)$$

To transform the costs  $C_{inj}$ , we define

$$\begin{aligned} \theta_{inj} = \ln(C_{inj}) &= \ln \delta_j^{0in} - h_j^{in} \sum_{r \in R} \delta_{rj}^{in} y_{rj} \\ &\forall (i \in I_n, j \in J_n, n \in N). \end{aligned} \quad (21)$$

Next, to convexify the objective function, we define

$$p_i^M p_{(in|i)}^A p_{(nj|in)}^O C_{inj} = e^{z_{inj}} \quad \forall (i \in I_n, j \in J_n, n \in N),$$

which, upon taking the natural logarithm of both sides and using (14), (16), (18), and (21), can be equivalently stated as

$$\begin{aligned} z_{inj} &= q_{1i}^M + q_{1(in|i)}^A + q_{1(nj|in)}^O + \theta_{inj} \\ &\forall (i \in I_n, j \in J_n, n \in N). \end{aligned} \quad (22)$$

In addition, we bound  $z_{inj}$  as follows:

$$\left\{ \begin{array}{l} z_{inj}^l \leq z_{inj} \leq z_{inj}^u \\ \text{where } z_{inj}^l = \ln(p_i^{Ml} p_{(in|i)}^{Al} p_{(nj|in)}^{Ol} C_{inj}^l) \\ \text{and } z_{inj}^u = \ln(p_i^{Mu} p_{(in|i)}^{Au} p_{(nj|in)}^{Ou} \delta_j^{0in}) \end{array} \right\}, \quad \forall (i \in I_n, j \in J_n, n \in N). \quad (23)$$

Accordingly, we now have a hyperrectangle ( $\Omega$ ) that bounds the intermediate variables as

$$\Omega = \{p_k^{Ll} \leq p_k^L \leq p_k^{Lu} \quad \forall L, (k \in K_L)\}, \quad (24)$$

with an induced hyperrectangle ( $\Phi$ ) that bounds the corresponding auxiliary variables as given by

$$\Phi = \left\{ \begin{array}{l} q_{1k}^{Ll} \leq q_{1k}^L \leq q_{1k}^{Lu}, \quad \forall L, (k \in K_L), \\ q_{2k}^{Ll} \leq q_{2k}^L \leq q_{2k}^{Lu}, \quad \forall L, (k \in K_L), \\ z_{inj}^l \leq z_{inj} \leq z_{inj}^u, \quad \forall (i \in I_n, j \in J_n, n \in N). \end{array} \right. \quad (25)$$

We also define the vectors  $q \equiv (q_{vk}^L, \forall L, v, (k \in K_L))$  and  $z \equiv (z_{inj}, \forall (i \in I_n, j \in J_n, n \in N))$ . Accordingly, we

shall refer to (25) as  $(q, z) \in \Phi$ . (We similarly refer to (24) as  $p \in \Omega$ .) This results in the following equivalent reformulation, **NETO**( $\Omega$ ), of problem NETO:

$$\text{(NETO}(\Omega)\text{): } \min \sum_{n \in N} \sum_{i \in I_n} \sum_{j \in J_n} e^{z_{inj}} \quad (26)$$

$$\text{subject to } \sum_{i \in I: s \in S_i^M} x_{si}^M + \sum_{n \in N: s \in S_n^T} x_{sn}^T \leq \xi_s \quad \forall s \in S, \quad (27)$$

$$\sum_{j \in J} y_{rj} \leq \psi_r \quad \forall r \in R, \quad (28)$$

$$\sum_{i \in I} \sum_{s \in S_i^M} a_{si} x_{si}^M + \sum_{n \in N} \sum_{s \in S_n^T} b_{sn} x_{sn}^T + \sum_{j \in J} \sum_{r \in R} c_{rj} y_{rj} \leq B, \quad (29)$$

$$q_{1i}^M - q_{2i}^M = \alpha_i^0 - \sum_{s \in S_i^M} \alpha_{si} x_{si}^M \quad \forall i \in I, \quad (30)$$

$$q_{1(i|in)}^A - q_{2(i|in)}^A = \beta_n^{0i} - \sum_{s \in S_n^A} \beta_{sn}^i x_{sn}^T \quad \forall (i \in I_n, n \in N), \quad (31)$$

$$q_{1(nj|in)}^O - q_{2(nj|in)}^O = \chi_j^{0in} - \sum_{s \in S_n^O} \chi_{sj}^{in} x_{sn}^T \quad \forall (i \in I_n, j \in J_n, n \in N), \quad (32)$$

$$z_{inj} = q_{1i}^M + q_{1(i|in)}^A + q_{1(nj|in)}^O + \theta_{inj} \quad \forall (i \in I_n, j \in J_n, n \in N), \quad (33)$$

$$\theta_{inj} = \ln \delta_j^{0in} - h_j^{in} \sum_{r \in R} \delta_{rj}^{in} y_{rj} \quad \forall (i \in I_n, j \in J_n, n \in N), \quad (34)$$

$$q_{1k}^L = \ln(p_k^L) \quad \forall L, (k \in K_L), \quad (35)$$

$$q_{2k}^L = \ln(1 - p_k^L) \quad \forall L, (k \in K_L), \quad (36)$$

$$p \in \Omega \cap P, \quad (q, z) \in \Phi, (x, y) \geq 0. \quad (37)$$

Note that the formulation **NETO**( $\Omega$ ) is linear except for the objective (26) (which is nonetheless convex) and constraints (35) and (36) (which are, however, nonconvex). We next discuss how to handle the latter nonconvexity in this reformulated problem.

**3.1.2. Convex Outer Approximation.** To address the nonconvexity within **NETO**( $\Omega$ ), we begin by developing a polyhedral outer approximation for each of the constraints in Equations (35) and (36) to construct a linearly constrained convex programming relaxation of **NETO**( $\Omega$ ), which we will subsequently embed within a branch-and-bound algorithm. The form of constraints (35) and (36) can be represented generically as follows:

$$\tau = \ln(\sigma), \quad \text{where } 0 < \sigma^l \leq \sigma \leq \sigma^u < 1. \quad (38)$$

The lower-bounding affine convex envelope of  $\tau$  over  $\sigma \in [\sigma^l, \sigma^u]$  is given by

$$\tau \geq \ln(\sigma^l) + \frac{\ln(\sigma^u) - \ln(\sigma^l)}{\sigma^u - \sigma^l} (\sigma - \sigma^l), \quad (39)$$

which is used together with four upper-bounding tangential supports to construct the desired outer approximation. The points of tangency include the interval endpoints and are equispaced along the  $\tau$ -axis to better control the approximation error:

$$\tau \leq \ln(\sigma_t) + \frac{\sigma - \sigma_t}{\sigma_t}, \quad \text{where } \sigma_t = e^{[\ln(\sigma_t) + (t/3)(\ln(\sigma^u) - \ln(\sigma^l))]}, \quad t = 0, 1, 2, 3. \quad (40)$$

When constraints (35) and (36) are replaced with outer approximations of the form (39) and (40), a convex program **CP**( $\Omega$ ) results. The following properties readily hold for **NETO**( $\Omega$ ) (referred to as *relaxation properties RP1–RP4*), given any (possibly further restricted) hyperrectangle  $\Omega$ , where  $\nu(P)$  denotes the optimal value for any optimization problem  $P$ .

(RP1) By construction,  $\nu[\text{CP}(\Omega)]$  is a lower bound on  $\nu[\text{NETO}(\Omega)]$ .

(RP2) Given  $(\bar{x}, \bar{y})$  as part of an optimal solution to **CP**( $\Omega$ ), let  $\hat{p}$  and  $\hat{C}$  be computed using the original **NETO** constraints (5)–(7) and (12), respectively, as follows:

$$\hat{p}_i^M = \frac{\bar{g}_i}{1 + \bar{g}_i} \quad \forall i \in I, \quad \text{where } \bar{g}_i = e^{\alpha_i^0 - \sum_{s \in S_i^M} (\alpha_{si} \bar{x}_{si}^M)}, \quad (41)$$

$$\hat{p}_{(in|i)}^A = \frac{\bar{g}_{(in|i)}}{1 + \bar{g}_{(in|i)}} \quad \forall (i \in I_n, n \in N), \quad \text{where } \bar{g}_{(in|i)} = e^{\beta_n^{0i} - \sum_{s \in S_n^A} (\beta_{sn}^i \bar{x}_{sn}^T)}, \quad (42)$$

$$\hat{p}_{(nj|in)}^O = \frac{\bar{g}_{(nj|in)}}{1 + \bar{g}_{(nj|in)}} \quad \forall (i \in I_n, j \in J_n, n \in N), \quad \text{where } \bar{g}_{(nj|in)} = e^{\chi_j^{0in} - \sum_{s \in S_n^O} (\chi_{sj}^{in} \bar{x}_{sn}^T)}, \quad (43)$$

$$\hat{C}_{inj} = \delta_j^{0in} e^{-h_j^{in} \sum_{r \in R} (\delta_{rj}^{in} \bar{y}_{rj})} \quad \forall (i \in I_n, j \in J_n, n \in N). \quad (44)$$

Then it can be verified that  $\hat{p} \in \Omega$  (see Sherali et al. 2008 for a similar construction). Thus, if  $\hat{p} \in P$ , we have that  $(\bar{x}, \bar{y}, \hat{p}, \hat{C})$  is feasible to **NETO**, and its objective value in (1) yields an upper bound on the optimal values for both problems **NETO** and **NETO**( $\Omega$ ).

(RP3) If an optimal solution to **CP**( $\Omega$ ) satisfies constraints (35) and (36), then it is also optimal to **NETO**( $\Omega$ ) with the same objective value.

(RP4) Given an optimal solution to **CP**( $\Omega$ ), if each of the  $p_k^L$  variables equals one of its bounds in the hyperrectangle  $\Omega$ , then the solution automatically satisfies constraints (35) and (36). By RP3, this solution



is also optimal to NETO( $\Omega$ ) with the same objective value.

The cases enumerated in RP3 and RP4 indicate situations when the lower bound (LB) equals the upper bound (UB) computed using RP1 and RP2, respectively. In general, whenever we have  $UB - LB \leq \epsilon$  for some tolerance  $\epsilon \geq 0$ , then  $(\bar{x}, \bar{y}, \hat{p}, \hat{C})$  is  $\epsilon$ -optimal to NETO( $\Omega$ ).

**3.1.3. Branch-and-Bound Algorithm A1.** Consider the following notation:

- $a \equiv$  index for node number in the branch-and-bound tree, where the node has the following characteristics:

- $\Omega^a \equiv$  the hyperrectangle for node  $a$ ;
- $w^a = (x^a, y^a, p^a, q^a, \theta^a, z^a) \equiv$  the solution to  $CP(\Omega^a)$ ;
- $LB_a = \nu[CP(\Omega^a)] \equiv$  the lower bound on NETO( $\Omega^a$ ) based on  $w^a$ ;

$(L_a, \Delta_a) \equiv$  the two-tuple (with  $L_a \in \{M, A, O\}$  and  $\Delta_a \in K_{L_a}$ ) that achieves the maximum absolute violation in the logarithmic identity (constraints (35) and (36)) as determined by

$$(L_a, \Delta_a) \in \arg \max \{ |(q_{lk}^L)^a - \ln(p_k^L)^a|, |(q_{2k}^L)^a - \ln(1 - (p_k^L)^a)| \quad \forall L, (k \in K_L); \quad (45)$$

$(p_{\Delta_a}^{L_a})^a \equiv$  the value of the intermediate variable  $p_{\Delta_a}^{L_a}$  in the solution  $w^a$ , indexed by  $\Delta_a \in K_{L_a}$ , where  $L_a \in \{M, A, O\}$ , which is used to partition the hyperrectangle at node  $a$  into two hyperrectangles. Specifically,  $(p_{\Delta_a}^{L_a})^a$  will replace the upper bound on  $p_{\Delta_a}^{L_a}$  in one hyperrectangle and the lower bound in the other;

- $\zeta^a = (x^a, y^a, \hat{p}^a, \hat{C}^a) \equiv$  the corresponding feasible solution to NETO, given that  $\hat{p}^a \in P$ , where  $\hat{p}^a$  is calculated using  $x^a$  and Equations (41)–(43), and

- $\hat{C}^a$  is calculated using  $y^a$  and Equation (44);

- $UB_a \equiv$  an upper bound on NETO computed using Equation (1), based on  $(\hat{p}^a, \hat{C}^a)$ , given that  $\hat{p}^a \in P$ .

- $s \equiv$  the stage of the branch-and-bound procedure.
- $A_s \equiv$  the set of active (nonfathomed) nodes in stage  $s$ .
  - $LB(s) \equiv$  the least lower bound for all active nodes  $a \in A_s$ ; i.e.,  $LB(s) \equiv \min_{a \in A_s} LB_a$ .
  - $a(s) \equiv$  the node selected for branching at stage  $s$ . This is a node that yields the least lower bound among the active nodes; i.e.,  $a(s) \in \arg \min_{a \in A_s} LB_a$ .
  - $\nu^* \equiv$  the incumbent objective value for NETO with corresponding solution  $\zeta^*$ .
  - $\epsilon \equiv$  the relative percent optimality tolerance for terminating the branch-and-bound procedure.

**Step 0. Initialization.** Let the incumbent solution  $\zeta^*$  correspond to setting  $x = 0$  and  $y = 0$  in problem NETO, and let  $\nu^*$  be the corresponding objective function value. Set  $s = 0$ ,  $a = 0$ ,  $A_s = \{0\}$ , and let  $\Omega^0$  be given by (24). Solve  $CP(\Omega^0)$  to find  $w^0$  and  $LB_0$ . Determine  $\zeta^0$ ,  $\hat{p}^0$ ,  $\hat{C}^0$ , and  $UB_0$ . If  $\hat{p}^0 \in P$  and  $UB_0 < \nu^*$ , then set  $\nu^* = UB_0$  and  $\zeta^* = \zeta^0$ . If  $LB_0 \geq \nu^*(1 - \epsilon)$ , then terminate the algorithm and accept the incumbent solution as  $\epsilon$ -optimal to NETO. Otherwise, determine  $(L_0, \Delta_0)$  and proceed to Step 1.

**Step 1. Node Selection and Branching Step.** Select a node for branching based on the least lower bound of active nodes; i.e.,  $a(s) \in \arg \min_{a \in A_s} LB_a$ . Branch on the selected node  $a(s)$  by creating two subnodes, indexed by  $a + 1$  and  $a + 2$  and update  $A_{s+1} = A_s \cup \{a + 1, a + 2\} - \{a(s)\}$ . Define  $\Omega^{a+1}$  and  $\Omega^{a+2}$  to represent a partitioning of  $\Omega^a$  by replacing  $p_{\Delta_a}^{L_a} \leftarrow (p_{\Delta_a}^{L_a})^a$  for  $\Omega^{a+1}$  and  $p_{\Delta_a}^{L_a} \leftarrow (p_{\Delta_a}^{L_a})^a$  for  $\Omega^{a+2}$ .

**Step 2. Bounding Step.** For nodes  $h = a + 1, a + 2$ , do the following: (a) solve  $CP(\Omega^h)$  to determine  $w^h$  and  $LB_h$ ; (b) if  $LB_h < \nu^*(1 - \epsilon)$ , then determine  $(L_h, \Delta_h)$ ; (c) compute  $\zeta^h$  and  $UB_h$ ; and (d) if  $UB_h < \nu^*$ , then let  $\nu^* = UB_h$  and  $\zeta^* = \zeta^h$ . Replace  $a \leftarrow a + 2$  and  $s \leftarrow s + 1$ .

**Step 3. Fathoming and Termination Check.** For all nodes  $a \in A_s$ , if  $LB_a \geq \nu^*(1 - \epsilon)$ , then update  $A_s \leftarrow A_s - \{a\}$ . If  $A_s = \{\emptyset\}$ , then terminate the algorithm and accept the incumbent as  $\epsilon$ -optimal for NETO. Otherwise, return to Step 1.

**PROPOSITION 1.** *The proposed branch-and-bound Algorithm A1, with  $\epsilon = 0$ , either terminates finitely with the incumbent solution for problem CP( $\Omega$ ) being optimal to problem NETO or an infinite sequence of stages is generated such that, along any infinite branch of the branch-and-bound tree, any accumulation point of the  $(x, y, p, C)$ -variable part of the convex programming relaxation solutions generated for the corresponding node subproblems solves problem NETO.*

**PROOF.** By the validity of the lower and upper bounds computed by the algorithm, the case of finite termination is clear. Hence, suppose that an infinite sequence of stages is generated. Consider any infinite branch of the branch-and-bound tree generated via the sequence of nested hyperrectangles  $\Omega^{a(s)}$  that correspond to a set of stages  $s$  in some index set  $S$ . Thus, we have

$$LB(s) = LB_{a(s)} = \nu[CP(\Omega^{a(s)})] \quad \forall s \in S. \quad (46)$$

For each node  $a(s)$ ,  $s \in S$ , let  $w^{a(s)}$  be the solution to  $CP(\Omega)$ . By taking any convergent subsequence, if necessary, using the compactness of the feasible region, assume without loss of generality, that  $\{w^{a(s)}, \Omega^{a(s)}\}_S \rightarrow (w^*, \Omega^*)$ . We must show that the solution  $(x^*, y^*, p^*, C^*)$ , which is a part of  $w^*$ , solves problem NETO.

Because  $LB_{a(s)}$  is the least lower bound at stage  $s$ , we have

$$V^* \equiv \lim_{\substack{s \rightarrow \infty, \\ s \in S}} LB_{a(s)} \leq \nu[\text{NETO}].$$

Furthermore, along this infinite branch of the enumeration tree, some  $p_k^L$  variable (say,  $p_k^L$ ) is partitioned infinitely over a subset  $S_1$  of the nodes  $S$  (so that  $(L_{a(s)}, \Delta_{a(s)}) = (\hat{L}, \hat{k}), \forall s \in S_1$ ). By virtue of the partitioning scheme, it is evident that in the limit we have either  $(p_k^L)^* = p_k^{Ll}$  or  $(p_k^L)^* = p_k^{Lu}$ . From the construction of the polyhedral outer approximation (39) and (40), we observe that when  $\sigma = \sigma^l$  or  $\sigma = \sigma^u$ , we have that  $\tau = \ln(\sigma)$ . Hence, in the limit as  $s \rightarrow \infty, s \in S$ , we get

$$(q_{1k}^L)^* = \ln[(p_k^L)^*] \quad \text{and} \quad (q_{2k}^L)^* = \ln[1 - (p_k^L)^*]. \quad (47)$$

However, for all  $s \in S_1$ , we have from (45) that

$$\begin{aligned} & \max\{|(q_{1k}^L)^{a(s)} - \ln(p_k^L)^{a(s)}|, |(q_{2k}^L)^{a(s)} - \ln(1 - (p_k^L)^{a(s)})|\} \\ & \leq \max\{|(q_{1k}^L)^{a(s)} - \ln(p_k^L)^{a(s)}|, |(q_{2k}^L)^{a(s)} - \ln(1 - (p_k^L)^{a(s)})|\} \\ & \quad \forall L, (k \in K_L). \end{aligned} \quad (48)$$

Taking limits as  $s \rightarrow \infty, s \in S_1$  in (48) and using (47) along with  $\{w^{a(s)}, \Omega^{a(s)}\}_S \rightarrow (w^*, \Omega^*)$ , it follows that  $w^*$  satisfies Equations (35) and (36). Hence,  $(x^*, y^*, p^*, C^*)$  is a feasible solution to problem NETO with the same objective value  $V^*$  as for NETO( $\Omega^*$ ) by RP4. Therefore, we also have  $\nu[\text{NETO}] \leq V^*$ . Combined with (46), we obtain  $V^* = \nu[\text{NETO}]$ , whereby  $(x^*, y^*, p^*, C^*)$  solves problem NETO.  $\square$

**COROLLARY 1.** For  $\epsilon > 0$ , the proposed algorithm applied to problem CP( $\Omega$ ) will converge to an  $\epsilon$ -optimal solution for problem NETO within a finite number of iterations.

**PROOF.** The proof follows directly from Proposition 1.  $\square$

### 3.2. Algorithm A2: Linear Programming Relaxations

For the second algorithm, we reformulate NETO( $\Omega$ ) to replace the convex objective function by a lower-bounding piecewise linear convex supporting function and combine this with the polyhedral outer approximation of the nonlinear constraints (35) and (36) as previously developed to derive a linear programming (LP) relaxation. Following this, we apply a modification to the recursive partitioning of hyperrectangles in a branch-and-bound framework to ensure a global optimal solution to NETO (within any prescribed  $\epsilon$ -optimality tolerance).

**3.2.1. Reformulation of NETO( $\Omega$ ).** We reformulate NETO( $\Omega$ ) by rewriting the objective function as

$$\min \sum_{n \in N} \sum_{i \in I_n} \sum_{j \in J_n} Z_{inj}, \quad (49)$$

where we incorporate within the constraint set the convex constraints

$$Z_{inj} \geq e^{z_{inj}} \quad \forall (i \in I_n, j \in J_n, n \in N). \quad (50)$$

**3.2.2. Outer Approximation to a Linear Program.** We now relax NETO( $\Omega$ ) by replacing (50) with four equispaced lower-bounding tangential supports, including those at the two interval endpoints:

$$\begin{aligned} Z_{inj} & \geq e^{z_{inj}^{t'}} (1 + z_{inj} - z_{inj}^{t'}), \quad \text{where} \\ z_{inj}^{t'} & = z_{inj}^l + \frac{t'}{3} (z_{inj}^u - z_{inj}^l), \quad t' = 0, 1, 2, 3. \end{aligned} \quad (51)$$

When constraints (35) and (36) are replaced with outer approximations of the form (39) and (40) and, furthermore, (50) is replaced with a lower-bounding support as in (51), a linear program LP( $\Omega$ ) results. The following relaxation properties RP1' to RP4' hold for LP( $\Omega$ ):

(RP1') By construction,  $\nu[\text{LP}(\Omega)]$  is a lower bound on  $\nu[\text{NETO}(\Omega)]$ .

(RP2') Given  $(\bar{x}, \bar{y})$  as part of an optimal solution to LP( $\Omega$ ), compute  $\hat{p}$  and  $\hat{C}$  using Equations (41)–(44). If  $\hat{p} \in P$ , then  $(\bar{x}, \bar{y}, \hat{p}, \hat{C})$  is feasible to NETO and hence yields an upper bound on the optimal values for both problems NETO and NETO( $\Omega$ ).

(RP3') If an optimal solution to LP( $\Omega$ ) satisfies constraints (35) and (36), and  $z_{inj} = z_{inj}^{t'}$  for some  $t' \in \{0, \dots, 3\}$  in (51) for each  $(i \in I_n, j \in J_n, n \in N)$ , then it is also optimal to NETO( $\Omega$ ) with the same objective value.

(RP4') Given an optimal solution to LP( $\Omega$ ), if each of the  $p_k^L$  variables equals one of its bounds in the hyperrectangle  $\Omega$  and if  $z_{inj} = z_{inj}^{t'}$  for some  $t' \in \{0, \dots, 3\}$  in (51) for each  $(i \in I_n, j \in J_n, n \in N)$ , then this solution is also optimal to NETO( $\Omega$ ) with the same objective value.

Again, for LP( $\Omega$ ), whenever the LB and the UB (computed using RP1' and RP2', respectively) satisfy  $UB - LB \leq \epsilon$ , then  $(\bar{x}, \bar{y}, \hat{p}, \hat{C})$  is  $\epsilon$ -optimal to problem NETO( $\Omega$ ).

**3.2.3. Branch-and-Bound Algorithm A2.** We maintain the same notation as before, with the following exceptions:

- $w^a = (x^a, y^a, p^a, q^a, \theta^a, z^a, Z^a) \equiv$  the solution to LP( $\Omega^a$ ).
- $LB_a = \nu[\text{LP}(\Omega^a)] \equiv$  the lower bound on NETO( $\Omega$ ) based on  $w^a$ .

The branch-and-bound algorithmic statement follows that for Algorithm A1, except that the linear program  $LP(\Omega^a)$  is solved at each node  $a$  in lieu of the nonlinear convex program  $CP(\Omega^a)$  to derive a lower bound. To state the main convergence result for this problem, define problem  $NETO'(\Omega)$  as the approximation of, or relaxation to, problem  $NETO(\Omega)$  in which the objective function (26) is replaced with (49) along with the supporting constraint (51). Then problem  $LP(\Omega)$  is the linearized relaxation of problem  $NETO'(\Omega)$  obtained by imposing the polyhedral outer approximations (39) and (40) to constraints (35) and (36). Furthermore, observe that given any  $\epsilon' > 0$ , by increasing the number of equispaced supports in (51) as necessary (e.g., to  $t'(\epsilon')$ ), we can assure that the maximum error between the objective function of  $NETO(\Omega)$  and its piecewise linear representation in  $NETO'(\Omega)$  is no more than  $\epsilon'$ . Consequently, the following result holds true.

**PROPOSITION 2.** *Given any  $\epsilon > 0$ , let  $NETO'(\Omega)$  be defined using  $t'(\epsilon')$  equispaced supports in (51), where  $0 < \epsilon' < \epsilon$ . Then the proposed branch-and-bound algorithm A2, executed with an optimality tolerance  $(\epsilon - \epsilon')$ , will converge to an  $\epsilon$ -optimal solution for problem  $NETO$  within a finite number of iterations.*

**PROOF.** See the Online Supplement (available at <http://joc.pubs.informs.org/ecompanion.html>).  $\square$

### 3.3. Algorithm A3: Mixed-Integer Programming Approach

For a third proposed approach, we utilize the linear approximation of the objective function from  $NETO'(\Omega)$  but apply a piecewise linear mixed-integer programming (MIP) approximation to the nonlinear relationships (35) and (36) to derive a linear mixed-integer program  $MIP(\Omega)$ , which can then be solved directly using standard software. Considering Equations (35) and (36) stated generically as in Equation (38), Proposition 3 motivates the equispacing of approximating segment breakpoints along the  $\tau$ -axis to maintain a fixed maximum error over the different segments, which can then be bounded by manipulating the number of segments,  $T$ .

**PROPOSITION 3.** *Given a segment of a piecewise linear approximation of the function  $\tau = \ln(\sigma)$ , constructed on  $\sigma \in [\sigma_1, \sigma_2]$ , where  $0 < \sigma^l \leq \sigma_1 < \sigma_2 \leq \sigma^u < 1$ , define  $w \equiv \ln(\sigma_2) - \ln(\sigma_1)$ . Then the maximum approximating error  $E^*$  over  $[\sigma_1, \sigma_2]$  depends only on the width  $w$  along the ordinal axis and is given by*

$$E^* = \ln \left[ \frac{e^w - 1}{w} \right] + \frac{w}{(e^w - 1)} - 1. \quad (52)$$

**PROOF.** Define  $E = \tau - \tilde{\tau}$ , where  $\tilde{\tau}$  is the approximating segment over  $[\sigma_1, \sigma_2]$ . Hence, we have

$$E = \ln(\sigma) - \left[ \left( \frac{\ln(\sigma_2) - \ln(\sigma_1)}{\sigma_2 - \sigma_1} \right) (\sigma - \sigma_1) + \ln(\sigma_1) \right]. \quad (53)$$

Maximizing the strictly concave function (53) using calculus yields

$$E^* = \ln \left[ \frac{\sigma_2 - \sigma_1}{\ln(\sigma_2) - \ln(\sigma_1)} \right] - 1 + \left\{ \frac{\sigma_1 [\ln(\sigma_2) - \ln(\sigma_1)]}{\sigma_2 - \sigma_1} \right\} - \ln(\sigma_1).$$

Substituting  $\ln(\sigma_2) - \ln(\sigma_1) = w$ , along with  $\sigma_2/\sigma_1 = e^w$  and simplifying, yields (52).  $\square$

Because the segments are equispaced on the  $\tau$ -axis, Proposition 3 asserts that we can control the maximum approximation error simply by adjusting the ordinal segment width,  $w$ . For any specified error tolerance  $\epsilon > 0$ , which lets  $w_\epsilon$  be the value of  $w$  that yields  $E^* = \epsilon$  in (52), we can therefore select the number of segments,  $T$ , according to

$$T = \left\lceil \frac{\ln(\sigma^u) - \ln(\sigma^l)}{w_\epsilon} \right\rceil. \quad (54)$$

**REMARK 1.** For illustration, when  $[\sigma^l, \sigma^u] = [0.0001, 0.9999]$ , we get  $T = 2$  in (54) for  $\epsilon \in [0.0001, 0.10]$ . When  $[\sigma^l, \sigma^u] = [10^{-7}, 1 - 10^{-7}]$ , Equation (54) yields  $T = 3$  for  $\epsilon \in [0.0001, 0.07]$ , and  $T = 2$  for  $\epsilon \in [0.08, 0.10]$ .

Considering Equations (35) and (36) stated generically as in Equation (38), a corresponding strong linearized MIP approximation of these constraints that yields a partial convex hull representation takes the form (Sherali 2001):

$$\tau = \sum_{t=1}^T [\tau^{t-1} \lambda^{t1} + \tau^t \lambda^{t2}] \quad \text{and} \quad \sigma = \sum_{t=1}^T [e^{\tau^{t-1}} \lambda^{t1} + e^{\tau^t} \lambda^{t2}],$$

where

$$\tau^t = \tau^l + \frac{t}{T} (\tau^u - \tau^l) \quad \text{for } t = 0, \dots, T, \\ \lambda^{t1} + \lambda^{t2} = \mu^t \quad \text{for } t = 1, \dots, T, \quad (55)$$

$$\sum_{t=1}^T \mu^t = 1, \quad (56)$$

$$(\lambda^{t1}, \lambda^{t2}) \geq 0 \quad \text{and} \quad \mu^t \in \{0, 1\} \quad \forall t = 1, \dots, T. \quad (57)$$

We found this MIP approximation representation to suffice in the present context because  $T$  is typically small ( $\approx 4$ ). However, Vielma et al. (2010) describe alternative representations that might be more suitable for contexts involving a relatively large number of piecewise linear segments. Hence, we can solve

MIP( $\Omega$ ) directly to obtain a near-optimal solution to problem NETO( $\Omega$ ), wherein the degree of optimality can be controlled via the granularity of the piecewise linear approximations to (35) and (36) and the objective function, as governed by  $T$  and  $t'$ , respectively. Note that, given an optimal solution to this MIP, we can derive a corresponding near-optimal solution to NETO by using Equations (41)–(44) in a manner similar to applying (RP2)', where (exact) feasibility is attained provided the solution satisfies  $p \in P$ . We henceforth refer to the application of this technique as Algorithm A3.

Observe that Algorithm A3 simply provides an approximating near-optimal solution to problem NETO. Determining a precise optimality gap for the resulting solution (provided it is feasible) via this approach itself is elusive. However, as demonstrated next, for the special case when  $P \equiv \mathbb{R}^{\bar{n}}$ , this approach can indeed derive a near-optimal feasible solution along with an optimality gap assurance.

**3.3.1. Algorithm A3S: Special Case When  $P \equiv \mathbb{R}^{\bar{n}}$ .** We now consider a fundamental special case of problem NETO( $\Omega$ ) given by (26)–(37) when  $P \equiv \mathbb{R}^{\bar{n}}$ . In this case, NETO( $\Omega$ ) is equivalent to relaxing constraints (35) and (36), maintaining only  $(q, z) \in \Phi$  and  $(x, y) \geq 0$  from (37) and adding the relationship

$$q_{2k}^L = \ln(1 - e^{q_{1k}^L}) \quad \forall L, (k \in K_L). \quad (58)$$

This follows because the latter is a valid inequality implied by constraints (35) and (36). Once  $(x, y, q, \theta, z)$  is determined via this revised model, we can compute the  $p_k^L$  variable values as  $p_k^L = e^{q_{1k}^L} \forall L, (k \in K_L)$ , which results in (35), (36), and  $p \in \Omega$  holding true. Henceforth, we will refer to this equivalent representation of NETO( $\Omega$ ) that dispenses with the  $p_k^L$  variables and represents (35) and (36) in the  $(q_1, q_2)$ -space via constraint (58) as NETO<sup>q</sup>( $\Omega$ ).

For the special case NETO<sup>q</sup>( $\Omega$ ) of NETO( $\Omega$ ) under the condition  $P \equiv \mathbb{R}^{\bar{n}}$ , similar to the approach of Algorithm A3, we construct a linear mixed-integer program MIP<sup>q</sup>( $\Omega$ ) by utilizing the tangential approximations (49) and (51) to the objective function and by replacing (58) with the following MIP approximation, as stated for the generic functional form  $q_2 = \ln(1 - e^{q_1})$ :

$$q_1 = \sum_{t=1}^T [q_1^{t-1} \lambda^{t1} + q_1^t \lambda^{t2}] \quad \text{and} \\ q_2 = \sum_{t=1}^T [\ln(1 - e^{q_1^{t-1}}) \lambda^{t1} + \ln(1 - e^{q_1^t}) \lambda^{t2}], \quad (59)$$

where

$$q_1^t = q_1^l + \frac{t}{T}(q_1^u - q_1^l) \quad \text{for } t = 0, \dots, T, \quad (60)$$

$$\text{along with Equations (55)–(57).} \quad (61)$$

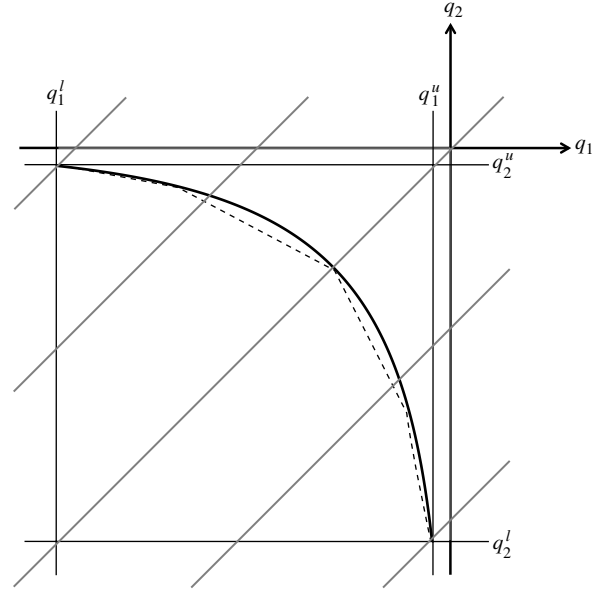


Figure 2 Inner Linearization for  $q_2 = \ln(1 - e^{q_1})$  Along with Contours of  $q_1 - q_2$

**PROPOSITION 4.**  $\nu[MIP^q(\Omega)]$  provides a lower bound on  $\nu[NETO]$ .

**PROOF.** Let  $(x^*, y^*, p^*, C^*)$  be an optimal solution to NETO, and let us construct a corresponding feasible solution  $\bar{\eta} = (\bar{x}, \bar{y}, \bar{q}, \bar{\theta}, \bar{z}, \bar{Z}, \bar{\lambda}, \bar{\mu})$  to MIP<sup>q</sup>( $\Omega$ ) as follows. Let  $(\bar{x}, \bar{y}) = (x^*, y^*)$ . Calculate  $q^*$  using Equations (35) and (36), with  $p \equiv p^*$ , so that (30)–(32) hold. Now, for each  $p_k^L$  variable, while holding the difference  $(q_{k1}^L)^* - (q_{k2}^L)^*$  fixed, decrease both  $(q_{k1}^L)^*$  and  $(q_{k2}^L)^*$  simultaneously along the (generic) contour for  $q_1 - q_2$ , as illustrated in Figure 2, until this contour intersects with the piecewise linear approximation, resulting in  $\bar{q}_{k1}^L \leq (q_{k1}^L)^*$  and  $\bar{q}_{k2}^L \leq (q_{k2}^L)^*$ , along with  $(\bar{\lambda}, \bar{\mu})$ , where  $(\bar{q}, \bar{\lambda}, \bar{\mu})$  is feasible to (59)–(61) defining MIP<sup>q</sup>( $\Omega$ ). Next, calculate  $\bar{\theta}$  using Equation (21), compute  $\bar{z}$  via Equation (22) using  $\bar{q}$  and  $\bar{\theta}$ , and then determine  $\bar{Z}$  according to  $\bar{Z}_{inj} = e^{\bar{z}_{inj}} \forall (i \in I_n, j \in J_n, n \in N)$ . Thus the resulting solution  $\bar{\eta}$  is feasible to MIP<sup>q</sup>( $\Omega$ ). Moreover, noting (33) and (26), because  $\bar{q}_{k1}^L \leq (q_{k1}^L)^* \forall L, (k \in K_L)$ , and because the objective function tangents represented by (49) and (51) underestimate (26), the corresponding objective value of this feasible solution  $\bar{\eta}$  to MIP<sup>q</sup>( $\Omega$ ) is lesser than or equal to  $\nu[NETO]$ . Hence,  $\nu[MIP^q(\Omega)] \leq \nu[NETO]$ .  $\square$

Therefore, upon solving MIP<sup>q</sup>( $\Omega$ ), we directly obtain a lower bound on  $\nu[NETO]$ . Moreover, by applying (RP2)' as before to the resulting optimal solution, we can construct a corresponding upper bounding feasible solution to NETO, and thus compute the associated relative optimality gap. We refer to this approach as Algorithm A3S. Both Proposition 4 and (RP2)' represent relatively unique characteristics of the A3S formulation vis-à-vis other applications

that utilize piecewise linear approximations (e.g., see Martin et al. 2006, where the focus is on nonseparable functions). Moreover, by suitably increasing the number of segments,  $T$ , in the MIP approximation, along with the number of tangential supports,  $(t + 1)$ , for the objective function to obtain sufficient granularity, we can accordingly derive a desired near-optimal solution to problem NETO.

REMARK 2. The Online Supplement details the construction of special-case Algorithms A1S and A2S for  $\text{NETO}^q(\Omega)$ , based on solving similar convex and linear relaxations  $\text{CP}^q(\Omega)$  and  $\text{LP}^q(\Omega)$ , respectively, at each node in the branch-and-bound tree. Both relaxations incorporate a lower-bounding constraint and upper-bounding tangential supports for constraint (58) in  $(q_1, q_2)$ -space over  $q \in \Phi$  and with  $\text{LP}^q(\Omega)$  further replacing the objective function (26) with (49) and the supporting constraint (51). Whereas Algorithms A1S and A2S also attain global optimal solutions, we suppress further discussion of these algorithms because they were significantly dominated by Algorithm A3S.

## 4. Algorithmic Enhancements

In this section, we propose certain branching variable selection and partitioning strategies, along with range reduction techniques, to improve the computational effectiveness of Algorithms A1 and A2.

### 4.1. Branching Variable Selection and Partitioning Strategies

We select a branching variable via the following rule, which measures the relative (as opposed to absolute) violations in the logarithmic identities (constraints (35) and (36)):

$$(L_a, \Delta_a) \in \arg \max \left\{ \left| \frac{(q_{1k}^L)^a - \ln(p_k^L)^a}{\ln(p_k^L)^a} \right|, \left| \frac{(q_{2k}^L)^a - \ln(1 - (p_k^L)^a)}{\ln(1 - (p_k^L)^a)} \right| \right\} \quad \forall L, (k \in K_L). \quad (62)$$

The convergence arguments for Algorithms A1, A1S, A2, and A2S remain identical under (62).

Having selected a branching variable, we split its interval at the arithmetic mean to partition  $\Omega^a$  at node  $a$  as follows:

$$\text{split the interval } [p_{\Delta_a}^{L_a}, p_{\Delta_a}^{L_a u}] \text{ at } (p_{\Delta_a}^{L_a} + p_{\Delta_a}^{L_a u})/2. \quad (63)$$

Similar to the benchmark partitioning strategy, (63) also ensures a finite number of partitions before an  $\epsilon$ -optimal solution is attained. For additional branching variable selection and partitioning strategies examined, along with related computational results, we refer the reader to the Online Supplement.

### 4.2. Range Reduction

We also implement range reduction strategies within our Algorithms A1 and A2 as recommended in different contexts by Ryoo and Sahinidis (1996) and Sherali and Tuncbilek (1997), which serve to strengthen the underlying relaxations by tightening the bounds defining  $\Omega$ . (For Algorithm A3, we apply the range reduction process for Algorithm A2 and then formulate the MIP based on the resulting tightened intervals.) Specifically, as a preprocessing step at any node  $a$ , the imposed interval for each  $p_k^L$  variable is updated by solving two linear programs that minimize and maximize  $p_k^L$  in turn over the feasible region. Any new bounds for a  $p_k^L$  variable will induce corresponding new bounds on the  $q_{1k}^L$  and  $q_{2k}^L$  variables by (20) as well as on the  $z$  variables by (23). Upon completion of the range reduction process, the polyhedral approximations are updated as per (39) and (40). In the case of  $\text{LP}(\Omega^a)$ , we further impose the objective function constraint  $\sum_{n \in N} \sum_{i \in I_n} \sum_{j \in J_n} Z_{inj} \leq \nu^*$ , where  $\nu^*$  is the incumbent objective value within each of the range reduction subproblems, and we also update the tangential supports determined by (51) based on the revised bounds on the  $z$  variables. This range reduction process is continued at each new node so long as a sufficient incremental tightening of the bounds results or until a fixed number of iterations is reached.

## 5. Computational Testing and Evaluation

We coded Algorithm A1 using C++ and SNOPT 7.2 and Algorithms A2 and A3/A3S using C++ and ILOG CPLEX 11.1, and we compared their performance against BARON 8.1.5 using CPLEX for LP subproblems and SNOPT for nonlinear programming subproblems. All runs were executed on a computer with an Intel 2.40 GHz Xeon Processor with 1.5 GB of RAM. We tested our models over the generic network structure depicted in Figure 1, using 10 randomly generated instances having up to 4 types of terrorist capabilities, 6 targets, 4 possible outcomes, and 4 resources of each class. The Online Supplement provides the detailed network attributes, data pertaining to the random generation of instance parameters, detailed experimental results from preliminary testing and sensitivity analyses discussed later in this section, and the general performance results of Algorithms A1S and A2S (neither of the latter significantly improved over Algorithms A1 and A2, respectively). Here, we shall briefly summarize our findings.

Preliminary tests prompted the following settings for each of Algorithms A1, A2, A3, and A3S: construct outer approximations for Equations (35) and (36) with  $(t + 1) = 4$  tangential supports, linearize

the objective function as in (51) with  $(t' + 1) = 32$  lower-bounding supports, and generate MIP approximations for the formulations in §3.3 using  $T = 4$  segments. Each algorithm invokes range reduction, as detailed in §4.2 for Algorithms A1 and A2. For Algorithms A3 and A3S, range reduction is performed using the respective Algorithms A2 and A2S (as detailed in the Online Supplement) for a maximum of 100 reduction loops before solving MIP( $\Omega$ ) or MIP<sup>q</sup>( $\Omega$ ), respectively. (On average, range reduction conserves 89.3%, 90.5%, and 94.2% of the effort for the A1-, A2-, and A3-based algorithms, respectively.) Furthermore, the branch-and-bound algorithms attained  $\epsilon$ -optimality most rapidly when selecting the branching variable according to Equation (62) and when partitioning the hyperrectangle utilizing Equation (63). Hence, all of our results reported below pertain to these parameter settings and algorithmic strategies. We ran all test instances with a relative optimality tolerance of  $\epsilon = 0.01$ , a limit of 1,001 nodes in the branch-and-bound enumeration tree, and a time limit of 1,800 CPU seconds (checked at the completion of any stage in the algorithmic process). The results are displayed in Table 2. For Algorithm A3, we report the *implied optimality gap*, which is the actual optimality gap for the solution produced by Algorithm A3 based on the greatest lower bound available from Algorithm A2 at its termination.

In Table 2, note that all of the proposed algorithms tested outperformed BARON with respect to optimality gap attained and computational time, with the exception of Algorithms A1, A3, and A3S on instance 2 (for which all computational times were less than 9 CPU seconds), and for Algorithm A1 on instance 9 (where BARON itself terminated without identifying a feasible solution). Restricting consideration to the four largest instances for which BARON obtained a feasible solution (see the bottom row of Table 2), all of the proposed algorithmic variants exhibited a significantly more robust and improved performance compared to BARON. As tested, only Algorithm A3S attained the specified optimality gap for all instances. Although applying  $t' = 63$  for Algorithm A2 on instances 7 and 8 and for Algorithm A3 on instance 7 results in attainment of the required optimality gap or implied optimality gap, respectively, we recommend Algorithm A3S as the prescribed solution technique among those considered to solve problem NETO, and we further examine the sensitivity of this procedure and the nature of NETO solutions to different model and algorithmic parameter settings.

### 5.1. Sensitivity Analyses

We also conducted sensitivity analyses using the 10 test instances with respect to three issues in the context of Algorithm A3S: (a) the effect of varying the

**Table 2 Selected Algorithmic Strategies with Range Reduction: Optimality Gap Attained (%), Nodes Explored, and CPU Time (Sec)**

Instance	A1	A2	A3	A3S	BARON
1	0.58	0.77	0.72	0.21	7.33
	3	3	—	—	266
	16.688	5.015	18.265	5.672	1,884.6
2	0.21	0.27	0.27	0.05	1.00
	1	1	—	—	1
	3.938	1.828	6.828	8.796	3.904
3	0.38	0.50	0.49	0.13	1.00
	1	1	—	—	84
	7.516	1.968	7.765	9.734	195.6
4	0.91	0.34	0.35	0.26	1.00
	1	3	—	—	61
	20.657	13.375	8.281	11.469	191.6
5	0.74	0.65	0.62	0.26	5.60
	3	3	—	—	141
	120.32	30.891	232.03	35.515	1,887.8
6	0.45	0.81	0.85	0.76	6.17
	5	5	—	—	33
	264.24	94.030	29.047	27.015	1,881.4
7	0.53	1.05	1.04	0.82	8.86
	3	1,001	—	—	21
	369.73	853.57	524.78	60.172	1,881.0
8	0.94	1.03	0.91	0.47	6.61
	7	1,001	—	—	21
	771.14	983.36	383.23	68.202	1,884.3
9	1.07	0.73	0.65	0.54	*
	17	7	—	—	39
	1,884.5	242.74	38.109	35.077	1,884.2
10	0.87	0.98	0.95	0.15	1.00
	1	1	—	—	9
	125.59	12.351	350.20	64.796	494.3
<b>Average:</b>	<b>0.62</b>	<b>0.71</b>	<b>0.69</b>	<b>0.35</b>	<b>4.29</b>
<b>Instances</b>	<b>2.78</b>	<b>224.33</b>	<b>—</b>	<b>—</b>	<b>70.78</b>
<b>1–8, 10</b>	<b>188.87</b>	<b>221.82</b>	<b>173.38</b>	<b>35.001</b>	<b>1,144.94</b>
<b>Average:</b>	<b>0.6975</b>	<b>0.9675</b>	<b>0.9375</b>	<b>0.55</b>	<b>5.66</b>
<b>Instances</b>	<b>4</b>	<b>502</b>	<b>—</b>	<b>—</b>	<b>21</b>
<b>6–8, 10</b>	<b>382.675</b>	<b>485.83</b>	<b>321.82</b>	<b>60.956</b>	<b>1,535.25</b>

Note. Implied optimality gap reported for A3.

\*No feasible solution found.

number of segments in the MIP approximation on the optimality gap attained, (b) the impact of varying the number of objective tangential supports on the same measure, and (c) the sensitivity of the value of the optimal solution to various resource parameters. The following is a summary of our findings (the Online Supplement provides detailed tabular results).

- As summarized in Table 3, increasing the number of MIP approximation segments generally improved the optimality gap attained for Algorithm A3S, although at a faster-than-linear increase in computational times for larger instances. Using  $T = 2$ , the Algorithm A3S did not attain  $\epsilon$ -optimal solutions for all instances, whereas with  $T = 16$ , it exceeded the specified time limit for 4 of the 10 instances.

**Table 3** Algorithm A3S with Varying  $T$ : Optimality Gap (%) and CPU Time (Sec)

	$T=2$	$T=4$	$T=8$	$T=16$
Averages	0.58 22.786	0.37 35.009	0.32 163.39	* *

\*Solution not obtained within 1,800 CPU seconds for 4 of 10 instances.

**Table 4** Algorithm A3S with Varying  $t'$ : Optimality Gap (%) and CPU Time (Sec)

	$t'=31$	$t'=63$	$t'=127$	$t'=255$
Averages	0.37 35.009	0.14 52.339	0.088 81.900	0.072 157.33

- An increase in  $t'$  consistently improved the value of the optimality gap attained. As displayed in Table 4, average computational times increased less than fivefold for an eightfold increase in  $t'$ , which was consistent across all instances.

- As evident from Tables 3 and 4, compared to increasing the number of MIP approximation segments, it is much more effective to tighten the linear approximation of the objective function with regard to the attained optimality gaps and computational times.

- In response to an increase in the countermeasure resources  $\xi_s, \forall s \in S$ , the objective function value decreased in near-linear proportions. Average performances are displayed in Table 5, where for our test instances the objective function values were reduced by approximately 66.33% on average from the corresponding values when no resources are applied. As it turns out, additional resources are applied where they are most effective, namely, near the points of inflection on the logit probability-resource relationships, ensuring the greatest marginal effect on probabilities with a near-linear relationship.

- Linearly proportional increases in the consequence-related resources  $\psi_r, \forall r \in R$ , caused a near-linear improvement in objective function values, as summarized in Table 6.

**Table 5** Effect of Varying  $\xi_s$  on the Optimal Solution Value (%)

	+10%	+20%	+30%	+40%
Average	-2.67	-5.45	-8.33	-11.32
Range	[-4.72, -1.53]	[-9.73, -3.09]	[-15.08, -4.65]	[-20.69, -6.25]

**Table 6** Effect of Varying  $\psi_r$  on the Optimal Solution Value (%)

	+10%	+20%	+30%	+40%
Average	-1.70	-3.42	-5.17	-6.94
Range	[-2.80, -0.61]	[-5.63, -1.21]	[-8.58, -1.81]	[-11.60, -2.40]

- The improvement in the objective function value for a proportional increase in the consequence-related resources  $\psi_r$  was only 61% to 64% as effective as an identical proportional increase in the countermeasure resources  $\xi_s$  on average, although this result did not hold uniformly. For instances 1, 3, and 4, proportional increases in  $\psi_r$  were actually more effective at reducing the optimal objective function value, indicating that a broader generalization is elusive.

## 6. Discussion and Recommendations

In this paper, we have studied the modeling and analysis of an optimization problem to combat terrorism using a nested event tree framework to address capability, intent, vulnerability, and consequence issues in the context of utilizing available resources to minimize the expected loss as a result of potential terrorist attacks. The importance of our model lies in its suitability for higher-level strategic planning for resource deployment. In addition to optimally allocating existing resources within budgetary constraints, our model has further value for conducting what-if scenario analyses, wherein a coordinating agency can assess the impact of adjusting budgets or resource levels, or it can seek to enhance effectiveness by considering the adjustment of resource levels between the different capability-, intent-, vulnerability-, and consequence-related domains. A principal component in applying our model is to adequately calibrate the resource-probability and resource-outcome logit model parameters. We suggest that there are sufficient data collected within recent decades to enable such a calibration (albeit potentially classified or requiring the cooperation of strategic partners), where the nature of the logit models enables separable calibration steps specific to each resource type. Furthermore, such calibration efforts might be facilitated by simulating response functions using game-theoretic models (e.g., see Zhuang and Bier 2007).

For future research, our model could be modified to better represent the current terrorist threat and the complexity of international relations by expanding it in scope to account for multiple terrorist organizations and for *partially aligned* interests between nations applying resources to combat terrorism. These considerations give rise to a cooperative game, thereby requiring alternative measures of optimality or equilibrium. Finally, we propose the application of our model to the resource prioritization phase of the U.S. National Infrastructure Protection Plan (Department of Homeland Security 2009) to validate priorities among and within the 18 sectors of critical infrastructure and key resources. Although the classified nature of the data, even in aggregated form, precludes addressing such a potential use here, such a study

might be of insightful value to the U.S. Department of Homeland Security as well as its federal, state, local, and private sector partners.

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### References

- Albores, P., D. Shaw. 2008. Government preparedness: Using simulation to prepare for a terrorist attack. *Comp. Oper. Res.* **35**(6) 1924–1943.
- Beim, G. K., B. F. Hobbs. 1997. Event tree analysis of lock closure risks. *J. Water Resources Planning Management* **123**(3) 169–178.
- Bier, V. M., N. Haphuriwat, J. Menoyo, R. Zimmerman, A. M. Culpen. 2008. Optimal resource allocation for defense of targets based on differing measures of attractiveness. *Risk Anal.* **28**(3) 763–770.
- Department of Homeland Security. 2009. *National Infrastructure Protection Plan: Partnering to Enhance Protection and Resiliency*. Washington, DC.
- Dillon, R. L., M. E. Paté-Cornell. 2005. Including technical and security risks in the management of information systems: A programmatic risk management model. *Systems Engrg.* **8**(1) 15–28.
- Golany, B., E. H. Kaplan, A. Marmur, U. G. Rothblum. 2009. Nature plays with dice—Terrorists do not: Allocating resources to counter strategic versus probabilistic risks. *Eur. J. Oper. Res.* **192**(1) 198–208.
- Homeland Security Council. 2007. *National Strategy for Homeland Security*. White House Office, Washington, DC. [http://www.dhs.gov/xlibrary/assets/nat\\_strat\\_homelandsecurity\\_2007.pdf](http://www.dhs.gov/xlibrary/assets/nat_strat_homelandsecurity_2007.pdf).
- Martin, A., M. Möller, S. Moritz. 2006. Mixed integer models for the stationary case of gas network optimization. *Math. Programming Ser. B* **105**(2–3) 563–582.
- Masse, T., S. O’Neil, J. Rollins. 2007. *The Department of Homeland Security’s risk assessment methodology: Evolution, issues, and options for Congress*. CRS Report for Congress, Congressional Research Service, Washington, DC.
- Mehr, A. F., I. Y. Tumer. 2006. Risk-based decision-making for managing resources during the design of complex space exploration systems. *J. Mech. Design* **128**(4) 1014–1022.
- National Counterterrorism Center. 2008. About the National Counterterrorism Center. Retrieved August 6, 2008: [http://www.nctc.gov/about\\_us/about\\_nctc.html](http://www.nctc.gov/about_us/about_nctc.html).
- National Security Council. 2006. *National Strategy for Combating Terrorism*. White House Office, Washington, DC. <http://georgewbush-whitehouse.archives.gov/nsc/nsct/2006/nsct2006.pdf>.
- Ryoo, H. S., N. V. Sahinidis. 1996. A branch-and-reduce approach to global optimization. *J. Global Optim.* **8**(2) 107–138.
- Scaparra, M. P., R. L. Church. 2008. A bilevel mixed-integer program for critical infrastructure protection planning. *Comp. Oper. Res.* **35**(6) 1905–1923.
- Sherali, H. D. 2001. On mixed-integer zero-one representations for separable lower-semicontinuous piecewise-linear functions. *Oper. Res. Lett.* **28**(4) 155–160.
- Sherali, H. D., C. H. Tuncbilek. 1997. Comparison of two reformulation-linearization technique based linear programming relaxations for polynomial programming problems. *J. Global Optim.* **10**(4) 381–390.
- Sherali, H. D., H. Wang. 2001. Global optimization of nonconvex factorable programming problems. *Math. Programming Ser. A* **89**(3) 459–478.
- Sherali, H. D., E. Dalkiran, T. S. Glickman. 2010. Selecting optimal alternatives and risk reduction strategies in decision trees. Working paper, Grado Department of Industrial and Systems Engineering, Virginia Polytechnic Institute and State University, Blacksburg.
- Sherali, H. D., J. Desai, T. S. Glickman. 2008. Optimal allocation of risk-reduction resources in event trees. *Management Sci.* **54**(7) 1313–1321.
- Stranlund, J. K., B. C. Field. 2006. On the production of homeland security under true uncertainty. Working Paper 2006–5, Department of Resource Economics, University of Massachusetts at Amherst, Amherst. <http://en.scientificcommons.org/38829130>.
- Vielma, J. S., S. Ahmed, G. Nemhauser. 2010. Mixed-integer models for nonseparable piecewise-linear optimization: Unifying framework and extensions. *Oper. Res.* **58**(2) 303–315.
- Willis, H. H., A. R. Morral, T. K. Kelly, J. J. Medby. 2005. *Estimating Terrorism Risk*. RAND Center for Terrorism Risk Management Policy, Washington, DC.
- Zhuang, J., V. M. Bier. 2007. Balancing terrorism and natural disasters—Defense strategy with endogenous attacker effort. *Oper. Res.* **55**(5) 976–991.